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The Krichever Map and Automorphic Line Bundles

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Abstract

A solution of the KP-hierarchy can be given by the τ-function or the Baker function associated to an element of the Grassmannian $\mathrm{Gr}(L^2(S^1))$ consisting of some subspaces of the space $L^2(S^1)$ of square-integrable functions on the unit circle S^1. The Krichever map associates an element $W \in \mathrm{Gr}(L^2(S^1))$ to a line bundle over a Riemann surface equipped with some additional data. We consider a line bundle over a modular curve associated to an automorphy factor J and prove that the elements of the image W of this bundle under the Krichever map can be characterized by a set of criteria involving J.

1 Introduction

Many well-known equations in mathematical physics such as the Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP), Boussinesq, Sine-Gordon, and nonlinear Schrödinger equations belong to the class of integrable nonlinear partial differential equations called soliton equations because they possess solitary waves, or solitons, as solutions. Soliton equations have been studied in numerous papers during the past few decades in connection with various topics in pure and applied mathematics. One way of systematically generating a large number of soliton equations is by using Lax equations, which are certain operator equations involving pseudodifferential operators (see e.g. [3, 5]). Thus, solutions of Lax equations determine solutions of the associated soliton equations. One of the important contributions of the Japanese school was the interpretation of solutions of Lax equations in terms of τ-functions (see e.g. [2]). Such solutions can also be expressed in terms of Baker functions, which can be written as quotients of values of τ-functions. A τ-function or a Baker function can be determined by using a certain infinite Grassmannian associated to a line bundle over a Riemann surface.

Let $\mathcal{A}$ be the algebra over $\mathbb{C}$ consisting of polynomials in formal symbols $\{u_i^{(j)}\}$ on which the differentiation operator $\partial = d/dx$ acts by

$$\partial(fg) = (\partial f)g + f(\partial g), \quad \partial u_i^{(j)} = u_i^{(j+1)} \tag{1.1}$$

for all $f, g \in \mathcal{A}$ and integers $i, j \geq 0$. Then a pseudodifferential operator associated to $\mathcal{A}$ is a formal sum of the form

$$D = \sum_{i=-\infty}^{m} D_i \partial^i \tag{1.2}$$

for some integer m with $D_i \in \mathcal{A}$ for all $i \leq m$. We denote by $\Psi\mathrm{DO}_{\mathcal{A}}$ the set of all pseudodifferential operators associated to $\mathcal{A}$. Then $\Psi\mathrm{DO}_{\mathcal{A}}$ has the structure of an algebra over $\mathbb{C}$ whose multiplication operation is defined by using (1.1). If $D \in \Psi\mathrm{DO}_{\mathcal{A}}$ is the pseudodifferential operator in (1.2), then we denote by

$$D_+ = \sum_{i=0}^{m} D_i \partial^i$$

the differential part of D. In order to describe the KP-hierarchy, we consider a pseudodifferential operator $L \in \Psi\mathrm{DO}_{\mathcal{A}}$ of the form

$$L = \partial + u_0 \partial^{-1} + u_1 \partial^{-2} + \cdots,$$

where $u_0, u_1, \ldots$ are the generators of a differential algebra $\mathcal{A}$ and are regarded as functions of the indeterminates $x_1, x_2, \ldots$ with $x_1 = x$. If we set $B_m = (L^m)_+$ for each positive integer m, then the KP-hierarchy is the set of partial differential equations produced by an operator equation of the form

$$\frac{\partial L}{\partial x_m} = [B_m, L]$$

for $m \geq 1$. The equations of this type also imply the zero curvature equations

$$\frac{\partial B_n}{\partial x_m} - \frac{\partial B_m}{\partial x_n} = [B_m, B_n]$$

for positive integers m and n. Then the KP-equation, for example, is obtained by specializing these equations to the case of $n = 3$ and $m = 2$. Note that the KdV and Boussinesq equations can be obtained from the KP-equation by simple reductions.

Let $H = L^2(S^1)$ be the space of square-integrable complex-valued functions on the unit circle S^1. In [9] Segal and Wilson considered the Grassmannian $\mathrm{Gr}(H)$ consisting of the subspaces of H satisfying certain conditions and discussed connections between elements of $\mathrm{Gr}(H)$ and solutions of soliton equations. Let W be an element of $\mathrm{Gr}(H)$, and let $h(z)$ be a holomorphic function on the closed disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ defined by

$$h(z) = \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)$$

with $t_k \in \mathbb{R}$ for each k. Then the associated Baker function $w_W(h, z)$ is a function of the form

$$w_W(h, z) = h(z)\left(1 + \sum_{i=-\infty}^{-1} a_i z^i\right)$$

with $a_i \in \mathcal{A}$ for each i such that the map $z \mapsto w_W(h,z)$ is an element of W. If $\phi = 1 + \sum_{i=-\infty}^{-1} a_i z^i$, then $L = \phi \partial \phi^{-1}$ is a solution of the KP-hierarchy. A solution of the KP-hierarchy can also be expressed in terms of the τ-function τ_W associated to such an element W of $\mathrm{Gr}(H)$ (see [3] and [9] for details).

One way of obtaining elements of $\mathrm{Gr}(H)$ is by the Krichever map, introduced by Krichever [4] (see also [9]) which associates an element $W \in \mathrm{Gr}(H)$ to a line bundle over a Riemann surface equipped with some additional data. In this paper, we consider a line bundle over the quotient $\Gamma\backslash\mathcal{H}$ of the Poincaré upper half plane by a discrete subgroup Γ of $SL(2,\mathbb{R})$ determined by an automorphy factor J for Γ and prove that the elements of the image W of this bundle under the Krichever map can be characterized by a set of criteria involving J.

2 Automorphic line bundles

In this section we construct a line bundle over the quotient of the Poincaré upper half plane by a discrete subgroup Γ of $SL(2,\mathbb{R})$ associated to an automorphy factor for Γ and discuss some of its properties.

Let $\mathcal{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ be the Poincaré upper half plane, and let $SL(2,\mathbb{R})$ be the group of real 2×2 matrices of determinant one. Then $SL(2,\mathbb{R})$ acts on $\mathcal{H}$ by linear fractional transformations, that is,

$$gz = \frac{az+b}{cz+d}$$

for all $z \in \mathcal{H}$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$, and let $J : \Gamma \times \mathcal{H} \to \mathbb{C}^\times$ with $\mathbb{C}^\times = \mathbb{C} - \{0\}$ be an automorphy factor of Γ. Thus J satisfies the cocycle condition

$$J(\gamma\gamma', z) = J(\gamma, \gamma' z) J(\gamma', z) \tag{2.1}$$

for all $z \in \mathcal{H}$ and $\gamma, \gamma' \in \Gamma$. Throughout the rest of this paper we assume that the discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$ is cocompact, which means that the associated Riemann surface $X = \Gamma\backslash\mathcal{H}$ is a compact. Such a Riemann surface X can be regarded as a complex projective algebraic curve and is known as a *modular curve.*

Given an automorphy factor J of Γ, we set

$$\gamma \cdot (z, \lambda) = (\gamma z, J(\gamma, z)\lambda) \tag{2.2}$$

for $\gamma \in \Gamma$ and $(z,\lambda) \in \mathcal{H} \times \mathbb{C}$.

Example. Given a nonnegative integer m, we define the map $j^m : \Gamma \times \mathcal{H} \to \mathbb{C}^\times$ by

$$j^m(\gamma, z) = (cz+d)^m \tag{2.3}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. Then it can be shown that j^m is an automorphy factor of Γ. Such an automorphy factor determines modular forms, which play an important role in number theory. Indeed, a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a *modular form for* Γ *of weight* m if

$$f(\gamma z) = j^m(\gamma, z) f(z) \tag{2.4}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$ (see e.g. [7]). Note that the usual definition of modular forms also includes a cusp condition, which is not needed in our case because Γ is cocompact.

Lemma 1. *The formula* (2.2) *defines an action of* Γ *on* $\mathcal{H} \times \mathbb{C}$.

Proof. Given elements $\gamma, \gamma' \in \Gamma$, $z \in \mathcal{H}$, and $\lambda \in \mathbb{C}$, using (2.1) and (2.2), we have

$$\begin{aligned}(\gamma\gamma') \cdot (z, \lambda) &= (\gamma\gamma' z, J(\gamma\gamma', z)\lambda)\\ &= (\gamma\gamma' z, J(\gamma, \gamma' z) J(\gamma', z)\lambda)\\ &= \gamma \cdot (\gamma' z, J(\gamma', z)\lambda) = \gamma \cdot (\gamma' \cdot (z, \lambda));\end{aligned}$$

hence the lemma follows. ■

We denote the quotient of the space $\mathcal{H} \times \mathbb{C}$ by the discrete group Γ with respect to the action in Lemma 1 by

$$\mathcal{L}_{\Gamma,J} = \Gamma \backslash \mathcal{H} \times \mathbb{C}. \tag{2.5}$$

Then the natural projection map $\mathcal{H} \times \mathbb{C} \to \mathcal{H}$ induces a surjective holomorphic map $\pi : \mathcal{L}_{\Gamma,J} \to X = \Gamma\backslash\mathcal{H}$ such that the inverse image of each element of X is isomorphic to $\mathbb{C}$. Thus $\mathcal{L}_{\Gamma,J}$ has the structure of a line bundle over the Riemann surface X. We denote by $\mathbf{\Gamma}(X, \mathcal{L}_{\Gamma,J})$ the space of holomorphic sections of the bundle $\mathcal{L}_{\Gamma,J}$ over X, that is, holomorphic maps $s : X \to \mathcal{L}_{\Gamma,J}$ such that $\pi \circ s = 1_X$ with 1_X being the identity map on X. We denote by $\varpi : \mathcal{H} \to X = \Gamma\backslash\mathcal{H}$ the natural projection map, and set

$$\mathcal{H}_Y = \varpi^{-1}(Y)$$

if Y is a subset of X.

Proposition 1. *Given a subset of* $V \subset X$*, the space* $\mathbf{\Gamma}(V, \mathcal{L}_{\Gamma,J})$ *of holomorphic sections of* $\mathcal{L}_{\Gamma,J}$ *over* V *is isomorphic to the space of holomorphic functions* $f : \mathcal{H}_V \to \mathbb{C}$ *on* $\mathcal{H}_V = \varpi^{-1}(V)$ *satisfying*

$$f(\gamma z) = J(\gamma, z) f(z) \tag{2.6}$$

for all $\gamma \in \Gamma$ *and* $z \in \mathcal{H}_V$.

Proof. Let $s : V \to \pi^{-1}(V) \subset \mathcal{L}_{\Gamma,J}$ be an element of $\mathbf{\Gamma}(V, \mathcal{L}_{\Gamma,J})$. Then for each $z \in \mathcal{H}_V = \varpi^{-1}(V)$ we have

$$s(\varpi z) = [(z, \lambda_z)] \in \pi^{-1}(V) = \Gamma\backslash\mathcal{H}_V \times \mathbb{C}$$

for some $\lambda_z \in \mathbb{C}$, where $[(z, \lambda_z)] \in \mathcal{L}_{\Gamma,J}$ denotes the Γ-orbit corresponding to the element $(z, \lambda_z) \in \mathcal{H} \times \mathbb{C}$. We define the function $f_s : \mathcal{H}_V \to \mathbb{C}$ by $f_s(z) = \lambda_z$ for all $z \in \mathcal{H}_V$. Using (2.2), we have

$$\begin{aligned}s(\varpi z) = s(\varpi(\gamma z)) &= [(\gamma z, \lambda_{\gamma z})] = [\gamma^{-1} \cdot (\gamma z, \lambda_{\gamma z})]\\ &= [(z, J(\gamma^{-1}, \gamma z) \cdot \lambda_{\gamma z})],\end{aligned}$$

for each $\gamma \in \Gamma$ and $z \in \mathcal{H}_V$. Hence it follows that

$$f_s(z) = J(\gamma^{-1}, \gamma z) \cdot \lambda_{\gamma z} = J(\gamma^{-1}, \gamma z) \cdot f_s(\gamma z) \tag{2.7}$$

for all $z \in \mathcal{H}_V$ and $\gamma \in \Gamma$. If e denotes the identity element in $SL(2, \mathbb{R})$, by (2.1) we have $J(e, z) = J(e^2, z) = J(e, z)^2$; hence $J(e, z) = 1 \in \mathbb{C}$. Thus we obtain

$$1 = J(\gamma^{-1}\gamma, z) = J(\gamma^{-1}, \gamma z) \cdot J(\gamma, z),$$

and therefore we have $J(\gamma^{-1}, \gamma z) = J(\gamma, z)^{-1}$. From this and (2.7) we see that f_s satisfies (2.6). On the other hand, if $f : \mathcal{H}_V \to \mathbb{C}$ is a holomorphic function satisfying (2.6), we define the map $s_f : V \to \pi^{-1}(V) \subset \mathcal{L}_{\Gamma,J}$ by

$$s_f(\varpi z) = [(z, f(z))]$$

for all $z \in \mathcal{H}_V$. This map is well-defined because, for each $\gamma \in \Gamma$ and $z \in \mathcal{H}$, we have

$$\begin{aligned} s_f(\varpi(\gamma z)) &= [(\gamma z, f(\gamma z))] = [(\gamma z, J(\gamma, z) f(z))] \\ &= [\gamma \cdot (z, f(z))] = [(z, f(z))] = s_f(\varpi z). \end{aligned}$$

Since the relation $\pi \circ s_f = 1_X$ obviously holds, s_f is an element of $\mathbf{\Gamma}(V, \mathcal{L}_{\Gamma,J})$; hence the proof of the proposition is complete. ■

Corollary 2. *The space $\mathbf{\Gamma}(X, \mathcal{L}_{\Gamma,J})$ of holomorphic sections of $\mathcal{L}_{\Gamma,J}$ over $X = \Gamma\backslash\mathcal{H}$ is isomorphic to the space of modular forms of weight m for Γ.*

Proof. This follows from Proposition 1 by letting $V = X$ and $\mathcal{H}_V = \mathcal{H}$. ■

3 Grassmannians

In this section, we review the Krichever map which associates an infinite Grassmannian W to a line bundle over a Riemann surface following the description in [9] (see also [3]). We then prove our main theorem, which provides a condition for a function on a unit circle to belong to such W determined by a line bundle of the type discussed in Section 2.

Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle centered at the origin in the complex plane $\mathbb{C}$, and let $H = L^2(S^1)$ be the Hilbert space consisting of the square-integrable complex-valued functions on S^1. Then the functions $z \mapsto z^k$ on S^1 for $k \in \mathbb{Z}$ form a basis of H. We consider the subspaces H_+ and H_- of H defined by

$$H_+ = \bigoplus_{k \geq 0} \mathbb{C}z^k, \quad H_- = \bigoplus_{k<0} \mathbb{C}z^k,$$

which determine the decomposition

$$H = H_+ \oplus H_-$$

of H. Let p_+ and p_- be the orthogonal projections of H onto H_+ and H_-, respectively.

Definition 1. The *Grassmannian* of H is the set $\mathrm{Gr}(H)$ of all subspaces W of H satisfying the following two conditions:

(i) Both the kernel and the cokernel of the restriction $p_+|_W : W \to H_+$ of the map p_+ to W are finite-dimensional.

(ii) The restriction $p_-|_W : W \to H_-$ of the map p_- to W is a compact operator.

Let $X = \Gamma\backslash\mathcal{H}$ be the compact Riemann surface considered in Section 2, and fix a point $x^\infty \in X$. Let ζ^{-1} be a local parameter in a neighborhood U^∞ of this point. Thus there is a neighborhood $\widetilde{U}^\infty$ of 0 in the complex plane $\mathbb{C}$ and an isomorphism $\alpha : U^\infty \to \widetilde{U}^\infty$ such that $\alpha(x^\infty) = \zeta(x^\infty)^{-1} = 0$ (or $\zeta(x^\infty) = \infty$) and $\alpha(x) = \zeta(x)^{-1}$ for all $x \in U^\infty$. We set

$$X^\infty = \{x \in X \mid |\zeta(x)^{-1}| \leq 1\} = \{x \in X \mid |\zeta(x)| \geq 1\}. \tag{3.1}$$

By rescaling the parameter ζ if necessary we may assume that X^∞ is contained in U^∞. We denote by $X^0 = \overline{X - X^\infty}$ the closure of $X - X^\infty$.

We now consider a holomorphic line bundle $\mathcal{L} = \mathcal{L}_{\Gamma,J}$ over the Riemann surface $X = \Gamma\backslash\mathcal{H}$ given by (2.5) associated to a cocompact discrete subgroup $\Gamma \subset SL(2,\mathbb{R})$ and an automorphy factor J of Γ. Let $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}$ be a trivialization of $\mathcal{L}$ over an open set $U \subset X$ containing X^∞. Then a holomorphic section $s : U \to \pi^{-1}(U)$ of $\mathcal{L}$ over U can be regarded as a holomorphic function $f_s : U \to \mathbb{C}$ on U such that

$$(\varphi \circ s)(x) = (x, f_s(x)) \in U \times \mathbb{C} \tag{3.2}$$

for all $x \in U$. Let W^0 be the set of analytic functions on S^1 which extend to holomorphic sections of $\mathcal{L}$ over X^0. Thus an analytic function $h : S^1 \to \mathbb{C}$ belongs to W^0 if and only if there is a section $s : X^0 \to \pi^{-1}(X_0)$ of $\mathcal{L}$ over X_0 such that

$$f_s(x) = h(\zeta^{-1}(x)) = h(\zeta(x)^{-1}) \tag{3.3}$$

for all $x \in X^0 \cap X^\infty$, where f_s is the function on X_0 corresponding to s by the relation (3.2). Then the L^2-closure W of W^0 is an element of $\mathrm{Gr}(H)$ (see [9] for details), and it is the element associated to the 5-tuple $(X, \mathcal{L}, x^\infty, \zeta, \varphi)$ under the Krichever map.

Theorem 3. *Let W be the element of $\mathrm{Gr}(H)$ determined by applying the Krichever map to 5-tuple $(X, \mathcal{L}, x^\infty, \zeta, \varphi)$ with $\mathcal{L} = \mathcal{L}_{\Gamma,J}$ described above. A function $f : S^1 \to \mathbb{C}$ belongs to W if and only if there are sequences $\{f_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ of holomorphic functions $f_n : S^1 \to \mathbb{C}$ and $h_n : \mathcal{H}_{X_0} \to \mathbb{C}$ satisfying the following conditions:*

(i) The sequence $\{f_n\}_{n=1}^\infty$ converges to f with respect to the L^2-norm.

(ii) For each positive integer n the function h_n satisfies

$$h_n(\gamma z) = J(\gamma, z) h_n(z) \tag{3.4}$$

for all $\gamma \in \Gamma$ and $z \in \mathcal{H}_{X^0} = \varpi^{-1}(X^0)$.

(iii) For each positive integer n the function f_n can be written in the form

$$f_n(\zeta(\varpi(z))^{-1}) = (\mathrm{Pr}_2 \circ \varphi)([z, h_n(z)])$$

for all $z \in \varpi^{-1}(X^0 \cap X^\infty)$, where $\mathrm{Pr}_2 : U \times \mathbb{C} \to \mathbb{C}$ is the natural projection map.

Proof. By the definition of W a function $f : S^1 \to \mathbb{C}$ is an element of W if and only if there is a sequence $\{f_n\}_{n=1}^{\infty}$ of elements f_n of W^0 which converges to f with respect to the L^2-norm. Thus it suffices to show that the functions f_n satisfying the conditions stated above are elements of W^0. Given a positive integer n, by (3.3) the function f_n belongs to W^0 if and only if there exists a holomorphic section $s_n : X^0 \to \pi^{-1}(X^0)$ of $\mathcal{L}$ over X^0 such that

$$f_{s_n}(x) = f_n(\zeta^{-1}(x))$$

for all $x \in S^1 = X^0 \cap X^\infty$, where f_{s_n} is the function on X^0 determined from s_n by using (3.2). Thus we have

$$(\varphi \circ s_n)(x) = (x, f_{s_n}(x)) = (x, f_n(\zeta^{-1}(x))$$

for all $x \in X^0 \cap X^\infty$, which can be written as

$$\varphi(s_n(\varpi z)) = (\varpi z, f_n(\zeta(\varpi z)^{-1})) \in (X^0 \cap X^\infty) \times \mathbb{C} \tag{3.5}$$

for all $z \in \varpi^{-1}(X^0 \cap X^\infty)$. By Proposition 1 the existence of the section s_n is equivalent to the existence of a holomorphic function $h_n : \mathcal{H}_{X^0} \to \mathbb{C}$ satisfying the condition (3.4). Furthermore, the functions f_n and s_n are related by

$$s_n(\varpi z) = [z, h_n(z)]$$

for all $z \in \mathcal{H}_{X^0}$ as described in the proof of Proposition 1. By combining this with (3.5) we obtain

$$\varphi([z, h_n(z)]) = (\varpi(z), f_n(\zeta(\varpi(z))^{-1}) \in (X^0 \cap X^\infty) \times \mathbb{C}$$

for all $z \in \varpi^{-1}(X^0 \cap X^\infty)$. Hence we have

$$f_n(\zeta(\varpi(z))^{-1}) = (\mathrm{Pr}_2 \circ \varphi)([z, h_n(z)]),$$

and therefore the proof of the theorem is complete. ■

Example. Let $M(2,\mathbb{R})$ be the algebra of 2×2 matrices over $\mathbb{R}$, and let $\mathcal{O}$ be an order in $M(2,\mathbb{R})$, i.e., a subring containing the identity element of $M(2,\mathbb{R})$. Then the set

$$\mathfrak{A} = \mathbb{Q} \cdot \mathcal{O} = \{rx \mid r \in \mathbb{Q},\ x \in \mathcal{O}\}$$

becomes an indefinite quaternion algebra over $\mathbb{Q}$, and we have

$$\mathfrak{A} \otimes_{\mathbb{Q}} \mathbb{R} = M(2,\mathbb{R}).$$

We fix a positive integer q, and set

$$\Gamma(\mathcal{O}, q) = \{x \in M(2,\mathbb{R}) \mid x\mathcal{O} = \mathcal{O},\ \det x = 1,\ x - 1 \in q\mathcal{O}\}.$$

Then $\Gamma(\mathcal{O}, q)$ is a discrete subgroup of $SL(2,\mathbb{R})$. We assume that $\mathcal{O}$ is chosen in such a way that $\mathfrak{A} = \mathbb{Q} \cdot \mathcal{O}$ is a division algebra. Then it is known that the corresponding quotient space $\Gamma(\mathcal{O}, q)\backslash\mathcal{H}$ is compact (cf. [7, Theorem 5.2.13]). We set

$$\Gamma = \Gamma(\mathcal{O}, q), \quad X = \Gamma\backslash\mathcal{H},$$

so that X is a compact Riemann surface. Given a nonnegative integer m, let j^m be the automorphy factor given by (2.3). We denote by $\mathcal{L}$ the associated line bundle constructed in Example 2. We consider a sequence $\{h_n\}_{n=1}^{\infty}$ of modular forms $h_n : \mathcal{H} \to \mathbb{C}$ for Γ of weight m which converges to a function $h : \mathcal{H} \to \mathbb{C}$ with respect to the L^2-norm, and define the functions $f, f_n : S^1 \to \mathbb{C}$ by

$$f(\zeta(\varpi(z))^{-1}) = (\mathrm{Pr}_2 \circ \varphi)([z, h(z)]), \tag{3.6}$$

$$f_n(\zeta(\varpi(z))^{-1}) = (\mathrm{Pr}_2 \circ \varphi)([z, h_n(z)])$$

for all $z \in \varpi^{-1}(X^0 \cap X^\infty)$ and $n \geq 1$. Then we have

$$\begin{aligned} f(\zeta(\varpi(z))^{-1}) &= (\mathrm{Pr}_2 \circ \varphi)([z, \lim_{n\to\infty} h_n(z)]) \\ &= \lim_{n\to\infty} (\mathrm{Pr}_2 \circ \varphi)([z, h_n(z)]) = \lim_{n\to\infty} f_n(\zeta(\varpi(z))^{-1}); \end{aligned}$$

hence the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f in the L^2-norm. On the other hand, since each h_n is a modular form for Γ of weight m, by (2.4) the function h_n satisfies (3.4) for $J = j^m$. Thus by Theorem 3 the function $f : S^1 \to \mathbb{C}$ given by (3.6) is an element of W.

4 Concluding remarks

As is well-known, pseudodifferential operators play an important role in the theory of soliton equations. On the other hand, pseudodifferential operators are also linked to the theory of modular forms, which is a major part of number theory (see e.g. [1, 6]). These observations suggest that there is at least an indirect relation between soliton equations and modular forms. Indeed, such relations have been explored in a number of papers (see e.g. [8, 10, 11]). In Example 3 we provided another, rather weak, connection between modular forms and solutions of soliton equations. It would be interesting to investigate a more direct link between those two areas.

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References

[1] Cohen P, Manin Y, and Zagier D, Automorphic Pseudodifferential Operators, *Algebraic Aspects of Nonlinear Systems*, Birkhäuser, Boston (1997), 17–47.

[2] Date E, Kashiwara M, Jimbo M, and Miwa T, Transformation Groups for Soliton Equations, Nonlinear integrable systems – classical theory and quantum theory (Kyoto, 1981), World Scientific, Singapore (1983), 39–119.

[3] Dickey L, Soliton Equations and Hamiltonian Systems, World Scientific, Singapore, 1991.

[4] Krichever A, Methods of algebraic geometry in the theory of non-linear equations, *Russian Math. Surveys* **32** (1977), 185–213.

[5] Kupershmidt B A, KP or mKP: Noncommutative Mathematics of Lagrangian, Hamiltonian, and Integrable Systems, Amer. Math. Soc., Providence, 2000.

[6] Lee M H, Hilbert Modular Pseudodifferential Operators, Proc. Amer. Math. Soc. **129** (2001), 3151–3160.

[7] Miyake T, Modular Forms, Springer-Verlag, Heidelberg, 1989.

[8] Mumford D, Tata Lectures on Theta II, Birkhäuser, Boston, 1984.

[9] Segal G and Wilson G, Loop Groups and Equations of KdV, Publ. Math. I.H.E.S. **61** (1985), 5–65.

[10] Takhtajan L, A Simple Example of Modular Forms as tau-Functions for Integrable Equations, Theor. and Math. Phys. **93** (1992), 1308–1317.

[11] Takhtajan L, Modular Forms as τ-Functions for Certain Integrable Reductions of the Yang-Mills Equations, Progress in Math., vol. 115, Birkhäuser, Boston (1993), 115–129.