

A Generalization of the Sine-Gordon Equation to $2 + 1$ Dimensions

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Abstract

The Singular Manifold Method (SMM) is applied to an equation in $2 + 1$ dimensions [13] that can be considered as a generalization of the sine-Gordon equation. SMM is useful to prove that the equation has two Painlevé branches and, therefore, it can be considered as the modified version of an equation with just one branch, that is the AKNS equation in $2 + 1$ dimensions. The solutions of the former split as linear superposition of two solutions of the second, related by a Bäcklund-gauge transformation. Solutions of both equations are obtained by means of an algorithmic procedure derived from these transformations.

1 Introduction

In recent papers, one of us (PGE) [7], [8], [9] has used the singular manifold method (SMM) [18], based on the Painlevé test [19], in order to obtain an unified point of view of the properties of nonlinear partial differential equations (PDE's). Our aim is to prove that practically all features of a PDE can be obtained from the singular manifold equations derived from the application of the SMM. Of particular interest is the case in which the equation has two Painlevé branches. In this case, the SMM needs to be improved to incorporate two singular manifolds [5], [7], [9]. Miura transformations arise in a very natural way that connects the equation under scrutiny with another equation (that can be considered as its modified version) with just one Painlevé branch.

Let us apply this procedure to the following non-linear system in $(2 + 1)$ dimensions:

$$\begin{aligned} 0 &= \eta_x + u^2 \\ 0 &= u_{xy} + 2u\eta_y + 4\omega \\ 0 &= u_t - \omega_x \end{aligned} \tag{1.1}$$

where $u = u(x, y, t)$, $\eta = \eta(x, y, t)$, $\omega = \omega(x, y, t)$. This system can also be trivially written as the following equation for u .

$$0 = 4u_t + u_{xxy} - 4u^2u_y - 4u_x \int uu_y dx \quad (1.2)$$

that appears in [2] as the modified version of a “breaking soliton” equation

$$4m_{xt} + m_{xxxy} + 8m_x m_{xy} + 4m_y m_{xx} = 0 \quad (1.3)$$

introduced by Calogero [3]. In the paper of Bogoyavlenskii [2], the Lax pair and a non-isospectral condition for the spectral parameter is presented as well as the Miura transformations between (1.2) and (1.3).

Equation (1.2) was derived by Kudryashov and Pickering [13] as a member of a (2 + 1) Schwarzian breaking soliton hierarchy. Rational solutions of it were obtained by these authors. The equation appears also in reference [4] as one of the equations associated to non-isospectral scattering problems.

Toda and Yu presented in 2000 [17] the equation

$$0 = 4W_t + W_{xxy} - 2\frac{W_x W_{xy}}{W} - \frac{W_y W_{xx}}{W} + 2\frac{W_x^2 W_y}{W^2} - \frac{W_x}{2} \int \left(\frac{W_x^2}{W^2} \right)_y dx \quad (1.4)$$

that is trivially related to (1.4) through the change

$$W = e^{2 \int u dx}$$

In this paper [17], equation (1.4) was named (2 + 1)-dimensional Schwarz-Korteweg-de Vries equation. Solutions of this equation have been obtained in [11] by means of the classical Lie method.

Let us notice that Eq (1.1) can be considered as a generalization to 2 + 1 dimensions of the sine-Gordon equation. Actually if we set $w = 0$ in Eq (1.1), the system reduces to

$$0 = q_{xy} - \sin q \quad (1.5)$$

where $q = q(x, y)$ is related with u and η through the changes:

$$\begin{aligned} u &= -\frac{i q_x}{2} \\ \eta_y &= -\frac{1}{2} \cos q \\ u_y &= -\frac{i}{2} \sin q \end{aligned} \quad (1.6)$$

Furthermore, (1.2) can be considered as a generalization to 2 + 1 dimensions of the well known AKNS (Ablowitz-Kaup-Newell-Segur) equation [1], [4], [16]. In [8], one of us studied the relationship between sine-Gordon and AKNS by means of the SMM. Our aim in the present work is to apply to (1.2) and (1.3) the method developed in [8]. Let us summarize the plan of the paper:

- In section 2, we notice that the Painlevé expansion for (1.1) has two different Painlevé branches which allows us to split the solutions of (1.1) as linear superposition of two solutions, related by an auto-Bäcklund transformation, of an equation with just one Painlevé branch that turns out to be (1.3).

- The singular manifold method (SMM) is used in section 3 to study 2+1-AKNS equation (1.3). We present the SMM as an unified tool that allow us to identify many of the properties of the equation. In particular, we recover the Lax pair and the nonisospectral condition [4] for the spectral parameter.

A crucial point of this paper is that the SMM can be applied to the Lax pair itself providing us transformations that map the Lax pair into a new one and giving us a simple and iterative method to construct solutions. These transformations have been called Bäcklund-gauge transformations by same authors [12] and sometimes [6] have been considered as Darboux transformations in the sense that they are transformations of the fields and eigenfunctions of the Lax pair. Nevertheless, they are not the usual binary Darboux transformations of the Schrödinger operator that appears, for instance, in [14] or [10].

- Section 4 deals again with the fact that equation (1.1) has two Painlevé branches. It means that we need to introduce two different singular manifolds (one for each branch) that are not independent because each one is related with one of the two different solutions of (1.3) that have been used to split the solutions of (1.1). This splitting allow us to reconstruct the Lax pair for (1.1) in terms of two eigenfunctions connected with the two singular manifolds. Bäcklund-gauge transformations and iterated solutions of (1.1) can be obtained through the same connection.

- Section (5) is devoted to use the above described method to get some particular solutions for (1.1) and (1.3).

- Conclusions are listed in the pertinent section.

2 Painlevé branches for (1.1)

It is not difficult to prove that (1.1) passes the Painlevé test for PDEs [13], [4], [16]. To check it, it is necessary to expand the fields u , η and ω as a local expansion in the neighborhood of a movable singular manifold $\phi(x, y, t)$. It means

$$\begin{aligned}
 u &= \sum_{j=0}^{\infty} u_j(x, y, t) [\phi(x, y, t)]^{j-1} \\
 \omega &= \sum_{j=0}^{\infty} \omega_j(x, y, t) [\phi(x, y, t)]^{j-1} \\
 \eta &= \sum_{j=0}^{\infty} \eta_j(x, y, t) [\phi(x, y, t)]^{j-1}
 \end{aligned} \tag{2.1}$$

The substitution of (2.1) in (1.1) provides easily that

$$\eta_0 = \phi_x, \quad u_0 = \pm \phi_x, \quad w_0 = \pm \phi_x$$

and there are resonances at $j = 2, 3, 4$ which means that the coefficients $u_2(x, y, t)$, $u_3(x, y, t)$ and $u_4(x, y, t)$ are arbitrary functions. It is a trivial exercise to check that the conditions at the resonances are satisfied. Therefore, (1.1) passes the Painlevé test. Nevertheless the \pm sign of u_0 indicates that the system has two Painlevé branches. As one of us (PGE) has shown for several cases ([5], [7], [9]), when an equation has two branches, it means that the solution can be considered as the linear superposition of two different solutions (related by a Bäcklund transformation) of an equation with just one branch. In our case, it means that we need to consider two new functions m and \hat{m} such that:

$$\begin{aligned} u &= m - \hat{m} \\ \eta &= m + \hat{m} \\ w &= \int (m_t - \hat{m}_t) dx \end{aligned} \quad (2.2)$$

The system (1.1) is now:

$$\begin{aligned} 0 &= m_x + \hat{m}_x + (m - \hat{m})^2 \\ 0 &= m_{xy} - \hat{m}_{xy} + 2(m - \hat{m})(m_y + \hat{m}_y) + 4 \int (m_t - \hat{m}_t) dx \end{aligned} \quad (2.3)$$

The inverse of Eq (2.2) is:

$$\begin{aligned} m &= \frac{u + \eta}{2} \\ \hat{m} &= \frac{-u + \eta}{2} \end{aligned}$$

By using $\eta_x = -u^2$, we can write:

$$\begin{aligned} m_x &= \frac{u_x - u^2}{2} \\ \hat{m}_x &= \frac{-u_x - u^2}{2} \end{aligned} \quad (2.4)$$

that allows us to write differential equations for m and \hat{m} . Direct calculation yields to:

$$4m_{xt} + m_{xxxxy} + 8m_x m_{xy} + 4m_y m_{xx} = 0 \quad (2.5)$$

$$4\hat{m}_{xt} + \hat{m}_{xxxxy} + 8\hat{m}_x \hat{m}_{xy} + 4\hat{m}_y \hat{m}_{xx} = 0 \quad (2.6)$$

that is the AKNS equation in 2 + 1 dimensions [2], [3].

In consequence, the simple ansatz (2.2) has provided us the following results:

a) we can split the solutions of Eq (1.1) as a **linear superposition** (2.2) of two different solutions m and \hat{m} that satisfy the same equation (2.5), (2.6).

b) This two solutions are related by the **auto-Bäcklund transformation** (2.3).

c) The inverse transformation associated to this splitting is (2.4). This is nothing but a **Miura transformation** between (1.1) and (2.5)-(2.6)[2].

3 Singular manifold method for Eq (1.3)

It is rather easy to apply the singular manifold method [18] for Eq (1.3). It suffices to check that the leading index is 1 and therefore a truncated Painlevé expansion for Eq (1.3) takes the form of an **auto-Bäcklund transformation** (2.3).

$$m' = m + \frac{\phi_x}{\phi} \quad (3.1)$$

where m and m' are two solutions of (1.3) and $\phi(x, y, t)$ is the **singular manifold**. Substitution of (3.1) in (1.3) yields to a polynomial in ϕ whose coefficients provide the expression of the derivatives of m in terms of the singular manifold in the following way (see Appendix A):

$$m_x = -\frac{v_x}{4} - \frac{v^2}{8} + \frac{\lambda(y, t)}{2} \quad (3.2)$$

$$m_y = -\frac{v_y}{2} - \lambda(y, t)q - r \quad (3.3)$$

where

$$\begin{aligned} v(x, y, t) &= \frac{\phi_{xx}}{\phi_x} \\ r(x, y, t) &= \frac{\phi_t}{\phi_x} \\ q(x, y, t) &= \frac{\phi_y}{\phi_x} \end{aligned} \quad (3.4)$$

and $\lambda(y, t)$ satisfies:

$$\lambda_t + \lambda\lambda_y = 0 \quad (3.5)$$

Singular manifold equations

Furthermore, the singular manifold ϕ should satisfy the equations:

$$0 = r_x + \frac{v_{xy} - vv_y}{4} + \lambda q_x + \frac{\lambda_y}{2} \quad (3.6)$$

$$v_t = (r_x + rv)_x \quad (3.7)$$

$$v_y = (q_x + qv)_x \quad (3.8)$$

(3.7) and (3.8) are a direct consequence of the compatibility between the definitions (3.4). Eq(3.6) is the **specific singular manifold equation**. It is interesting to point out that λ appears as the consequence of an integration in x . We thank A. Pickering for his observation that λ is not necessarily a constant but a function of y and t that satisfies (3.5), [4].

Lax pair

The Lax pair for (1.3) is easily obtained from (3.2)-(3.3). Eq (3.2) can be considered as a Riccati equation for v . Therefore it can be linearized by introducing a function $\psi(x, y, t)$ such that

$$v = 2 \frac{\psi_x}{\psi} \quad (3.9)$$

that combined with (3.4) means that

$$\phi_x = \psi^2 \quad (3.10)$$

With this definition of ψ , (3.2) can be written as:

$$0 = \psi_{xx} + (2m_x - \lambda)\psi \quad (3.11)$$

The same substitution can be used in the singular manifold equation Eq(3.6) that combined with Eq(3.3), Eq(3.7) and Eq(3.8) yields to

$$0 = \psi_t + \lambda\psi_y + m_y\psi_x + \frac{1}{2}(\lambda_y - m_{xy})\psi \quad (3.12)$$

Compatibility condition of Eq(3.11)-(3.12) is the AKNS-(2+1) equation (1.3) together with the nonisospectral condition (3.5). It means that (3.11) and (3.12) are the Lax pair for (1.3).

Bäcklund-gauge transformations

According to the results of previous papers, [5], [7], [8], [9], [12], the singular manifold method can be used to derive transformations [12], for the solutions and eigenfunctions of (3.11)-(3.12). It requires to consider the iterated solution m'

$$m' = m + \frac{\phi_{1,x}}{\phi_1} \quad (3.13)$$

where the subindex 1 refers to the fact that ϕ_1 is the singular manifold attached to an eigenfunction ψ_1 corresponding to an eigenvalue λ_1 . It means that:

$$\begin{aligned} \phi_{1,x} &= \psi_1^2 \\ 0 &= \psi_{1,xx} + (2m_x - \lambda_1)\psi_1 \\ 0 &= \psi_{1,t} + \lambda_1\psi_{1,y} + m_y\psi_{1,x} + \frac{1}{2}(\lambda_{1,y} - m_{xy})\psi_1 \end{aligned} \quad (3.14)$$

Next step requires to consider m' as a new seed solution that allows us to perform a new iteration

$$m'' = m' + \frac{\phi'_{2,x}}{\phi'_2} \quad (3.15)$$

where ϕ'_2 is a singular manifold related through

$$\phi'_{2,x} = (\psi'_2)^2 \quad (3.16)$$

with an eigenfunction ψ'_2 of m' with eigenvalue λ_2 .

$$\begin{aligned} 0 &= \psi'_{2,xx} + (2m'_x - \lambda'_2)\psi'_2 \\ 0 &= \psi'_{2,t} + \lambda'_2\psi'_{2,y} + m'_y\psi'_{2,x} + \frac{1}{2}(\lambda'_{2,y} - m'_{xy})\psi'_2 \end{aligned} \quad (3.17)$$

The crucial point is to consider (3.16) and (3.17) as a nonlinear system of PDE's for the fields m' , ψ'_2 and ϕ'_2 [12], [7]. It means that the truncated Painlevé expansion (3.13) should be accompanied with the corresponding expansions for ψ'_2 and ϕ'_2 . This is:

$$\begin{aligned} \psi'_2 &= \psi_2 + \frac{\Theta}{\phi_1} \\ \phi'_2 &= \phi_2 + \frac{\Delta}{\phi_1} \end{aligned} \quad (3.18)$$

where ψ_2 is an eigenfunction for m with eigenvalue λ_2 .

$$\begin{aligned} \phi_{2,x} &= \psi_2^2 \\ 0 &= \psi_{2,xx} + (2m_x - \lambda_2)\psi_2 \\ 0 &= \psi_{2,t} + \lambda_2\psi_{2,y} + m_y\psi_{2,x} + \frac{1}{2}(\lambda_{2,y} - m_{xy})\psi_2 \end{aligned} \quad (3.19)$$

Substitution of (3.13) and (3.18) in (3.17) provides (see Appendix B):

$$\phi_{1,x} = \psi_1^2 \quad (3.20)$$

$$\Theta = -\psi_1\Omega$$

$$\Delta = -\Omega^2$$

$$\Omega = \frac{\psi_1\psi_{2,x} - \psi_2\psi_{1,x}}{\lambda_2 - \lambda_1} \quad (3.21)$$

In consequence, two eigenfunctions ψ_1 and ψ_2 of the seed field m with eigenvalues λ_1 and λ_2 allow us to construct the following transformations of the Lax pair

$$m' = m + \frac{\psi_1^2}{\int \psi_1^2 dx} \quad (3.22)$$

$$\psi'_2 = \psi_2 - \psi_1 \frac{\psi_1\psi_{2,x} - \psi_2\psi_{1,x}}{(\lambda_2 - \lambda_1) \int \psi_1^2 dx} \quad (3.23)$$

in which the new field m' and its eigenfunction ψ'_2 are constructed by using ψ_1 and ψ_2 only. It constitutes an iterative method to obtain new solutions m' arising from a previous known solution m .

Notice that the iteration (3.22) is constructed not with the eigenfunction ψ_1 as in the binary Darboux transformations [14], [10], but with the singular manifold ϕ_1 that is related to the eigenfunction through $\phi_{1,x} = \psi_1^2$ [12].

τ -function

Eq (3.22) can be iterated by using (3.15), that combined with (3.18) and (3.21) yields to:

$$m'' = m + \frac{\tau_x}{\tau} \quad (3.24)$$

where

$$\tau = \phi_1\phi_2 - \Omega^2 \quad (3.25)$$

4 Singular manifold method for Eq (1.1)

According with the above results, the solutions m and \hat{m} of (2.5) and (2.6) have the following singular manifolds expansions

$$\begin{aligned} m' &= m + \frac{\phi_x}{\phi} \\ \hat{m}' &= \hat{m} + \frac{\hat{\phi}_x}{\hat{\phi}} \end{aligned} \quad (4.1)$$

It means, using (2.2), that the expansions for u , η and ω are:

$$\begin{aligned} u' &= u + \frac{\phi_x}{\phi} - \frac{\hat{\phi}_x}{\hat{\phi}} \\ \eta' &= \eta + \frac{\phi_x}{\phi} + \frac{\hat{\phi}_x}{\hat{\phi}} \\ \omega' &= \omega + \frac{\phi_t}{\phi} - \frac{\hat{\phi}_t}{\hat{\phi}} \end{aligned} \quad (4.2)$$

Coupling of the singular manifolds

Nevertheless, m and \hat{m} are not independent because there are related by the Bäcklund transformation (2.3).

By substituting (4.1) in (2.4), we get:

$$\frac{\phi_x}{\phi} \frac{\hat{\phi}_x}{\hat{\phi}} = A \frac{\phi_x}{\phi} + B \frac{\hat{\phi}_x}{\hat{\phi}} \quad (4.3)$$

where

$$\begin{aligned} A &= u + \frac{1}{2} \frac{\phi_{xx}}{\phi_x} \\ B &= -u + \frac{1}{2} \frac{\hat{\phi}_{xx}}{\hat{\phi}_x} \end{aligned} \quad (4.4)$$

Two component Lax pair

We can introduce eigenfunctions ψ and $\hat{\psi}$ for m and \hat{m} that, according to (3.10), should be:

$$\phi_x = \psi^2, \quad \hat{\phi}_x = \hat{\psi}^2 \quad (4.5)$$

With these eigenfunctions the Lax pair for m and \hat{m} will be:

$$\begin{aligned} 0 &= \psi_{xx} + (2m_x - \lambda)\psi \\ 0 &= \psi_t + \lambda\psi_y + m_y\psi_x + \frac{1}{2}(\lambda_y - m_{xy})\psi \end{aligned} \quad (4.6)$$

$$\begin{aligned}
0 &= \hat{\psi}_{xx} + (2\hat{m}_x - \hat{\lambda})\hat{\psi} \\
0 &= \hat{\psi}_t + \hat{\lambda}\hat{\psi}_y + \hat{m}_y\hat{\psi}_x + \frac{1}{2}(\hat{\lambda}_y - \hat{m}_{xy})\hat{\psi}
\end{aligned} \tag{4.7}$$

where λ and $\hat{\lambda}$ are spectral parameters that satisfy (3.5). With the aid of (4.5), (4.4) can be written as:

$$A = u + \frac{\psi_x}{\psi} \tag{4.8}$$

$$B = -u + \frac{\hat{\psi}_x}{\hat{\psi}} \tag{4.9}$$

Substitution of (4.1) and (4.2) in the Miura transformations (2.4) provides, with the aid of (4.3)-(4.9), the following result:

$$\begin{aligned}
\lambda &= \hat{\lambda} \\
AB &= \lambda
\end{aligned} \tag{4.10}$$

Furthermore, if we combine (4.6) with (4.8) and (2.4), we get:

$$A_x - A^2 + 2A\frac{\psi_x}{\psi} - \lambda = 0 \tag{4.11}$$

A similar combination between (4.7) with (4.9) and (2.4) provides:

$$B_x - B^2 + 2B\frac{\hat{\psi}_x}{\hat{\psi}} - \lambda = 0 \tag{4.12}$$

By using (4.10) in (4.11) and (4.12), the result is:

$$A = \sqrt{\lambda}\frac{\hat{\psi}}{\psi} \tag{4.13}$$

$$B = \sqrt{\lambda}\frac{\psi}{\hat{\psi}} \tag{4.14}$$

By combining (4.8-4.9) and (4.13-4.14), we get:

$$\begin{aligned}
\psi_x &= -u\psi + \sqrt{\lambda}\hat{\psi} \\
\hat{\psi}_x &= u\hat{\psi} + \sqrt{\lambda}\psi
\end{aligned} \tag{4.15}$$

Substitution of (4.15) in the temporal part of the Lax pairs (4.6)-(4.7) can be written as:

$$\begin{aligned}
0 &= \psi_t + \lambda\psi_y + \frac{\sqrt{\lambda}}{2}(u_y + \eta_y)\hat{\psi} + \frac{\lambda_y}{2}\psi + \omega\psi \\
0 &= \hat{\psi}_t + \lambda\hat{\psi}_y + \frac{\sqrt{\lambda}}{2}(-u_y + \eta_y)\psi + \frac{\lambda_y}{2}\hat{\psi} - \omega\hat{\psi}
\end{aligned} \tag{4.16}$$

(4.15) and (4.16) are a two component Lax pair for (1.1).

Notice that the coupling condition (4.4) can be written, with the aid of (4.5) and (4.13)-(4.14) as:

$$\psi\hat{\psi} = \sqrt{\lambda}(\phi + \hat{\phi}) \tag{4.17}$$

Bäcklund-gauge transformations and τ functions

The induced Bäcklund-gauge transformations for (4.15) and 4.16) are rather easy to construct. Let $(\psi_1, \hat{\psi}_1)$, $(\psi_2, \hat{\psi}_2)$ be eigenfunctions of (4.15) and 4.16) with spectral parameters λ_1 and λ_2 respectively. The corresponding singular manifolds are obviously:

$$\begin{aligned}\phi_{1,x} &= \psi_1^2, & \hat{\psi}_{1,x} &= \hat{\psi}_1^2 \\ \phi_{2,x} &= \psi_2^2, & \hat{\psi}_{2,x} &= \hat{\psi}_2^2\end{aligned}$$

If we use ψ_1 and $\hat{\psi}_1$ to perform a truncated expansion like (4.2)

$$\begin{aligned}u' &= u + \frac{\phi_{1,x}}{\phi_1} - \frac{\hat{\phi}_{1,x}}{\hat{\phi}_1} \\ \eta' &= \eta + \frac{\phi_{1,x}}{\phi_1} + \frac{\hat{\phi}_{1,x}}{\hat{\phi}_1} \\ \omega' &= \omega + \frac{\phi_{1,t}}{\phi_1} - \frac{\hat{\phi}_{1,t}}{\hat{\phi}_1}\end{aligned}\tag{4.18}$$

the truncated expansion for the eigenfunctions of the iterated fields would be:

$$\begin{aligned}\psi'_2 &= \psi_2 - \psi_1 \frac{\Omega}{\phi_1} \\ \hat{\psi}'_2 &= \hat{\psi}_2 - \hat{\psi}_1 \frac{\hat{\Omega}}{\hat{\phi}_1}\end{aligned}\tag{4.19}$$

where

$$\begin{aligned}\Omega &= \frac{\psi_1 \psi_{2,x} - \psi_2 \psi_{1,x}}{\lambda_2 - \lambda_1} \\ \hat{\Omega} &= \frac{\hat{\psi}_1 \hat{\psi}_{2,x} - \hat{\psi}_2 \hat{\psi}_{1,x}}{\lambda_2 - \lambda_1}\end{aligned}\tag{4.20}$$

Functions τ and $\hat{\tau}$ can be trivially defined as:

$$\begin{aligned}\tau &= \phi_1 \phi_2 - \Omega^2 \\ \hat{\tau} &= \hat{\phi}_1 \hat{\phi}_2 - \hat{\Omega}^2\end{aligned}\tag{4.21}$$

The second iteration of (4.17) provides:

$$\begin{aligned}u'' &= u + \frac{\tau_x}{\tau} - \frac{\hat{\tau}_x}{\hat{\tau}} \\ \eta'' &= \eta + \frac{\tau_x}{\tau} + \frac{\hat{\tau}_x}{\hat{\tau}} \\ \omega'' &= \omega + \frac{\tau_t}{\tau} - \frac{\hat{\tau}_t}{\hat{\tau}}\end{aligned}\tag{4.22}$$

5 Particular Solutions

The above described method can be easily used to obtain solutions for m , \hat{m} as well as for u . We will show some simple cases.

$$5.1 \quad u = U(y, t) \implies m = \frac{U(y, t) - (U(y, t))^2 x}{2}, \quad \hat{m} = \frac{-U(y, t) - (U(y, t))^2 x}{2}$$

Now (1.2) implies that $U(y, t)$ satisfies:

$$U_t - U^2 U_y = 0$$

- If we call ψ_i and $\hat{\psi}_i$ the solutions of (4.15)-(4.16) attached to an spectral parameter λ_i ($i = 1, 2$) we have:

$$\psi_i = \sqrt{(K_i(y, t) - U(y, t))} e^{K_i(y, t)x + \frac{H_i(y, t)}{2}}$$

$$\hat{\psi}_i = \sqrt{(K_i(y, t) + U(y, t))} e^{K_i(y, t)x + \frac{H_i(y, t)}{2}}$$

where

$$K_i(y, t) = \sqrt{\lambda_i(y, t) + (U(y, t))^2}$$

and $H_i(y, t)$ is a function that satisfies:

$$H_{i,t} + \lambda_i H_{i,y} + \lambda_{i,y} + U U_y = 0$$

- Integration of (4.5) together with (4.17) and (3.6) yields to

$$\phi_i = \frac{1}{2K_i(y, t)} \left(R_i(y, t) + [(K_i(y, t) - U(y, t))] e^{2K_i(y, t)x + H_i(y, t)} \right)$$

$$\hat{\phi}_i = \frac{1}{2K_i(y, t)} \left(-R_i(y, t) + [(K_i(y, t) + U(y, t))] e^{2K_i(y, t)x + H_i(y, t)} \right)$$

where $R_i(y, t)$ are functions that satisfy:

$$R_{i,t} + \lambda R_{i,y} - R_i U U_y = 0$$

and the iterated solutions are:

$$m' = \frac{U(y, t) - (U(y, t))^2 x}{2} + \frac{(\phi_i)_x}{\phi_i}$$

$$\hat{m}' = \frac{-U(y, t) - (U(y, t))^2 x}{2} + \frac{(\hat{\phi}_i)_x}{\hat{\phi}_i}$$

$$u' = U(y, t) + \frac{(\phi_i)_x}{\phi_i} - \frac{(\hat{\phi}_i)_x}{\hat{\phi}_i}$$

- For the second iteration, we need to use (4.20)-(4.21). The result is:

$$\begin{aligned} \frac{4K_1 K_2}{R_1 R_2} \tau = 1 + \frac{K_1 - U}{R_1} e^{2K_1 x + H_1} + \frac{K_2 - U}{R_2} e^{2K_2 x + H_2} + \\ + \left(\frac{K_2 - K_1}{K_2 + K_1} \right)^2 \frac{(K_1 - U)(K_2 - U)}{R_1 R_2} e^{2K_1 x + H_1} e^{2K_2 x + H_2} \end{aligned}$$

$$\frac{4K_1K_2}{R_1R_2} \hat{\tau} = 1 - \frac{K_1 - U}{R_1} e^{2K_1x+H_1} - \frac{K_2 - U}{R_2} e^{2K_2x+H_2} +$$

$$+ \left(\frac{K_2 - K_1}{K_2 + K_1} \right)^2 \frac{(K_1 - U)(K_2 - U)}{R_1R_2} e^{2K_1x+H_1} e^{2K_2x+H_2}$$

We have solutions for the second iteration through the expressions:

$$m'' = \frac{U(y, t) - (U(y, t))^2 x}{2} + \frac{\tau_x}{\tau}$$

$$\hat{m}'' = \frac{-U(y, t) - (U(y, t))^2 x}{2} + \frac{\hat{\tau}_x}{\hat{\tau}}$$

$$u'' = U(y, t) + \frac{\tau_x}{\tau} - \frac{\hat{\tau}_x}{\hat{\tau}}$$

5.2 $u = 0$, $\lambda_i = \text{constant}$

This case is contained in the previous one. Now, we have:

$$\lambda_i = k_i^2, \quad K_i = k_i$$

$$H_{i,t} + k_i^2 H_{i,y} = R_{i,t} + k_i^2 R_{i,y} = 0$$

In particular, if we choose

$$R_i = 1 + e^{c_0(y - k_i^2 t)}$$

$$e^{H_i} = 1 + a_0 e^{c_0(y - k_i^2 t)}$$

where a_0 and c_0 are arbitrary constants, we get dromionic behavior.

5.3 $u = U(y, t)$, $\lambda = 0$

- The solutions of (4.15)-(4.16) can be chosen now as:

$$\psi = 0$$

$$\hat{\psi} = e^{U(y,t)x + \frac{H(y)}{2}}$$

where $H(y)$ is an arbitrary function of y and $U(y, t)$ obviously satisfies:

$$U_t - U^2 U_y = 0$$

- Expressions for ϕ and $\hat{\phi}$ can be easily obtained through (4.5), together with (4.17) and (3.6). The result is:

$$\phi = R(y)$$

$$\hat{\phi} = \hat{R}(y) + \frac{e^{2U(y,t)x + H(y)}}{2U(y, t)}$$

being $R(y)$ and $\hat{R}(y)$ arbitrary functions of y .

- Iterated solutions can be now constructed as:

$$\begin{aligned} m' &= \frac{U(y, t) - (U(y, t))^2 x}{2} \\ \hat{m}' &= \frac{-U(y, t) - (U(y, t))^2 x}{2} + \frac{2U(y, t)e^{2U(y, t)x+H(y)}}{2\hat{R}(y)U(y, t) + e^{2U(y, t)x+H(y)}} \\ u' &= U(y, t) - \frac{2U(y, t)e^{2U(y, t)x+H(y)}}{2\hat{R}(y)U(y, t) + e^{2U(y, t)x+H(y)}} \end{aligned}$$

6 Conclusions

- The Painlevé test of (1.1) implies that the system has two Painlevé branches. It suggests that the solutions can be written as linear superposition of solutions of an equation with just one branch. We have proved that this equation is precisely AKNS in (2+1) dimensions. This above splitting allows us to get two Miura transformations between (1.1) and (1.3).
- The singular manifold method is applied to (1.3) to derive its Lax pair. When SMM is applied to the Lax pair itself we get Bäcklund-gauge transformations that allow us to derive an iterative method to construct solutions.
- SMM is applied to (1.1) with the aid of the above results for (1.3). The induced Bäcklund-gauge transformations for (1.1), as well as the iterated solutions are obtained through them.
- We close the paper in the last section with a rich collection of exact solutions.

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Appendix A

Let us substitute (3.1) into (1.9) (We have used MAPLE to handle the calculation). We obtain a polynomial in ϕ and, by imposing that all the coefficients are zero, we have:

- Coefficient in ϕ^{-3}

$$0 = 4m_y + 8qm_x + 2v_y + 2qv_x + qv^2 + 4r \quad (A1)$$

- Coefficient in ϕ^{-2}

$$\begin{aligned} 0 = & -8m_{xy} - 4qm_{xx} - 12vm_y - 24vqm_x - 16q_x m_x - 8r_x - 7vqv_x - \\ & -12rv - 2v^2q_x - 3qv^3 - 4v_{xy} - 6vv_y - qv_{xx} - 4q_x v_x \end{aligned} \quad (A2)$$

- Coefficient in ϕ^{-1}

$$0 = 4r_{xx} + 8m_x vq_x + 3v_x v^2 q + 3v_x vq_x + 8r_x v + 4rv^2 + v^3 q_x +$$

$$\begin{aligned}
& +v^4q + 8m_{yx}v + 4m_{xx}q_x + 4m_yv^2 + 8m_xv^2q + 4m_{xx}qv + v_{xxy} + \\
& +v_{xx}q_x + 3vv_{xy} + 3v_xv_y + 3v_yv^2 + 4rv_x + 4m_yv_x + 8m_xv_y + v_{xx}qv
\end{aligned} \tag{A3}$$

• Coefficient in ϕ^0

$$0 = 4m_{xx}m_y + 8m_{xy}m_x + 4m_{xt} + m_{xxx}y \tag{A4}$$

where, according to (3.4), we have used the following substitutions:

$$\phi_{xx} = v\phi_x$$

$$\phi_t = r\phi_x$$

$$\phi_y = q\phi_x$$

Let us do now the following combinations:

1) From (A1), we have:

$$m_y = -r - \frac{qv^2}{4} - 2qm_x - \frac{v_y}{2} - \frac{q}{2}v_x \tag{A5}$$

2) By substituting (A5) in (A2), we have:

$$q(3vv_x + 12m_{xx} + 3v_{xx}) = 0 \tag{A6}$$

that can be integrated with respect to x providing:

$$m_x = -\frac{v_x}{4} - \frac{v^2}{8} + \frac{\lambda}{2} \tag{A7}$$

where we have introduced $\lambda = \lambda(y, t)$ as a constant with respect to the integration in x .

3) Substitution of (A7) in (A5) yields to:

$$m_y = -\frac{v_y}{2} - r - \lambda q \tag{A8}$$

4) Compatibility between (A7) and (A8) implies that:

$$r_x + \frac{v_{xy}}{4} - \frac{v}{4}v_y + \frac{\lambda_y}{2} + \lambda q_x = 0 \tag{A9}$$

5) If we substitute (A7) and (A8) in (A3), it is trivially satisfied with the aid of (A9).

6) The same substitution, when applied to (A4), yields to:

$$2\lambda\lambda_y + 2\lambda_t = 0 \tag{A10}$$

Appendix B

1) If we substitute (3.18) in (3.16), we get:

$$0 = \phi_{2,x} - \psi_2^2 + \frac{1}{\phi_1} (\Delta_x - 2\psi_2\Theta) + \frac{1}{\phi_1^2} (-\Delta\phi_{1,x} - \Theta^2) \quad (B1)$$

and by using the fact that $\phi_{i,x} = \psi_i^2$, $i = 1, 2$, we obtain:

$$0 = \frac{1}{\phi_1} (\Delta_x - 2\psi_2\Theta) + \frac{1}{\phi_1^2} (-\Delta\psi_1^2 - \Theta^2) \quad (B2)$$

The coefficient in ϕ_1^{-2} provides:

$$\Delta = -\frac{\Theta^2}{\psi_1^2} \quad (B3)$$

and by substituting (B3) in (B2) we get:

$$\Theta_x = \Theta \frac{\psi_{1,x}}{\psi_1} - \psi_1^2 \psi_2 \quad (B4)$$

2) The substitution of (3.13) and (3.18) in the spatial part of the Lax pair (3.17) gives us:

$$\begin{aligned} 0 = & \psi_{2,xx} + (2m_x - \lambda_2)\psi_2 + \frac{1}{\phi_1} (\Theta_{xx} + 2m_x\Theta + 2\psi_2\phi_{1,xx} - \lambda_2\Theta) \\ & + \frac{1}{\phi_1^2} (-2\Theta_x\phi_{1,x} + \Theta\phi_{1,xx} - 2\psi_2\phi_{1,x}^2) \end{aligned} \quad (B5)$$

The coefficient in ϕ_1^0 means that ψ_2 is an eigenfunction for m with eigenvalue λ_2 .

$$0 = \psi_{2,xx} + (2m_x - \lambda_2)\psi_2 \quad (B6)$$

If we substitute $\phi_{1,x} = \psi_1^2$ and (B4) in (B5), we have:

$$0 = \frac{1}{\phi_1} (\psi_1\psi_2\psi_{1,x} - \psi_1^2\psi_{2,x} + \lambda_1\Theta - \lambda_2\Theta) \quad (B7)$$

By solving (B7) with respect to Θ , the result is:

$$\Theta = -\psi_1 \left(\frac{\psi_1\psi_{2,x} - \psi_2\psi_{1,x}}{\lambda_2 - \lambda_1} \right) \quad (B8)$$

if we define:

$$\Omega = \frac{\psi_1\psi_{2,x} - \psi_2\psi_{1,x}}{\lambda_2 - \lambda_1} \quad (B9)$$

we have from (B8)

$$\Theta = -\psi_1\Omega \quad (B10)$$

and

$$\Delta = -\Omega^2 \quad (B11)$$

from (B3).

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