Geometrical Formulation of the Conformal Ward Identity

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Abstract

In this paper we use deep ideas in complex geometry that proved to be very powerful in unveiling the Polyakov measure on the moduli space of Riemann surfaces and lead to obtain the partition function of perturbative string theory for 2, 3, 4 loops. Indeed a geometrical interpretation of the conformal Ward identity in two dimensional conformal field theory is proposed: the conformal anomaly is interpreted as a deformation of the complex structure of the basic Riemann surface. This point of view is in line with the modern trend of geometric quantizations that are based on deformations of classical structures. Then, we solve the conformal Ward identity by using this geometrical formalism.

1 Introduction

Two-dimensional conformal fields theories on Riemann surfaces without boundaries are powerful tools to deal with string theory. In particular the dependence on the background geometry has been used to develop effective actions for two-dimensional quantum gravity [1]. This has led to exciting developments in non critical string theory [2] and may shed some light on the quantization programme of higher dimensional gravity. Moreover the quantum theory of the string can be expressed in two different versions. In the canonical quantization it appears as the representation theory of Heisenberg, Virasoro and Kac-Moody algebras. In the quantization formalism of Polyakov, which is geometric and thus treats global objects, the integration over the matter field is Gaussian but the integration over the zweibein (the field corresponding to the graviton and defining the geometry of the two-dimensional gravity) [2] is non trivial and leads to two different settings depending on the gauge. In the conformal gauge, obtained after transforming the zweibein by diffeomorphism and Weyl rescalings into a flat reference gauge, the functional integration analysis leads to the Liouville theory the action of which yields, out of critical

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dimension, a measure of the violation of the conformal symmetry at the quantum level. In the light-cone gauge, which has a single non vanishing metric mode called the Wess-Zumino field and is represented geometrically by the Beltrami differential, the theory takes a local form. The resulting action is the Polyakov action which is a holomorphic functional of the Beltrami differential and is reparametrization invariant. However its variation under the Weyl rescaling produces the conformal anomaly the strength of which is measured by the central charge of the model under consideration up to a conventional normalization. The form of such an anomaly is universal.

2 The Beltrami differential

By definitin a collection ϕ of functions ϕ_α defined on each domain A_α of a Riemann surface Σ by

$$\phi_\alpha : A_\alpha \to \Sigma, \phi_\alpha = \phi \circ z_\alpha \tag{2.1}$$

is called a (p,q)-differential on the surface Σ if it is invariant under holomorphic change of coordinates:$(A_\alpha, z_\alpha) \to (A_\beta, z_\beta)$. It is writen locally as

$$\phi = \phi_{pq}\left(z, \overline{z}\right) dz^p d\overline{z}^q. \tag{2.2}$$

As an example the Beltrami differential μ is a $(-1,1)$-differential; $\mu = \mu_{\overline{z}}^{z} d\overline{z} \otimes \partial$ which is interpreted as a C^∞ section of the fibre bundle $k^{-1} \otimes k$, where k is the holomorphic cotangent bundle on the surface Σ. Geometrically the Beltrami differentials parametrize complex structures on the bidimensional Riemann surface Σ on which the model is constructed. Due to this fact, the transition from a reference complex structure $(z, \overline{z})$ parametrized by $\mu = 0$, to another one $\left(f\left(z, \overline{z}\right), \overline{f}\left(z, \overline{z}\right)\right)$ parametrized by $\mu\left(z, \overline{z}\right) \neq 0$,where the conformal invariance is maintained; $\partial f / \partial \overline{f} = 0$ is called a μ-quasiconformal transformation. This latter is defined by the Beltrami equation

$$\left(\overline{\partial} - \mu \partial\right) f = 0, \tag{2.3}$$

with $\partial \equiv \frac{\partial}{\partial z}$ and $\overline{\partial} \equiv \frac{\partial}{\partial \overline{z}}$. Moreover let us consider for the sense-preserving diffeomorphism f on a domain A of a Riemann surface Σ the derivative $\partial_\alpha f$ in the direction α:

$$\partial_\alpha f = \partial f + e^{-2i\alpha} \overline{\partial} f. \tag{2.4}$$

Then we have

$$\max_\alpha \left|\partial_\alpha f\right| = \left|\partial f\right| + \left|\overline{\partial} f\right| \tag{2.5}$$

$$\min_\alpha \left|\partial_\alpha f\right| = \left|\partial f\right| - \left|\overline{\partial} f\right|, \tag{2.6}$$

where || denotes the absolute value, and the dilatation quotient

$$D_f \equiv \frac{\max_\alpha |\partial_\alpha f|}{\min_\alpha |\partial_\alpha f|} \tag{2.7}$$

is finite. Hence we can write

$$D_f \leq K \tag{2.8}$$

for every $z \in A$. On the other hand the Jacobian $J_f = |\partial f|^2 - \left|\overline{\partial} f\right|^2$ for this sense-preserving diffeomorphism is positive. Then $\partial f \neq 0$, and we can form the quotient

$$\mu(z, \overline{z}) \equiv \frac{\overline{\partial} f(z, \overline{z})}{\partial f(z, \overline{z})} \tag{2.9}$$

The function μ, so defined, is called the analytic dilatation of the diffeomorphism f. Since f is continuous μ is Borel-measurable function and from (2.8) we see that $|\mu(z, \overline{z})| \leq \frac{K-1}{K+1} \prec 1$. The definition of complex dilatation leads us to consider the Beltrami equation

$$\left(\overline{\partial} - \mu \partial\right) f = 0, \tag{2.10}$$

where μ is measurable and $\|\mu\|_\infty \prec 1$. If f is conformal μ vanishes identically and the Beltrami equation becomes the Cauchy-Riemann equation $\overline{\partial} f = 0$.

3 Classical Ward identity

Ward identities are relations between Green's functions resulting from initial classical invariance. They are the basic means providing insight into the quantum structure of gauge theories. The use of Ward identities in Yang-Mills theory investigates gauge dependence (this is the case of theories with composite fields). In quantum general gauge theories (both renormalized and non renormalized ones) Ward identities underly the proof of the existence of Noether charge operators with the algebraic properties required for the analysis of unitary conditions. For a two-dimensional quantum field theory the number of the present exterior fields in the theory is the same as the number of Ward identities constraining the model. In particular for a two-dimensional conformal model constructed on a bidimensional Riemann surface that is endowed with complex structures there are two conformal Ward identities (one is the complex conjugate of the other). Moreover exterior fields are interpreted as exterior sources of the energy-momentum tensors. This is the basic statement of the Polyakov conjecture for a two-dimensional conformal model [3]. Moreover, in the Beltrami parametrization scheme, these exterior sources are identified as Beltrami differentials μ and $\overline{\mu}$ (the complex conjugate of μ). Indeed the classical conformal Ward identity is expressed as:

$$\left(\delta_\xi \mu \frac{\delta}{\delta \mu} + \delta_\xi \overline{\mu} \frac{\delta}{\delta \overline{\mu}}\right) S_C = 0, \tag{3.1}$$

where S_C is the classical action of an effective 2d conformal model, $\xi = \xi^z\partial + \xi^{\overline{z}}\overline{\partial}$ ($\in Vect(\Sigma)$) is a vector field on a tangent space of a bi-dimensional Riemann surface Σ and δ_ξ is the diffeomorphism symmetry generator. Then one can verify that the Ward identity encodes the conformal invariance of the classical two-dimensional conformal model the effective action of which is S_C. In a complex structure $(z, \overline{z})$ of this Riemann surface Σ the transform of a Beltrami differential μ, under an infinitesimal diffeomorphism, is given by

$$\delta_\xi \mu = W_{-1} H^z, \tag{3.2}$$

where $W_{-1} \equiv \overline{\partial} - \mu\partial + \mu\partial\mu$ and $H^z \equiv \xi^z + \mu^z_{\overline{z}}\xi^{\overline{z}}$. One can verify that equation (3.2) enables us to rewrite the classical conformal Ward identity as follows:

$$W_2 \frac{\delta S_C}{\delta \mu} + \overline{\mu W_2}\,\frac{\delta S_C}{\delta \overline{\mu}} = 0. \tag{3.3}$$

$W_2 \equiv \overline{\partial} - \mu\partial - 2\mu\partial\mu$ is called the Ward operator. Taking into account the holomorphic factorization of the action; $S_C(\mu,\, \overline{\mu}) = S(\mu) + \overline{S}(\overline{\mu})$ with $\overline{S}(\overline{\mu}) = \overline{S(\mu)}$, we get the following relation

$$W_2 \frac{\delta S}{\delta \mu} + \overline{\mu W_2}\,\frac{\delta \overline{S}}{\delta \overline{\mu}} = 0. \tag{3.4}$$

Then, as $|\mu| \prec 1$ [4], it is easy to show that this equation leads to the well-known classical Ward identity:

$$W_2 \frac{\delta S\,(\mu)}{\delta \mu} = 0, \tag{3.5}$$

where $\Theta(z, \overline{z}) \equiv \delta S\,(\mu)\,/\delta\mu\rfloor\mu = \overline{\mu} = 0$ is the classical effective energy-momentum tensor of the two- dimensional model under consideration. Geometrically speaking this latter equation is interpreted as a particular case (j=2) of an exact μ-holomorphy condition that is satisfied by a j-differential f_j (a (j, o)-differential) [5]:

$$W_j f_j = 0. \tag{3.6}$$

$W_j \equiv \overline{\partial} - \mu\partial - j\partial\mu$ is the generalized Ward operator the zero modes of which are j-differentials.

4 Quantum Ward identity

At the quantum level the classical action is extended to the vertex functional $\Gamma(\mu,\, \overline{\mu})$ which determines the Green functions of the model:

$$\Gamma = S_C + \hbar\Gamma^{(1)}. \tag{4.1}$$

$\Gamma^{(1)}$ depends only on μ and $\overline{\mu}$ and is generated by correlation functions of the classical energy-momentum tensor. This latter is a non local distribution that diverges quadratically in the sense of the classical power- counting [6]. Then the classical Ward identity is extended to its quantum version which is satisfied by the functional Γ modulo an inhomogeneous term of μ and $\overline{\mu}$. Such an anomaly exists and is unique [7]. It is the reflection of the Weyl-Lorentz anomaly of the metric scheme modulo a local counter-term [7]. An integral form of this anomaly is given, on the complex plane, by the following:

$$\int_p A(C,\mu) = \int_p dmC\partial^3\mu \tag{4.2}$$

where $A(C,\mu)$ is a differential three-form in the bigraded algebra of local cochains defined on the connected diffeomorphism algebra [8]. $C^z = c^z + \mu_{\overline{z}}^{z} c^{\overline{z}}$ is a suitable combination of the corresponding ghost fields to the diffeomorphisms parameters $\xi^z, \xi^{\overline{z}}$ in the BRST formalism and $dm(z) \equiv \frac{d\overline{z}\wedge dz}{2i}$ is the two-dimensional measure expressed in the coordinates $(z,\overline{z})$.

Now the Legendre transform of the generating functional Γ is the connected Green functional which is expressed as:

$$Z^c[J_\phi, \mu,\ \overline{\mu}] = \int_p dmJ_\phi\phi - \Gamma[\phi,\ \mu,\ \overline{\mu}], \tag{4.3}$$

where ϕ is a collection of fields of the model and J_ϕ are the associated exterior sources. Then the classical Ward identity (3.5) is translated, at the quantum level, to the following anomalous Ward identity:

$$W_2 \frac{\delta Z_v^c(\mu)}{\delta\mu} = \frac{k}{12\pi}(\partial^3\mu)(z). \tag{4.4}$$

$Z_v^c[\mu,\ \overline{\mu}] \equiv Z^c[J_\phi, \mu,\ \overline{\mu}]_{\rceil\phi=0}$ and k is the central charge of the model. As we can verify from this latter equation the anomaly measures the non holomorphic character of the energy-momentum tensor derived from the functional Z_v^c. On the other hand the transition from the classical level to the quantum one is expressed geometrically by a quasiconformal transformation, the dilatation coefficient of which is the Beltrami differential μ [4], from the reference complex structure $(z,\overline{z})$ defined by $\mu = 0$ to another one $(Z,\overline{Z})$ determined by the Beltrami equation:

$$(\overline{\partial} - \mu\partial)Z = 0. \tag{4.5}$$

Then in this geometrical setup the classical level is characterized by an exact μ-holomorphy condition for the classical energy-momentum tensor (a 2-differential). However the quantum level is characterized by a deformed one for the quantum energy-momentum tensor.

5 Conformally covariant Ward identity

To preserve the conformal covariance of the diffeomorphism anomaly and then to get manifest the conformal covariance of the modell a projective connection is required. This latter parametrizes a projective structure that is associated to a complex structure [4].

5.1 Conformally covariant anomaly

As a non trivial element of the 1-cohomological space of linear applications on the diffeomorphism algebra the above diffeomorphism anomaly can be expessed locally (up to a sign) as [8]:

$$A(C,\mu) = \frac{-k}{24\pi}(C\partial^3\mu - \mu\partial^3 C). \tag{5.1}$$

Its transformation law under a conformal change of coordinates

$$z \mapsto \omega(z) \tag{5.2}$$

is given by

$$A_\omega = A_z + 2\zeta_z(\omega)(C\partial\mu - \mu\partial C)(z), \tag{5.3}$$

where $\zeta_z(\omega) \equiv \partial^2 \ln\partial\omega - \frac{1}{2}(\partial\ln\partial\omega)^2$ is the Schwarzian derivative of the function $\omega(z,\overline{z})$ with respect to the variable z. In particular the Schwarzian derivative of a Möbius transformation ω; $\omega \in SL(2,p)$ is zero. Then the above expression of the anomaly becomes conformally invariant in a projective atlas (because, in this case, the second term in the right hand side of equation (5.3) vanishes). On a general complex atlas we consider the following form of the anomaly:

$$\tilde{A}(C,\mu) = \frac{k}{24\pi}[C\partial^3\mu - \mu\partial^3 C + 2R(C\partial\mu - \mu\partial C)], \tag{5.4}$$

where $R(z,\overline{z})$ is any complex function for the moment. However in order to get the anomaly (5.4) conformally covariant and precisely a $(-1,-1)$-tensor with respect to the conformal change of coordinates (5.2), that is

$$\tilde{A}_\omega = (\overline{\partial\omega})^{-1}(\partial\omega)^{-1}\tilde{A}_z, \tag{5.5}$$

and by taking into account the following transformation laws with respect to the same conformal change of coordinates,

$$dm(\omega) = |\partial\omega|^2 dm(z), \tag{5.6}$$

$$C_\omega = \partial\omega C_z, \tag{5.7}$$

$$\mu_\omega = (\overline{\partial\omega})^{-1}\partial\omega\mu_z, \tag{5.8}$$

one can verify that the function $R(z,\overline{z})$ should be a projective connection [4, 9]:

$$W_2 R = \partial^3\mu. \tag{5.9}$$

It is a deformed μ-holomorphic condition for the projective connection R by the diffeomorphism anomaly. Also it is easy to show that the transformation law, with respect to the conformal change (5.2), of the projective connection R is given by:

$$R_\omega = (\partial\omega)^{-2}(R_z - \zeta_z(\omega)). \tag{5.10}$$

Moreover one can verify that the particular case of this projective connection is the Schwarzian derivative $\zeta_z(\omega)$ which also satisfies the same equation

$$W_2\zeta_z(\omega) = \partial^3\mu. \tag{5.11}$$

Then one can verify that the conformally covariant form of the anomaly (5.4) can be expressed as:

$$\tilde{A}(C,\mu) = \frac{k}{24\pi}\left(CL_3^R(\mu) - \mu L_3^R(C)\right), \tag{5.12}$$

where $L_3^R \equiv \partial^3 + 2R\partial + \partial R$ (the covariant form of the operator ∂^3). It is called the third Bol's operator. Moreover one can show (on any Riemann surface Σ) the relation:

$$\int_\Sigma dm\mu L_3^R(C) = -\int_\Sigma dm\ CL_3^R(\mu) \tag{5.13}$$

and hence the integrated form of the diffeomorphism anomaly can be written as:

$$\tilde{A}(C,\mu) = \frac{k}{12\pi}\int_\Sigma dm\ CL_3^R(\mu). \tag{5.14}$$

5.2 Conformally covariant Ward identity

The Polyakov conjecture for a two-dimensional conformal model states that, on the complex plane, the formal series $Z_v^c(\mu)$ is resumed by the following Wess-Zumino-Polyakov action [8]:

$$\Gamma_{WZP} = \frac{-k}{24\pi}\int_p dm\mu\partial^2 \ln\partial Z, \tag{5.15}$$

where the local coordinate $Z(z,\overline{z})$ satisfies the Beltrami equation (4.5). Then we have

$$\Gamma_{WZP}(\mu) = \frac{-k}{12\pi}Z_v^c(\mu). \tag{5.16}$$

Moreover one can verify that the functional derivation of the action (5.15) with respect to the Beltrami differential μ is given by:

$$\frac{\delta\Gamma_{WZP}}{\delta\mu} = \frac{-k}{12\pi}\zeta_z(Z). \tag{5.17}$$

Hence, by using the above equation, we get the conformally covariant form of the quantum Ward identity:

$$\overline{\partial}T_{zz}(z,\overline{z}) = L_3^T(\mu), \tag{5.18}$$

where $T_{zz} \equiv \delta Z_v^c/\delta\mu$ is the quantum energy-momentum tensor of an effective two-dimensional conformal model. This is the analogue of the deformed μ-holomorphy equation that is satisfied by the Schwarzian derivative given before. Then we get its solution as:

$$T_{zz}(z,\overline{z}) = \zeta_z(Z) + f_{zz}(z,\overline{z}), \tag{5.19}$$

where $f_{zz}(z,\overline{z})$ satisfies the exact μ-holomorphy equation

$$W_2 f_{zz} = 0. \tag{5.20}$$

Moreover the transformation law of the function T under the holomorphic change of coordinates $z \mapsto \omega(z)$, which is
given by

$$T_{\omega\omega} = (\partial\omega)^2 (T_{zz} - \zeta_z(\omega)), \tag{5.21}$$

shows that T is not a tensor with respect to two-dimensional conformal transformations. On the other hand, as the classical Ward identity implies that the classical energy-momentum tensor $\Theta_{zz}(z,\overline{z})$ is a 2-differential ((2,0)-differential as defined in the introduction), we can reexpress the quantum energy-momentum tensor in terms of $\Theta_{zz}(z,\overline{z})$ as follows:

$$T_{zz}(z,\overline{z}) = \zeta_z(Z) + \Theta_{zz}(z,\overline{z}). \tag{5.22}$$

Then this conformal Ward identity's solution tells us that the quantum corrections to an effective classical two-dimensional conformal model are generated by the Schwarzian derivative of a quasiconformal transformation on a Riemann surface on which the model is considered. Moreover the transition from the classical level to the quantum one, $\Theta_{zz} \mapsto T_{zz}$, is geometrically interpreted as the passage from a reference atlas to its transform by this quasiconformal transformation.

6 Iterative solution of the conformally covariant Ward identity

Now we rewrite the local form of the conformal Ward identity as follows:

$$\overline{\partial} T = \mu\partial T + 2T\partial\mu + \frac{k}{12\pi} L_3^R(\mu). \tag{6.1}$$

To determine an iterative solution of such equation,on any two-dimensional Riemann surface Σ without boundary, as a Neumann series in powers of the Beltrami differential we define a Cauchy-Riemann kernel N on this surface by the following: for any complex valued function f we have

$$\left(\overline{\partial}^{-1} f\right)(z) \equiv \int_\Sigma dm(w) N(w,z) f(w). \tag{6.2}$$

We rewrite equation (6.1) as:

$$\overline{\partial}\check{T} = D\overline{\partial}\check{T} + L_3^R(\mu), \tag{6.3}$$

where $\check{\mathrm{T}} \equiv \frac{k}{12\pi} T$ and $\mathrm{D} \equiv (\mu\partial + 2\partial\mu)\overline{\partial}^{-1}$. As the Beltrami differential satisfies the ellipticity condition: $\mu \in C^\infty(\Sigma)$, $|\mu| \prec 1$ the Cauchy integral (6.2) enables us to get, in the complex structure $(z, \overline{z})$, a conformally covariant iterative solution of the conformal Ward (6.1) identity in powers of the Beltrami differential μ:

$$\check{T}_0 = \int_\Sigma dm_1 N_{01} L_3^R(1) - \int_\Sigma dm_1 \mu_1 V_{01} \check{T}_1, \tag{6.4}$$

where we have used the notations:

$$\begin{aligned} i &\equiv z_i & (6.5)\\ \check{T}_i &\equiv \check{T}(z_i) & (6.6)\\ N_{i-1i} &\equiv N(i-1, i) & (6.7)\\ V_{i-1i} &\equiv 2\partial_i N_{i-1i} + N_{i-1i}\partial_i & (6.8)\\ dm_i &\equiv \frac{d\overline{z}_i \wedge dz_i}{2i}. & (6.9) \end{aligned}$$

Then, at any order of the perturbative series say $(n-1)$ in any local coordinates system (z_{n-1}) of the same atlas on the surface Σ, we get

$$\check{T}_{n-1} = \int_\Sigma dm_n \mu_n (\partial_n^3 N_{n-1n} + V_{n-1n} R_n) - \int_\Sigma dm_n V_{n-1n} \check{T}_n, \tag{6.10}$$

where R_n is the iterative solution of the μ-holomorphy equation that is satisfied by the projective connection R on any Riemann surface Σ. On the complex plane this solution was given in [4] as:

$$R_n = \sum_{k=1}^{+\infty} (-1)^{-k} \int_p \Pi_{j=1}^{k+n} (dm_{j+n} \mu_{j+n}) \partial_{k+n}^3 A_{k+n-1}^p, \tag{6.11}$$

where $A_0^p \equiv N_{01}^p$ is the Cauchy kernel on the complex plane and $A_k^p = (2\partial_{k-1} A_{k-1}^p + A_{k-1}^p \partial_k) N_{kk+1}^p$. Finally we express the quantum energy-momentum tensor as the sum of the perturbative series

$$T_0 = \frac{k}{12\pi} \sum_{n=1}^{+\infty} (-1)^n \int_\Sigma \Pi_{i=1}^n (dm_i \mu_i V_{i-1i})(\partial_n^3 V_{n-1n} + R_n). \tag{6.12}$$

7 Conclusion

We have expressed geometrically the two-dimensional conformal Ward identity as a μ-holomorphy equation on a complex Riemann surface Σ on which the model is considered. This geometrical setup enables us to get the exact solution of this conformal Ward identity as a μ-holomorphic function up to the classical solution. Moreover we have exhibited the conformal covariance character of this identity by expressing this latter in terms of conformally covariant operators like the third Bol's operator. Finally we have developed the iterative solution of this identity as a Neumann series in powers of the the function μ on any complex Riemann surface without boundary by supposing the existence of a Cauchy kernel the explicit expression of which on this surface was not given.

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