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Integral Equation Approach for the Propagation of TE-Waves in a Nonlinear Dielectric Cylindrical Waveguide

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Abstract

We consider the propagation of TE-polarized electromagnetic waves in cylindrical dielectric waveguides of circular cross section filled with lossless, nonmagnetic, and isotropic medium exhibiting a local Kerr-type dielectric nonlinearity. We look for axially-symmetric solutions and reduce the problem to the analysis of the associated cubic-nonlinear equation. We show that the solution in the form of a TE-polarized electromagnetic wave exists and can be obtained by iterating a cubic-nonlinear integral equation. We derive the associated dispersion equation and prove that it has a root that determines this solution.

1 Introduction

The propagation of electromagnetic waves in a cylindrical dielectric waveguide of circular cross section filled with a linear medium is a relevant topic of classical electromagnetics [1], [2]. Nonlinear cylindrical dielectric waveguides were investigated by several authors [3]–[8]. However, from the mathematical viewpoint, the study is not complete because the analysis of the dispersion equation is still missing (to the best of our knowledge). In this paper, we study electromagnetic waves propagating in a cross-sectionally bounded dielectric, nonmagnetic waveguide filled with a medium exhibiting a local Kerr-type dielectric nonlinearity. The problem is reduced to a cubic-nonlinear ordinary differential equation of the second order and then to a nonlinear integral equation with the kernel in the form of Green's function for the Bessel equation. The existence of the propagating TE-waves is proved using the method of contraction mapping.

In Section 2 we specify the problem. In Section 3 we derive a nonlinear integral equation and present an iterative solution including a sufficient condition for its existence. Section 4 is devoted to the analysis of dispersion equations. We give the sufficient conditions for the existence of solutions for the exact and iterate dispersion equations and prove the convergence of the iterate eigenvalues to the exact eigenvalues.

2 Statement of the problem

We consider the wave propagation in a cylindrical dielectric waveguide with the circular cross section $W = \{(x, y) : \rho = \sqrt{x^2 + y^2} < R\}$. The waveguide is homogenous in the z-direction. The permittivity ϵ of the waveguide medium has a nonlinear dependence on the electric field according to the Kerr law, so that

$$\epsilon = \begin{cases} \epsilon_2 + a|\mathbf{E}|^2, & 0 \le \rho \le R, \\ \epsilon_1, & \rho > R, \end{cases} \tag{2.1}$$

where $\mathbf{E}$ denotes the electric field in the waveguide and a, $\epsilon_1 > 0$, $\epsilon_2 > 0$ are real constants. The medium is nonmagnetic with $\mu = \mu_0$ being the free-space permeability. The electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell's equations

$$\mathrm{rot}\boldsymbol{H} = -i\omega\epsilon\boldsymbol{E}, \tag{2.2}$$

$$\mathrm{rot}\boldsymbol{E} = i\omega\mu_0\boldsymbol{H}, \tag{2.3}$$

the continuity of the tangential components on the interface, and the radiation condition, according to which the amplitudes of the field components decay exponentially at infinity.

In the cylindrical coordinates (ρ, φ, z) Maxwell's equations have the form

$$\frac{1}{\rho}\frac{\partial E_z}{\partial \varphi} - \frac{\partial E_\varphi}{\partial z} = i\omega\mu_0 H_\rho, \tag{2.4}$$

$$\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} = i\omega\mu_0 H_\varphi, \tag{2.5}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho E_\varphi) - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \varphi} = i\omega\mu_0 H_z, \tag{2.6}$$

$$\frac{1}{\rho}\frac{\partial H_z}{\partial \varphi} - \frac{\partial H_\varphi}{\partial z} = -i\omega\epsilon E_\rho, \tag{2.7}$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} = -i\omega\epsilon E_\varphi, \tag{2.8}$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\varphi) - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \varphi} = -i\omega\epsilon E_z. \tag{2.9}$$

We consider the case of TE-polarization and assume [9] that $\mathbf{E}=\{0; E_\varphi; 0\}$, $\mathbf{H}=\{H_\rho; 0; H_z\}$. As a result, equations (2.4)–(2.9) are reduced to

$$-\frac{\partial E_\varphi}{\partial z} = i\omega\mu_0 H_\rho, \tag{2.10}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\varphi) = i\omega\mu_0 H_z, \tag{2.11}$$

$$\frac{1}{\rho}\frac{\partial H_z}{\partial\varphi} = 0, \tag{2.12}$$

$$\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial\rho} = -i\omega\epsilon E_\varphi, \tag{2.13}$$

$$-\frac{1}{\rho}\frac{\partial H_\rho}{\partial\varphi} = 0. \tag{2.14}$$

It follows from (2.12) and (2.14) that $H_z = H_z(\rho, z)$ and $H_\rho = H_\rho(\rho, z)$ do not depend on φ. Equations (2.10) and (2.11) yield

$$H_\rho = -\frac{1}{i\omega\mu_0}\frac{\partial E_\varphi}{\partial z}, \quad H_z = \frac{1}{i\omega\mu_0}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\varphi). \tag{2.15}$$

Insertion of H_ρ and H_z into (2.13) leads to

$$\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\varphi)\right) + \frac{\partial^2 E_\varphi}{\partial z^2} + \omega^2\epsilon\mu_0 E_\varphi = 0. \tag{2.16}$$

We look for solutions to this equation in the form of axially-symmetric waves $E_\varphi(\rho, z, \gamma) = u(\rho, \gamma)e^{i\gamma z}$, where γ is a real spectral parameter. Thus, (2.16) can be written as

$$(\frac{1}{\rho}(\rho u)')' + (\omega^2\epsilon\mu_0 - \gamma^2)u = 0, \tag{2.17}$$

where the prime denotes the differentiation with respect to ρ. Taking into account that $\epsilon = \epsilon_1$ outside the waveguide, we obtain the Bessel equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u - k_1^2 u = 0, \quad \rho > R, \tag{2.18}$$

where $k_1^2 = \gamma^2 - \omega^2\epsilon_1\mu_0$.

Inside the waveguide, where $\epsilon = \epsilon_2 + a|\mathbf{E}|^2$, we obtain a cubic-nonlinear second-order differential equation

$$u'' + \frac{1}{\rho}u' - \frac{1}{\rho^2}u + k_2^2 u + \alpha u^3 = 0, \quad 0 \le \rho \le R, \tag{2.19}$$

where $\alpha = \omega^2 a\mu_0$, $k_2^2 = \omega^2\epsilon_2\mu_0 - \gamma^2$, and $u(\rho, \gamma)$ is a real function. The continuity conditions on the interface are $[E_\varphi]_{\rho=R} = 0$ and $[H_z]_{\rho=R} = 0$, which lead to the conditions

$$[u]_{\rho=R} = 0, \quad [u']_{\rho=R} = 0, \tag{2.20}$$

where $[u]_{\rho=R} = u(R-0) - u(R+0)$. The spectral parameter is γ.

Let us formulate *problem P*: to find nontrivial bounded functions $u(\rho, \gamma)$ continuously differentiable on a semi-infinite interval $\rho > 0$ and the corresponding values of γ such that $u(\rho, \gamma)$ satisfies equations (2.18), (2.19) and the continuity (transmission) conditions (2.20).

Taking into account the radiation conditions, we choose the solution of the Bessel equation (2.18) in the form

$$u = C_1 K_1(k_1\rho), \quad \rho > R,$$

where K_1 is the Macdonald function and C_1 denotes an arbitrary real nonzero constant. The radiation conditions hold because $K_1(k_1\rho) \to 0$ exponentially as $\rho \to \infty$ for positive k_1.

We introduce the dimensionless variables and parameters $\tilde{\rho} = k_0\rho$, $\tilde{z} = k_0 z$, $\tilde{R} = k_0 R$, $\tilde{\epsilon} = \epsilon/\epsilon_0$, $\tilde{k}_2 = \sqrt{\tilde{\epsilon}_2 - (\tilde{\gamma})^2}$, $\tilde{k}_1 = \sqrt{(\tilde{\gamma})^2 - \tilde{\epsilon}_1}$ $(\tilde{\epsilon}_2 > \tilde{\epsilon}_1)$, $\tilde{\gamma} = \gamma/k_0$, $\tilde{\alpha} = aC_1^2/\epsilon_0$, $\tilde{u} = u/C_1$, and $k_0^2 = \omega^2\epsilon_0\mu_0$. Below, we omit the tildes and consider the problem in the normalized form. In particular,

$$u = K_1(k_1\rho), \quad \rho > R \tag{2.21}$$

in the normalized form.

3 Nonlinear integral equation and its solutions

Equation (2.19) can be written in the form (with $k = k_2$)

$$(\rho u')' + (k^2\rho - \frac{1}{\rho})u + \alpha\rho u^3 = 0; \tag{3.1}$$

the (linear) Bessel equation is written as

$$\rho u'' + u' + (k^2\rho - \frac{1}{\rho})u = 0. \tag{3.2}$$

Represent the latter in the operator form

$$Lu = 0, \quad L = \rho\frac{d^2}{d\rho^2} + \frac{d}{d\rho} + (k^2\rho - \frac{1}{\rho}). \tag{3.3}$$

Using standard methods [10] one can construct Green's function G for the boundary value problem

$$LG = -\delta(\rho - s),$$
$$G|_{\rho=0} = G'|_{\rho=R} = 0 \quad (0 \le s \le R)$$

in the form

$$G(\rho, s) = \frac{\pi}{2}\left[\frac{J_1(k\rho)J_1(ks)}{J_1'(kR)}N_1'(kR) - J_1(k\rho_<)N_1(k\rho_>)\right], \quad 0 \le \rho,\, s \le R, \tag{3.4}$$

where

$$\rho_< = \min\{\rho, s\}, \quad \rho_> = \max\{\rho, s\}. \tag{3.5}$$

In the operator form equation (2.19) reads

$$Lu + \alpha B(u) = 0, \quad B(u) = \rho u^3. \tag{3.6}$$

Using the second Green's formula

$$\int_0^R (vLu - uLv)d\rho = \int_0^R (v(\rho u')' - u(\rho v')')d\rho = R(u'(R)v(R) - v'(R)u(R)) \tag{3.7}$$

and setting $v = G$, we have

$$\int_0^R (GLu - uLG)d\rho = R(u'(R-0)G(R,s) - G'(R,s)u(R-0)) = Ru'(R-0)G(R,s). \tag{3.8}$$

Expressing the left-hand side by using (3.6)

$$\int_0^R (GLu - uLG)d\rho = -\alpha \int_0^R GB(u)d\rho + u(s), \tag{3.9}$$

we obtain an integral representation of the solution $u(s)$ to (2.19) in the interval $[0, R]$

$$u(s) = \alpha \int_0^R G(\rho, s)\rho u^3(\rho)d\rho + Ru'(R-0)G(R,s), \quad 0 \le s \le R. \tag{3.10}$$

Taking into account the transmission condition $u'(R-0) = u'(R+0)$ and formula (2.21), we transform equation (3.10) to obtain

$$u(s) = \alpha \int_0^R G(\rho, s)\rho u^3(\rho)d\rho + f(s), \quad 0 \le s \le R, \tag{3.11}$$

where

$$f(s) = Rk_1K_1'(k_1R)G(R,s) \tag{3.12}$$

and

$$G(R,s) = \frac{1}{k_2R}\frac{J_1(k_2s)}{J_1'(k_2R)}. \tag{3.13}$$

Note that $f(s)$ does not depend on u. The dispersion relation

$$u(R+0) = \alpha \int_0^R G(\rho, R)\rho u^3(\rho)d\rho + Rk_1K_1'(k_1R)G(R,R) \tag{3.14}$$

follows from the transmission conditions $u(R-0) = u(R+0)$ applied to (3.11).

We abbreviate $N(\rho, s) = \alpha G(\rho, s)\rho$ and consider an integral equation in $C[0, R]$

$$u(s) = \int_0^R N(\rho, s)u^3(\rho)\, d\rho + f(s) \tag{3.15}$$

assuming that $f \in C[0,R]$ $(J_1'(kR) \neq 0)$. The kernel $N(\rho, s)$ is continuous in the square $0 \leq \rho,\ s \leq R$.

The linear integral operator acting in $C[0,R]$

$$Nw = \int_0^R N(\rho, s) w(\rho)\, d\rho \tag{3.16}$$

is bounded with respect to the norm

$$\|N\| = \max_{s\in[0,R]} \int_0^R |N(\rho, s)|\, d\rho. \tag{3.17}$$

The nonlinear operator $B_0(u) = u^3(\rho)$ is bounded and continuous in $C[0,R]$. Hence, the nonlinear operator

$$F(u) = \int_0^R N(\rho, s) u^3(\rho)\, d\rho + f(s) \tag{3.18}$$

is completely continuous on each bounded subset in $C[0,R]$.

Below we need the solution of the equation

$$r - \|N\| r^3 = \|f\| \tag{3.19}$$

with

$$\|f\| = \max_{s\in[0,R]} |f(s)|\,. \tag{3.20}$$

The function

$$y(r) = r - \|N\| r^3 \tag{3.21}$$

has only one positive point of maximum $r_{max} = \dfrac{1}{\sqrt{3\|N\|}}$, where $y_{max} = y(r_{max}) = \dfrac{2}{3\sqrt{3\|N\|}}$. Subject to the condition

$$\|f\| < \frac{2}{3}\frac{1}{\sqrt{3\|N\|}} \tag{3.22}$$

equation (3.19) has two nonnegative solutions r_- and r^+, $r_- \leq r^+$, which satisfy the inequalities

$$\|f\| \leq r_- \leq \frac{1}{\sqrt{3\|N\|}}, \tag{3.23}$$

$$\frac{1}{\sqrt{3\|N\|}} \leq r^+ \leq \frac{1}{\sqrt{\|N\|}}. \tag{3.24}$$

Equation (3.19) has the roots

$$r_- = -2\sqrt{\frac{1}{3\|N\|}}\cos\left(\frac{\arccos(\frac{3\sqrt{3}}{2}\|f\|\sqrt{\|N\|})}{3} - \frac{2\pi}{3}\right), \tag{3.25}$$

$$r^+ = -2\sqrt{\frac{1}{3\|N\|}}\cos\left(\frac{\arccos(\frac{3\sqrt{3}}{2}\|f\|\sqrt{\|N\|})}{3} + \frac{2\pi}{3}\right). \tag{3.26}$$

If $\|f\| = 0$, then $r_- = 0$ and $r^+ = \frac{1}{\sqrt{\|N\|}}$. If condition (3.22) holds then

$$r_- < \frac{1}{\sqrt{3\|N\|}}. \tag{3.27}$$

If $\|f\| = \dfrac{2}{3}\dfrac{1}{\sqrt{3\|N\|}}$, then $r_- = r^+ = \dfrac{1}{\sqrt{3\|N\|}}$.

We have proved the following statement.

Lemma 1. *If the condition (3.22) holds then equation (3.19) has two nonnegative solutions r_- and r^+ such that $r_- < r^+$.*

Using Schauder's principle [12], one can prove that for each $f \in S_{\widehat{\rho}}(0) \subset C[0, R]$, where $\widehat{\rho} = \frac{2}{3}\frac{1}{\sqrt{3\|N\|}}$, there exists a solution $u(\rho, \gamma)$ for (3.15) inside the ball $S^+ = S_{r^+}(0)$.

Lemma 2. *If $\|f\| \le \frac{2}{3}\frac{1}{\sqrt{3\|N\|}}$, then equation (3.15) has at least one solution and $\|u\| \le r^+$.*

Proof. Since $F(u)$ is completely continuous, it is necessary to verify that F maps the ball S^+ into itself. Assume that $u \in S^+$. Using (3.18), (3.16), and (3.17) we obtain

$$\|F(u)\| \le \|N\|\|u\|^3 + \|f\| \le \|N\|(r^+)^3 + \|f\| = r^+.$$

It means that $FS^+ \subset S^+$. ■

Next we prove that if (3.22) holds then (3.15) has a unique solution inside the ball $S_- = S_{r_-}$. We introduce the kernel $N_0(\rho, s) = \rho G(\rho, s)$ such that $N(\rho, s) = \alpha N_0(\rho, s)$ and $\|N\| = |\alpha|\|N_0\|$. Note that $N_0(\rho, s)$ does not depend on α.

Theorem 1. *If $|\alpha| < A^2$, where*

$$A = \frac{2}{3}\frac{1}{\|f\|\sqrt{3\|N_0\|}} \tag{3.28}$$

and

$$\|N_0\| = \max_{s\in[0,R]}\int_0^R |\rho G(\rho, s)|\, d\rho$$

($A > 0$ does not depend on α), then (3.6) has a unique solution u and this solution is a continuous function: $u \in C[0, R]$, $\|u\| \le r_-$.

Proof. If $u \in S_-$, then

$$\|F(u)\| \leq \|N\|\|u^3\| + \|f\| \leq \|N\|r_-^3 + \|f\| = r_-.$$

According to Schauder's theorem, (3.15) has at least one solution if $f \in S_{\widehat{\rho}}(0)$. If $u_1, u_2 \in S_-$, then

$$\|F(u_1) - F(u_2)\| = \|\int_0^R N(\rho, s)(u_1^3(\rho) - u_2^3(\rho))d\rho\| \leq 3\|N\|r_-^2\|u_1 - u_2\|.$$

It follows from (3.28) that condition $|\alpha| < A^2$ is equivalent to (3.22). If $f(s)$ satisfies (3.22) then (3.27) holds and we have $3\|N\|r_-^2 < 1$.

Therefore, under the condition (3.22), F maps S_- into itself. Also F is a contraction on S_-. Thus (3.15) has the unique solution in S_-. ■

Below we need a statement about the dependence of solutions of equation (3.15) on parameters.

Theorem 2. *Let N and f in (3.15) be continuous functions of parameter $\lambda \in \Lambda_0$, $N(\lambda, \rho, s) \subset C(\Lambda_0 \times [0, R] \times [0, R])$, and $f(\lambda, s) \subset C(\Lambda_0 \times [0, R])$ on a real segment Λ_0. Assume also that the inequality*

$$\|f(\lambda)\| < \frac{2}{3}\frac{1}{\sqrt{3\|N(\lambda)\|}}, \quad \lambda \in \Lambda_0, \tag{3.29}$$

is valid. Then equation (3.15) has one and only one solution $u(\rho, \lambda)$ for each $\lambda \in \Lambda_0$ which is a continuous function of parameter λ, $u(\rho, \lambda) \subset C(\Lambda_0 \times [0, R])$.

Proof. Consider the equation

$$u(s, \lambda) = \int_0^R N(\lambda, \rho, s)u^3(\rho, \lambda)d\rho + f(s, \lambda). \tag{3.30}$$

Existence and uniqueness of solutions $u(\lambda)$ follow from Theorem 1. We will prove the continuous dependence of the solutions on parameter λ.

From (3.25), it follows that $r_-(\lambda)$ is a continuous function of λ on segment Λ_0. Set $r_0 = \max_{\lambda\in\Lambda_0} r_-(\lambda)$; this function has a maximum at the point λ_0 with $r_-(\lambda_0) = r_0$.

Next, set $Q = \max\limits_{\lambda\in\Lambda_0}(3r_-^2(\lambda)\|N(\lambda)\|)$; this function has a maximum at the point $\widehat{\lambda} \in \Lambda_0$ with $Q = 3r_-^2(\widehat{\lambda})\|N(\widehat{\lambda})\|$, and $Q < 1$ under the condition (3.29). Assume first that $\|u(\lambda)\| \geq \|u(\lambda + \Delta\lambda)\|$. Then we have

$$|u(s, \lambda + \Delta\lambda) - u(s, \lambda)| = |\int_0^R N(\lambda + \Delta\lambda, \rho, s)u^3(\rho, \lambda + \Delta\lambda)d\rho$$

$$- \int_0^R N(\lambda, \rho, s)u^3(\rho, \lambda)d\rho + (f(s, \lambda + \Delta\lambda) - f(s, \lambda))|$$

$$\leq \int_0^R |N(\lambda+\Delta\lambda,\rho,s)-N(\lambda,\rho,s)|\,|u(\rho,\lambda+\Delta\lambda)|^3 d\rho+\int_0^R |N(\lambda,\rho,s)|\,|u^3(\rho,\lambda+\Delta\lambda)-u^3(\rho,\lambda)|d\rho$$

$$+|f(s,\lambda+\Delta\lambda)-f(s,\lambda)| \leq \|u(\lambda+\Delta\lambda)\|^3\int_0^R |N(\lambda+\Delta\lambda,\rho,s)-N(\lambda,\rho,s)|d\rho$$

$$+\|u(\lambda+\Delta\lambda)-u(\lambda)\|(\|u(\lambda+\Delta\lambda)\|^2+\|u(\lambda+\Delta\lambda)\|\|u(\lambda)\|+\|u(\lambda)\|^2)\times$$

$$\times\int_0^R |N(\lambda,\rho,s)|d\rho+\|f(\lambda+\Delta\lambda)-f(\lambda)\|$$

$$\leq r_0^3\|N(\lambda+\Delta\lambda)-N(\lambda)\|$$

$$+\|u(\lambda+\Delta\lambda)-u(\lambda)\|3r_-^2(\lambda)\|N(\lambda)\|+\|f(\lambda+\Delta\lambda)-f(\lambda)\|.$$

Hence,

$$\|u(\lambda+\Delta\lambda)-u(\lambda)\| \leq r_0^3\|N(\lambda+\Delta\lambda)-N(\lambda)\|+$$

$$+\|u(\lambda+\Delta\lambda)-u(\lambda)\|3r_-^2(\lambda)\|N(\lambda)\|+\|f(\lambda+\Delta\lambda-f(\lambda)\|,$$

and we obtain

$$\|u(\lambda+\Delta\lambda)-u(\lambda)\| \leq \frac{1}{1-3r_-^2(\lambda)\|N(\lambda)\|}(r_0^3\|N(\lambda+\Delta\lambda)-N(\lambda)\|+\|f(\lambda+\Delta\lambda-f(\lambda)\|)$$

and

$$\|u(\lambda+\Delta\lambda)-u(\lambda)\| \leq \frac{1}{1-Q}(r_0^3\|N(\lambda+\Delta\lambda)-N(\lambda)\|+\|f(\lambda+\Delta\lambda)-f(\lambda)\|), \quad (3.31)$$

where Q and r_0 do not depend on λ.

Let $\|u(\lambda)\| < \|u(\lambda+\Delta\lambda)\|$. Then all estimates are valid if we replace λ by $\lambda+\Delta\lambda$ and $\lambda+\Delta\lambda$ by λ respectively. Thus estimate (3.31) is also valid. ■

Approximate solutions u_n to the integral equation (3.15) represented in the form $u = F(u)$ can be determined using the iteration procedure

$$u_{n+1} = F(u_n) = \alpha\int_0^R G(\rho,s)\rho u_n^3 d\rho + f, \quad n = 0,1,\ldots. \quad (3.32)$$

Remark: It is useful to choose $u_0 = f$, i.e. equal to the solution of the linear problem (with $\alpha = 0$ in Eq. (2.19)), where the eigenvalue γ^2 has to be determined by the nonlinear dispersion relation (cf. Section 4).

The sequence u_n converges uniformly to the solution u of (3.15) because $F(u)$ is a contraction [11]. The rate of convergence of the algorithm can also be estimated.

Proposition 1. *The sequence of approximations u_n of equation (3.15) defined by (3.32) exists and converges uniformly with respect to the $C[0,R]$-norm to the (unique) exact solution u of this equation. The estimate*

$$\|u_n-u\| \leq \frac{q^n}{1-q}\|f\|, \quad n\to\infty, \quad (3.33)$$

holds, where $q = 3\|N\|r_-^2 < 1$ *is the coefficient of contraction of* F.

It follows from (3.32) that $u_n = u_n(\lambda)$ is a continuous function with respect to λ.

4 Existence of solution to the dispersion equation

It follows from (3.13) and the properties of cylindrical functions that

$$G(R,R) = \frac{1}{k_2 R}\frac{J_1(k_2 R)}{J_1'(k_2 R)}.$$

Substituting this formula into (3.14), and taking into account (2.21) we can write the dispersion relation (3.14) in the form

$$\Phi(\lambda, R; u) = g(\lambda, R) - \alpha F_1(\lambda, R; u) = 0, \tag{4.1}$$

where

$$g(\lambda, R) = k_2 R K_1(k_1 R) J_0(k_2 R) + k_1 R K_0(k_1 R) J_1(k_2 R), \tag{4.2}$$

$$F_1(\lambda, R; u) = \int_0^R J_1(k_2\rho)\rho u^3(\rho) d\rho. \tag{4.3}$$

Zeros of the function $\Phi(\lambda, R; u)$ are the eigenvalues $\gamma^2 = \lambda$ associated to nontrivial solutions of the problem P. The following statement gives sufficient conditions for the existence of these eigenvalues.

Let j_{0m}, j_{1m} and j'_{1m} $(m = 1, 2, \dots)$ be (positive) zeros of Bessel functions J_0, J_1, and J_1', respectively. List the values of the zeros

$$\begin{array}{lllllllll}
j'_{11} & = & 1.841\dots, & j_{01} & = & 2.405\dots, & j_{11} & = & 3.832\dots,\\
j'_{12} & = & 5.331\dots, & j_{02} & = & 5.520\dots, & j_{12} & = & 7.016\dots,\\
j'_{13} & = & 8.536\dots, & j_{03} & = & 8.654\dots, & j_{13} & = & 10.173\dots,\\
 & \vdots & & & \vdots & & & \vdots &
\end{array}$$

Denote

$$\begin{aligned}
\lambda_{1m} &= \epsilon_2 - j_{1m}^2/R^2, \quad \lambda_{2m} = \epsilon_2 - j_{0m}^2/R^2,\\
\Lambda_i &= [\lambda_{1i}, \lambda_{2i}], \quad \Lambda = \bigcup_{i=1}^{m} \Lambda_i, \quad m = 1, 2, \dots,
\end{aligned}$$

and prove the following

Theorem 3. *Let* ϵ_1, ϵ_2 *and* α *satisfy the conditions* $\epsilon_2 > \epsilon_1 > 0$ *and* $0 < |\alpha| < \alpha_0$, *where*

$$\alpha_0 = \min\{A_1^2, A_2\},$$

$$A_1 = \min_{\lambda\in\Lambda} A(\lambda), \quad A_2 = \frac{\min\limits_{l=1,2,\,1\le i\le m} |g(\lambda_{li})|}{0.3R^2[\max\limits_{\lambda\in\Lambda} r_-(\lambda)]^3}, \tag{4.4}$$

and the inequality

$$\lambda_{1m} > \epsilon_1 \tag{4.5}$$

for a certain $m \ge 1$. Then there exist at least m values λ_i, $i = 1, \ldots, m$, $\lambda_{1i} < \lambda_i < \lambda_{2i}$ such that problem P has a nontrivial solution.

Proof. Let $i \ge 1$ be a fixed index. It is well known that $j_{0i} < j_{1i} < j_{0,i+1}$ and $j'_{1i} < j_{1i} < j'_{1,i+1}$. Hence, we have $j_{0i} < j_{1i} < j'_{1,i+1}$. There exists only one zero $j'_{1i} \in (j_{1,i-1}, j_{1i})$, where $j_{10} = 0$. Futhermore, from the alternation of zeros of functions $J_0(x)$ and $J_2(x)$ we obtain $\mathrm{sign} J_0(j_{2i}) = (-1)^i$ and $\mathrm{sign} J_2(j_{0i}) = (-1)^{i+1}$, where j_{2i} are positive zeros of Bessel function $J_2(x)$. It follows from $2J'_1(x) = J_0(x) - J_2(x)$ that $j'_{1i} \in (j_{2,i-1}, j_{0i})$ $(j_{20} = 0)$. Since $j'_{1i} < j_{0i} < j_{1i} < j_{1,i+1}$ $(i \ge 1)$, Green's function (3.4) exists for $\lambda \in \Lambda$. It follows from (3.28) and properties of Green's function that $A = A(\lambda)$ is a continuous function with respect to $\lambda \in \Lambda$. Set $A_1 = \min\limits_{\lambda\in\Lambda} A(\lambda)$ and take $|\alpha| < A_1^2$. According to Theorem 1, there exists the unique solution $u = u(\lambda)$ of equation (3.6) for each $\lambda \in \Lambda$. This solution is a continuous function and $\|u\| \le r_- = r_-(\lambda)$. Set $r_0 = \max\limits_{\lambda\in\Lambda} r_-(\lambda)$. Using the inequality $|J_1(x)| \le 0.6$, which is valid for nonnegative x, and estimating the integral in (4.3) we obtain $|F_1(\lambda, R; u)| \le 0.3R^2{r_0}^3$.

The Macdonald functions $K_0(x)$ and $K_1(x)$ are positive for positive x, $g(\lambda)$ is continuous, and $g(\lambda_{1i})g(\lambda_{2i}) < 0$, $i = 1, \ldots, m$. Therefore, the equation $g(\lambda) = 0$ has a root λ_{0i} on interval Λ_i, $\lambda_{1i} < \lambda_{0i} < \lambda_{2i}$.

Denote $M_1 = \min\limits_{1\le i\le m} |g(\lambda_{1i})|$, $M_2 = \min\limits_{1\le i\le m} |g(\lambda_{2i})|$, and $M = \min\{M_1, M_2\}$; $M > 0$ and does not depend on α.

If $|\alpha| \le \frac{M}{0.3R^2{r_0}^3}$ then $(g(\lambda_{1i}) - \alpha F_1(\lambda_{1i}))(g(\lambda_{2i}) - \alpha F_1(\lambda_{2i})) < 0$. Since $g(\lambda) - \alpha F_1(\lambda, R; u)$ is also a continuous function, the equation $g(\lambda) - \alpha F_1(\lambda, R; u) = 0$ has a root λ_i on interval Λ_i, $\lambda_{1i} < \lambda_i < \lambda_{2i}$. We can choose $\alpha_0 = \min\{A_1^2, \frac{M}{0.3R^2{r_0}^3}\}$. ■

Remarks: $\|f\|$ and A_1 implicitly depend on α via the nonlinear dispersion relation. In this sense A_2 depends on α both explicitly and implicitly. Note that $\lim_{\alpha\to 0} r_- = \|f\| > 0$.

The condition (4.5) implies $R^2 > j_{11}^2/(\epsilon_2 - \epsilon_1)$. Thus, the radius R cannot be arbitrarily small (this is similar to the existence of a 'cut-off' radius in the linear case). In view of this fact, the sufficient conditions for the existence of a nontrivial solution of problem P require the smallness of the nonlinearity parameter α, radius R, and the material parameter ϵ_2 of the waveguide.

Theorem 4. *Assume that ϵ_1, ϵ_2, and α satisfy the conditions $\epsilon_2 > \epsilon_1 > 0$ and $0 < |\alpha| < \alpha_0$, where α_0 is given by (4.4), and the condition (4.5) for a certain $m \ge 1$. Then for each $n \ge 0$ there exist at least m values $\lambda_i^{(n)}$, $i = 1, \ldots, m$, satisfying $\lambda_{1i} < \lambda_i^{(n)} < \lambda_{2i}$ that are roots of the equation*

$$k_2^{(n)} R K_1(k_1^{(n)} R) J_0(k_2^{(n)} R) + k_1^{(n)} R K_0(k_1^{(n)} R) J_1(k_2^{(n)} R) = \alpha \int_0^R J_1(k_2^{(n)} \rho) \rho u_n^3(\rho) d\rho, \tag{4.6}$$

where $k_1^{(n)} = \sqrt{\lambda^{(n)} - \epsilon_1}$, $k_2^{(n)} = \sqrt{\epsilon_2 - \lambda^{(n)}}$ *and* u_n *is determined according to (3.32).*

Proof. For each $n \geq 0$, functions u_n are continuous according to (3.32). Therefore it is sufficient to repeat the proof of Theorem 3, in which u should be replaced by u_n and to check the condition $\|u_n\| \leq r_- = r_-(\lambda)$. This inequality is valid because all iterations belong to the ball S_*. ■

Theorem 5. *Let* λ_i *and* $\lambda_i^{(n)}$ *be, respectively, an exact and an approximate eigenvalue of problem* P *on the interval* $[\lambda_{1i}, \lambda_{2i}]$ *(*λ_i *and* $\lambda_i^{(n)}$ *are roots of the dispersion equations (4.1) and (4.6), respectively,* $i \geq 1$*). Then* $|\lambda_i^{(n)} - \lambda_i| \to 0$ *as* $n \to \infty$*.*

Proof. Consider the functions

$$\Phi(\lambda, R; u) = g(\lambda, R) - \alpha F_1(\lambda, R; u), \Phi_n(\lambda, R; u_n) = g(\lambda, R) - \alpha F_1(\lambda, R; u_n). \quad (4.7)$$

Then

$$\begin{aligned} |\Phi(\lambda, R; u) - \Phi_n(\lambda, R; u_n)| &= |\alpha||F_1(\lambda, R; u) - F_1(\lambda, R; u_n)| \\ &= |\alpha| \left| \int_0^R J_1(k_2 R)\rho(u^3 - u_n^3)d\rho \right| \\ &\leq |\alpha|\|u - u_n\|(\|u\|^2 + \|u\|\|u_n\| + \|u_n\|^2) \int_0^R |J_1(k_2 R)|\, \rho d\rho \\ &\leq |\alpha| \frac{q^n}{1-q} \|f\| 3r_-^2 T, \end{aligned} \quad (4.8)$$

where $T = \int_0^R |J_1(k_2 R)|\, \rho d\rho$ and all other quantities were defined above.

If Λ_0 is an intervall that does not contain point j'_{1i}, then we have

$$\max_{\lambda \in \Lambda_0} |\Phi(\lambda, R; u) - \Phi_n(\lambda, R; u_n)| \leq |\alpha| \frac{Q^n}{1-Q} T_0, \quad (4.9)$$

where $T_0 = \max_{\lambda \in \Lambda_0}\{\|f(\lambda)\| 3r_-^2(\lambda)T(\lambda)\}$ and $Q < 1$.

Subject to the conditions of Theorems 3 and 4, there exist solutions λ_i and $\lambda_i^{(n)}$ of the exact and approximate dispersion equations $\Phi(\lambda, R; u) = 0$ and $\Phi_n(\lambda, R; u_n) = 0$ $(n \geq 0)$. Also, in the proof of Theorems 3 and 4, it was established that continuous functions $\Phi(\lambda, R; u)$ and $\Phi_n(\lambda, R; u_n)$ change signs at the endpoints of the interval $[\lambda_{1i}, \lambda_{2i}]$. The proof follows now from estimate (4.9). ■

5 Concluding remarks

(i) The above analysis can be applied straightforwardly to the polarization case $\mathbf{E} = \{0; 0; E_z\}$, $\mathbf{H} = \{H_\rho; H_\varphi; 0\}$.

(ii) It would be worthwhile to perform a stability analysis of the various modes resulting from the discussion of the dispersion relation. It is well known [13] that the power flow integral $P = \int_0^\infty d\rho \rho \overline{S}_z(\rho, \gamma)$, where $\overline{S}_z$ denotes the time-averaged z-component of the Poynting vector, can serve for a stability analysis in special cases. $\overline{S}_z$ can be calculated by means of the iterate solutions $u_n(\rho, \gamma)$.

(iii) Since the approach is rather general, nonlinearities that model physics more realistically than equation (2.1) (e.g. saturating and higher order nonlinearities) can be investigated.

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