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Two New Classes of Isochronous Hamiltonian Systems

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Abstract

An *isochronous* dynamical system is characterized by the existence of an *open* domain of initial data such that *all* motions evolving from it are *completely periodic* with a *fixed* period (independent of the initial data). Taking advantage of a recently introduced trick, two new *Hamiltonian* classes of such systems are identified.

1 Introduction and main results

An *isochronous* dynamical system is characterized by the existence of an *open* domain of initial data such that *all* motions evolving from it are *completely periodic* with a *fixed* period (independent of the initial data). Of course this *open* domain of initial data has nonvanishing measure, namely it possesses the full dimensionality of such initial data. Recently a trick – actually, merely a convenient change of (independent and dependent) variables – has been introduced [1], that has the potential to deform quite a large class of dynamical systems and evolution PDEs so that the deformed evolutions are *isochronous*. This approach has been investigated in various contexts [2] [13] [18] [14] [8] [3] [21] [19] [4] [5] [6] [7] [15] [16] [12] [9] [10] [17] [11] [20] [26]. In this paper we use it once more to identify two *new* classes of such *Hamiltonian* dynamical systems.

The first class is characterized by the Newtonian equations of motion

$$\ddot{z}_n + \omega^2 z_n = -\left[\dot{z}_n^2 + \omega^2 \left(c_n + z_n^2\right)\right]^{1/2} \frac{\partial F(\underline{z})}{\partial z_n}, \quad n = 1, ..., N, \tag{1.1}$$

where the function $F(\underline{z})$ of the N-vector $\underline{z} \equiv (z_1, ..., z_N)$ is only required to be *analytic* in its N arguments and to be a sum of functions scaling as the negative even integer powers,

$$F(\underline{z}) = \sum_{k=1}^{K} F^{(-2k)}(\underline{z}), \tag{1.2a}$$

$$F^{(-2k)}(\alpha \underline{z}) = \alpha^{-2k} F^{(-2k)}(\underline{z}). \tag{1.2b}$$

In these Newtonian equations (1.1), and throughout, superimposed dots denote differentiations with respect to the *real* independent variable t ("time"), the N dependent variables $z_n \equiv z_n(t)$ are *complex*, the constant ω is *real* (without significant loss of generality, *positive*), and the N constants c_n are *arbitrary* (possibly *complex*, possibly *vanishing*); and in the scaling relations (1.2), and throughout, K is an arbitrary *positive* integer.

The second class of *isochronous* Hamiltonian systems is characterized by the Newtonian equations of motion

$$\begin{aligned} & \ddot{z}_n + 2\,i\,\mu\,\omega\,\dot{z}_n + (1-\mu^2)\,\omega^2\,z_n \\ = & -[\dot{z}_n + i\,(1+\mu)\,\omega\,z_n]^{\frac{1}{2}(1-\mu)}\,[\dot{z}_n - i\,(1-\mu)\,\omega\,z_n]^{\frac{1}{2}(1+\mu)}\,\frac{\partial F(\underline{z})}{\partial z_n}\,, \\ n = & \; 1,...,N, \end{aligned} \tag{1.3a}$$

where μ is a *rational* number different from negative unity,

$$\mu = \frac{p}{q}\,, \quad \mu \neq -1\,, \tag{1.3b}$$

with p and q coprime *integers* (and, for definiteness, q *positive*, $q > 0$). [A generalization of this system, characterized by the replacement of the single *rational* number μ by N arbitrary, but *rational*, numbers μ_n, is reported below, see (1.8), and discussed in Section 4]. The function $F(\underline{z})$ of the N-vector $\underline{z} \equiv (z_1, ..., z_N)$ is again required to be *analytic* in its N arguments and is now characterized by the following scaling property:

$$F(\underline{z}) = F^{(0)}(\underline{z}) + \sum_{k=1}^{K} F^{(a_k)}(\underline{z}), \tag{1.4a}$$

$$F^{(a_k)}(\alpha\,\underline{z}) = \alpha^{a_k}\,F^{(a_k)}(\underline{z}), \quad k = 1,...,K; \qquad F^{(0)}(\alpha\,\underline{z}) = F^{(0)}(\underline{z}) + A(\alpha), \tag{1.4b}$$

where

$$a_k = -\frac{2\,k}{1+\mu}\,, \quad k = 1,...,K. \tag{1.4c}$$

And, in order that *isochronicity* be guaranteed, one more condition is required:

$$\mu > -3\,, \quad \text{or} \quad \mu < -(1+2\,K)\,. \tag{1.5}$$

Note that these two classes, (1.1) with (1.2) and (1.3) with (1.4), coincide in the special case $\mu = 0$, $c_n = 0$, $n = 1,...,N$.

In the following Section 2 we demonstrate the *Hamiltonian* character of both these Newtonian equations of motion, (1.1) and (1.3) (and also of the generalized version (1.8), see below), by showing that they are special cases of a general class of Newtonian equations of motion obtainable in a standard manner from a *Hamiltonian*; and we moreover exhibit some interesting examples of these Newtonian equations of motion. In the subsequent Section 3 we prove that, provided the function $F(\underline{z})$ satisfies the scaling properties detailed above (see (1.2)), the first class of Newtonian equations of motion (1.1) (in fact, a more general, albeit not necessarily Hamiltonian, class of evolution equations that include the class (1.1)), features a (complex) *open* domain of initial data $\underline{z}(0)$, $\underline{\dot{z}}(0)$, having of course full dimensionality in the (complex) phase space of such initial data, such that *all* the motions evolving out of it are *completely antiperiodic* with period $\frac{T}{2} = \frac{\pi}{\omega}$, hence *completely periodic* with period

$$T = \frac{2\pi}{\omega}, \tag{1.6}$$

$$\underline{z}(t + \frac{T}{2}) = -\underline{z}(t), \quad \underline{z}(t+T) = \underline{z}(t). \tag{1.7a}$$

The analogous result in the case of the Newtonian equations of motion (1.3) demonstrates the existence of an open domain of initial data $\underline{z}(0)$, $\dot{\underline{z}}(0)$, having of course again full dimensionality in the (complex) phase space of such initial data, such that *all* the motions evolving out of it are *completely* $(-\mu)$-*antiperiodic* with period $\frac{T}{2}$, hence *completely periodic* with period $\frac{qT}{2}$ respectively qT if $p+q$ is *odd* respectively *even* (see (1.3b)),

$$\underline{z}(t + \frac{T}{2}) = -\exp(-\pi\, i\, \mu)\underline{z}(t), \quad \underline{z}(t + \frac{q\,T}{2}) = (-)^{p+q+1}\, \underline{z}(t)\ . \tag{1.7b}$$

This result is proven in Section 4, where we actually show that the Newtonian equations of motion

$$\begin{aligned} & \ddot{z}_n + 2\, i\, \mu_n\, \omega\, \dot{z}_n + (1-\mu_n^2)\, \omega^2\, z_n \\ = & -[\dot{z}_n + i\,(1+\mu_n)\,\omega\, z_n]^{\frac{1}{2}(1-\mu_n)}\, [\dot{z}_n - i\,(1-\mu_n)\,\omega\, z_n]^{\frac{1}{2}(1+\mu_n)}\, \frac{\partial F(\underline{z})}{\partial z_n}\ , \\ n = & \ 1, ..., N, \end{aligned} \tag{1.8}$$

where the N constants μ_n are *rational* numbers, that clearly reduce to (1.3a) if $\mu_n = \mu$, and which are also shown to be *Hamiltonian* in the following Section 2, are as well *isochronous* provided the function $F(\underline{z})$ satisfies the scaling property

$$F(\underline{z}) = \sum_{k=1}^{K} F^{(a_{1k},...,a_{Nk})}(\underline{z})\ , \tag{1.9a}$$

$$F^{(a_{1k},...,a_{Nk})}(\alpha_1\, z_1, ..., \alpha_N\, z_N) = \left[\prod_{n=1}^{N} (\alpha_{nk})^{a_{nk}}\right]^{\frac{1}{N}} F^{(a_{1k},...,a_{Nk})}(\underline{z})\ , \quad k = 1, ..., K, \tag{1.9b}$$

with

$$\frac{1}{N} \sum_{n=1}^{N} a_{nk}\ (1+\mu_n) = -2\, k\ , \quad n = 1, ..., N, \quad k = 1, ..., K\ . \tag{1.9c}$$

[These scaling properties reduce of course to (1.4) if $\mu_n = \mu$ and $a_{nk} = a_k$]. But, as shown in Section 4, to guarantee *isochronicity* an additional condition is required: there should exist N numbers A_n such that the following $N\,K$ inequalities hold:

$$\begin{aligned} & -A_n + \frac{1}{N} \sum_{m=1}^{N} A_m\, a_{mk} \quad \geq \quad k + \frac{1}{2}\,(1-\mu_n), \\ & n = 1, ..., N, \quad k = 1, ..., K. \end{aligned} \tag{1.10}$$

These last conditions, however, are not too restrictive. For instance, by setting

$$A_n = -\frac{1}{2} b \, (1+\mu_n) + c \,, \quad n = 1, ..., N \tag{1.11a}$$

with b and c two arbitrary constants, they become (via (1.9c))

$$b \left[k + \frac{1}{2} \, (1+\mu_n) \right] + c \, (1 - \overline{a}_k) \;\; \geq \;\; k + \frac{1}{2} \, (1 - \mu_n) \,, \\ n = 1, ..., N, \quad k = 1, ..., K \,, \tag{1.11b}$$

where we introduced the convenient definition

$$\overline{a}_k = \frac{1}{N} \sum_{n=1}^{N} a_{nk} \,. \tag{1.11c}$$

And it is clear that these conditions, (1.11b), can be satisfied (at least) in any one of the following cases (and we indicate in each case in square brackets via which assignment of the two arbitrary constants b, c):

$$\text{case (i):} \quad \mu_n > -3, \; n = 1, ..., N \,, \qquad \left[b \geq \max \left(\frac{2\,k+1-\mu_n}{2\,k+1+\mu_n} \right) \,, \quad c = 0 \right] , \tag{1.12a}$$

$$\text{case (ii):} \quad \mu_n < -(1+2\,K) \,, \; n = 1, ..., N \,, \qquad \left[b \leq \min \left(\frac{2\,k+1-\mu_n}{2\,k+1+\mu_n} \right) \,, \quad c = 0 \right] , \tag{1.12b}$$

$$\text{case (iii):} \quad \overline{a}_k < 1, \; k = 1, ..., K \,, \qquad \left[b = 0, \; c \geq \max \left(\frac{k + \frac{1}{2} \, (1-\mu_n)}{1 - \overline{a}_k} \right) \right] , \tag{1.12c}$$

$$\text{case (iv):} \quad \overline{a}_k > 1, \; k = 1, ..., K \,, \qquad \left[b = 0, \; c \leq \min \left(\frac{k + \frac{1}{2} \, (1-\mu_n)}{1 - \overline{a}_k} \right) \right] . \tag{1.12d}$$

Note that the first two of these four cases correspond to the condition (1.5) when $\mu_n = \mu$. Also note that the last two of these four cases imply that these conditions can always be satisfied if $K = 1$, unless $\overline{a}_k = 1$, in which case the sufficient conditions for *isochronicity* are provided by the two cases (i) and (ii) (namely, the rational numbers μ_n are either *all* smaller, or *all* larger, than -3).

Final remarks are reported in the last Section 5.

2 A class of Hamiltonian systems

In this section we introduce a Hamiltonian $H(\underline{p}, \underline{z})$ that yields, in the standard manner, Newtonian equations of motions that include all those – (1.1), (1.3) and (1.8) – reported above, and we exhibit interesting examples of the Newtonian equations of motion (1.1) and (1.3).

This Hamiltonian reads

$$H(\underline{p},\underline{z}) = \sum_{n=1}^{N} [\varphi_n(p_n)\, g_n(z_n)] + F(\underline{z}), \tag{2.1}$$

hence it yields the Hamiltonian equations

$$\dot{z}_n = \varphi_n'(p_n)\, g_n(z_n) \ , \quad n = 1, ..., N, \tag{2.2a}$$

$$\dot{p}_n = -\varphi_n(p_n)\, g_n'(z_n) - \frac{\partial F(\underline{z})}{\partial z_n} \ , \quad n = 1, ..., N. \tag{2.2b}$$

Here and throughout appended primes denote differentiations with respect to the argument of the function they are appended to.

By t-differentiating the first, (2.2a), of these Hamiltonian equations and by then using both of them one gets

$$\begin{aligned} \ddot{z}_n &= \left\{ \left[\varphi_n'(p_n)\right]^2 - \varphi_n''(p_n)\, \varphi_n(p_n) \right\} g_n'(z_n)\, g_n(z_n) - \varphi_n''(p_n)\, g_n(z_n) \frac{\partial F(\underline{z})}{\partial z_n} \ , \\ n &= 1, ..., N. \end{aligned} \tag{2.3}$$

It is then clear that, via (2.2a), the assignment

$$\varphi_n(p) = \cosh(p), \qquad g_n(z) = \left(z^2 + c_n\right)^{1/2} \tag{2.4}$$

yields the Newtonian equation of motion (1.1), which is thereby shown to be implied by the Hamiltonian (2.1) with (2.4).

To obtain the Newtonian equations of motion (1.8) (of which (1.3) are a subcase) we set

$$g_n(z) = z \ , \quad n = 1, ..., N, \tag{2.5}$$

as well as

$$\varphi_n''(p)\, \varphi_n(p) = \left[\varphi_n'(p)\right]^2 + 2\, i\, \omega\, \mu_n\, \varphi_n'(p) + (1 - \mu_n^2)\, \omega^2, \quad n = 1, ..., N, \tag{2.6a}$$

or equivalently

$$\varphi_n''(p)\, \varphi_n(p) = \left[\varphi_n'(p) + i\ (1 + \mu_n)\ \omega\right]\ \left[\varphi_n'(p) - i\ (1 - \mu_n)\ \omega\right], \quad n = 1, ..., N. \tag{2.6b}$$

Each of these (decoupled) ODEs can be easily integrated once (after dividing out by the right-hand side and by $\varphi_n(p)$, and multiplying by $\varphi_n'(p)$), getting thereby

$$\begin{aligned} \varphi_n(p) &= \left[\varphi_n'(p) + i\ (1 + \mu_n)\ \omega\right]^{\frac{1}{2}(1+\mu_n)} \left[\varphi_n'(p) - i\ (1 - \mu_n)\ \omega\right]^{\frac{1}{2}(1-\mu_n)}, \\ n &= 1, ..., N \ , \end{aligned} \tag{2.6c}$$

where, without significant loss of generality, we set to unity an *a priori* arbitrary multiplicative integration constant. This relation (2.6c) entails, via (2.6b) and again (2.6c), the following (explicit) expression of $\varphi_n''(p)$ in terms of $\varphi_n'(p)$,

$$\begin{aligned} \varphi_n''(p) &= \left[\varphi_n'(p) + i\ (1 + \mu_n)\ \omega\right]^{\frac{1}{2}(1+\mu_n)} \left[\varphi_n'(p) - i\ (1 - \mu_n)\ \omega\right]^{\frac{1}{2}(1-\mu_n)}, \\ n &= 1, ..., N. \end{aligned} \tag{2.6d}$$

And the insertion of (2.5), (2.6a) and (2.6d) in (2.3) yields, via (2.2a) with (2.5), precisely (1.8), the *Hamiltonian* character of which is thereby demonstrated. Note that we managed to do this without being generally able to provide an explicit expression of the functions $\varphi_n(p)$ (because the ODEs (2.6c) cannot be generally explicitly integrated).

Let us end this section by exhibiting two interesting instances of the Newtonian equations of motion (1.1) with (1.2) respectively (1.3) with (1.4).

The following choice of the function $F(\underline{z})$ is consistent with the scaling property (1.2) and it moreover entails that the right-hand side of the Newtonian equations of motion (1.1) only feature *two-body* forces:

$$F(\underline{z}) = \frac{1}{4} \sum_{m,n=1,m\neq n}^{N} \sum_{k=1}^{K} \frac{g_{nm}^{(k)}}{k\,(z_n - z_m)^{2k}} \ . \tag{2.7a}$$

Indeed the corresponding Newtonian equations of motion (1.1) read

$$\ddot{z}_n + \omega^2 z_n = \left[\dot{z}_n^2 + \omega^2 \left(c_n + z_n^2\right)\right]^{1/2} \sum_{m=1,m\neq n}^{N} \sum_{k=1}^{K} \frac{g_{nm}^{(k)}}{(z_n - z_m)^{2k+1}}, \quad n = 1, ..., N. \tag{2.7b}$$

Here the "coupling constants" $g_{nm}^{(k)}$ are *arbitrary* (possibly *complex*), except for the obvious (see (2.7a)) symmetry property $g_{nm}^{(k)} = g_{mn}^{(k)}$. The Hamiltonian that yields these Newtonian equations of motion is of course given by (2.1) with (2.4) and (2.7a).

To provide a neat (and quite explicit, see below) example of (1.8), or rather of (1.3), we note that, with the assignment $\mu = 1$, the ODE (2.6c) can be solved to yield

$$\varphi_n(p) = 2\,i\,\omega + \exp(p) \ , \tag{2.8a}$$

where, without significant loss of generality, we set to unity the *a priori* arbitrary constant multiplying the exponential in the right-hand side. Hence the corresponding Hamiltonian (see (2.1) with (2.5)) can in this case be explicitly exhibited:

$$H(\underline{p}, \underline{z}) = \sum_{n=1}^{N} \left[2\,i\,\omega + \exp(p_n)\right]\, z_n + F(\underline{z}). \tag{2.8b}$$

The corresponding Newtonian equations of motion read of course (see (1.3))

$$\ddot{z}_n + 2\,i\,\omega\,\dot{z}_n = -\dot{z}_n \frac{\partial F(\underline{z})}{\partial z_n}, \quad n = 1, ..., N, \tag{2.8c}$$

where the function $F(\underline{z})$ must satisfy now the scaling property (1.4) with

$$a_k = -k, \quad k = 1, ..., K. \tag{2.8d}$$

An assignment of $F(\underline{z})$ that satisfies this condition and that moreover entails that the right-hand side of these Newtonian equations of motion (2.8c) only feature two-body forces reads

$$F(\underline{z}) = \frac{1}{2} \sum_{m,n=1,m\neq n}^{N} \sum_{k=1}^{K} \frac{g_{nm}^{(k)}}{k\,(z_n - z_m)^{k}}. \tag{2.9a}$$

Indeed the corresponding Newtonian equations of motion (2.8c) then read

$$\ddot{z}_n + 2\,i\,\omega\,\dot{z}_n = \dot{z}_n \sum_{m=1, m\neq n}^{N} \sum_{k=2}^{K} \frac{g_{nm}^{(k)}}{(z_n - z_m)^k}, \quad n = 1, ..., N. \tag{2.9b}$$

Here the "coupling constants" $g_{nm}^{(k)}$ are again *arbitrary* (possibly *complex*), except for the obvious (see (2.9a)) symmetry property $g_{nm}^{(k)} = g_{mn}^{(k)}$. Note that these Newtonian equations of motion, (2.9b), are *translation-invariant*. They of course obtain from the Hamiltonian (2.8b) with (2.9a). Note their similarity, as well as their difference, from the (also *Hamiltonian*, and also *isochronous*) Newtonian equations of motion

$$\ddot{z}_n - i\,\Omega\,\dot{z}_n = \dot{z}_n \sum_{m=1,\, m\neq n}^{N} \dot{z}_m \left\{ \frac{g_{nm}^{(0)}}{z_n - z_m} + \sum_{\ell=1}^{L} g_{nm}^{(\ell)}\,(z_n - z_m)^{a_\ell} \right\}, \quad n = 1, ..., N, \tag{2.9c}$$

where L is an arbitrary *nonnegative* integer (for $L = 0$ it is understood that the sum over ℓ be set to zero), the coupling constants $g_{nm}^{(j)}$ $(j = 0, 1, ..., L;\ n, m = 1, ..., N)$ are *arbitrary* (possibly *complex*) except for the symmetry requirement $g_{nm}^{(j)} = g_{mn}^{(j)}$, and also *arbitrary* (possibly *complex*) are the exponents a_ℓ (except for the obvious restriction $a_\ell \neq -1$): see eq. (17b) of Ref. [12].

3 Isochronicity of the first class of dynamical systems

In this section we prove that the dynamical system characterized by the Newtonian equations of motion

$$\ddot{z}_n + \omega^2\, z_n = -\left[\dot{z}_n^2 + \omega^2\,\left(c_n + z_n^2\right)\right]^{1/2} f_n(\underline{z})\,, \quad n = 1, ..., N \tag{3.1}$$

is *isochronous* (in the sense defined above), provided the functions $f_n(\underline{z})$ are analytic in their N arguments and satisfy the scaling property

$$f_n(\underline{z}) = \sum_{k=1}^{K} f_n^{(-2\,k-1)}(\underline{z})\,, \quad n = 1, ..., N, \tag{3.2a}$$

$$f_n^{(-2\,k-1)}(\alpha\,\underline{z}) = \alpha^{-2\,k-1}\, f_n^{(-2\,k-1)}(\underline{z})\,, \quad n = 1, ..., N. \tag{3.2b}$$

Clearly these Newtonian equations include as a subcase the Newtonian equations of motion (1.1) with (1.2), to which they reduce if

$$f_n^{(-2\,k-1)}(\underline{z}) = \frac{\partial\, F^{(-2\,k)}(\underline{z})}{\partial\, z_n}\,, \quad n = 1, ..., N. \tag{3.3}$$

Note the consistency of these assignments with the scaling properties (1.2b) and (3.2b).

The starting point of the proof is the following change of (dependent and independent) variables ("the trick"):

$$z_n(t) = \exp\left(-i\,\omega\,t\right)\,\zeta_n(\tau)\,, \quad n = 1, ..., N\,, \tag{3.4a}$$

$$\tau = \frac{\exp(2\,i\,\omega\,t) - 1}{2\,i\,\omega} \;, \tag{3.4b}$$

which clearly implies the following relations among the "initial data" $\underline{z}(0)$, $\underline{\dot{z}}(0)$ and $\underline{\zeta}(0)$, $\underline{\zeta}'(0)$:

$$\underline{z}(0) = \underline{\zeta}(0) \;, \quad \underline{\dot{z}}(0) = \underline{\zeta}'(0) - i\,\omega\,\underline{\zeta}(0) \;. \tag{3.5}$$

We now note that the relation (3.4b) among t and τ implies that, as the *real* variable t evolves over the half-period $\frac{T}{2} = \frac{\pi}{\omega}$ (see (1.6)), the *complex* variable τ travels in the complex τ-plane counterclockwise full round over the circle C the diameter of which, of length $\frac{1}{\omega}$, lies on the upper imaginary axis, with its lower end at the origin ($\tau = 0$). Hence, if the function $\zeta_n(\tau)$ is a *holomorphic* function of the complex variable τ in the (closed) disk D enclosed by that circle C, then (see (3.4)) the function $z_n(t)$ has the *periodicity* (indeed, the *isochronicity*) properties (1.7a). We now obtain the evolution equations (see (3.6) below) satisfied by the functions $\zeta_n(\tau)$ that correspond via (3.4) to the Newtonian equations of motion (1.1), and we then show that, under the hypotheses indicated above (see in particular (3.2)), these evolution equations imply that there exists an *open* set of initial data $\underline{\zeta}(0)$, $\underline{\zeta}'(0)$ (corresponding via (3.5) to an *open* set of initial data $\underline{z}(0)$, $\underline{\dot{z}}(0)$) such that the functions $\zeta_n(\tau)$ are *holomorphic* in a circular disk of *arbitrarily* large radius centered at the origin ($\tau = 0$) in the complex τ-plane, hence *a fortiori* in the closed disk D.

The evolution equations satisfied by the functions $\zeta_n(\tau)$ are easily obtained from (1.1) with (3.2) via (3.4):

$$\zeta_n'' = -\left[\omega^2\,c_n - 2\,i\,\omega\,\zeta_n'\,\zeta_n + (1 + 2\,i\,\omega\,\tau)\,\left(\zeta_n'\right)^2\right]^{1/2} \cdot$$
$$\cdot \sum_{k=1}^{K} (1 + 2\,i\,\omega\,\tau)^{k-1}\, f_n^{(-2\,k-1)}\left(\underline{\zeta}\right) \;, \quad n = 1, ..., N. \tag{3.6}$$

To make contact with the standard notation (see in particular Section 12.21 of [25]), we now set

$$w_n(\tau) = \frac{\zeta_n(\tau) - \zeta_n(0)}{\alpha} \;, \quad w_{N+n}(\tau) = \frac{\zeta_n'(\tau) - \zeta_n'(0)}{\beta} \;, \quad n = 1, ..., N \;, \tag{3.7}$$

where α and β are two *positive* constants (that we shall conveniently assign below). Note that this definition of the quantities $w_\ell(\tau)$ entails that they vanish at the origin,

$$w_\ell(0) = 0, \quad \ell = 1, ..., 2\,N. \tag{3.8}$$

Via this definition (3.7) the second-order (N-vector) ODE (3.6) gets reformulated as the following first-order system of $2\,N$ coupled ODEs:

$$w_n' = \varphi_n \;, \quad w_{N+n}' = \varphi_{N+n} \;, \quad n = 1, ..., N \;, \tag{3.9}$$

with

$$\varphi_n = \frac{\beta}{\alpha}\,(v_n + w_{N+n}) \;, \quad n = 1, ..., N \;, \tag{3.10a}$$

$$\varphi_{N+n} = -\alpha^{-\frac{5}{2}}\,\beta^{-\frac{1}{2}}\,\left\{\frac{\omega^2\,c_n}{\alpha\,\beta} - 2\,i\,\omega\,\left(r_n + w_n\right) + \frac{\beta}{\alpha}\,\left(1 + 2\,i\,\omega\,\tau\right)\,\left(v_n + w_{N+n}\right)^2\right\}^{1/2} \cdot$$

$$\cdot \sum_{k=1}^{K} \left(1 + 2\,i\,\omega\,\tau\right)^{k-1}\,\alpha^{2\,(1-k)}\,f_n^{(-2\,k-1)}\left(r_m + w_m\right)\;,\quad n = 1, ..., N\;, \tag{3.10b}$$

where we did also conveniently set

$$\zeta_n(0) = \alpha\,r_n\;,\quad \zeta_n'(0) = \beta\,v_n\;,\quad n = 1, ..., N\;, \tag{3.11}$$

and we took advantage of the scaling relations (3.2b). Note that, for notational convenience, in the right-hand side of (3.10b) we replaced the N-vector argument $\underline{r} + \underline{w}$ of the functions $f_n^{(-2\,k-1)}$ with its components $r_m + w_m$ (on the understanding that the index m always ranges from 1 to N).

We now use the Theorem [25] according to which the $2\,N$ functions $w_j(\tau)$ are *holomorphic* in τ (at least) in a circular disk, centered in the complex τ-plane at the origin ($\tau = 0$, where the initial conditions (3.8) are assigned), the radius ρ of which is bounded below by the formula

$$\rho \geq \vartheta\,\left\{1 - \exp\left[-\frac{w}{(2\,N + 1)\,\vartheta\,M(\vartheta, w)}\right]\right\}\;, \tag{3.12}$$

where the *positive* quantities ϑ and w are characterized by the requirement that the $2\,N$ functions $\varphi_\ell \equiv \varphi_\ell(\tau; w_1, ..., w_{2N})$, $\ell = 1, ..., 2\,N$ (see (3.10)) be *holomorphic* in the $2\,N + 1$ variables τ and w_j, $j = 1, ..., 2\,N$ provided

$$|\tau| \leq \vartheta;\quad |w_j| \leq w,\quad j = 1, ..., 2\,N, \tag{3.13}$$

and the *positive* quantity $M(\vartheta, w)$ is defined by the formula

$$M(\vartheta, w) = \max_{|\tau|\leq\vartheta;\,|w_j|\leq w, j=1,...,2\,N;\;\ell=1,...,2\,N} |\varphi_\ell(\tau; w_1, ..., w_{2N})|\;. \tag{3.14}$$

Note that the lower bound (3.12) holds of course *a fortiori* if we *overestimate* the quantity $M(\vartheta, w)$, as we shall indeed do in the following.

We now set (for instance)

$$\alpha = \varepsilon^{-4},\quad \beta = \varepsilon^2,\quad \vartheta = \varepsilon^{-1}, \tag{3.15a}$$

and we hereafter treat the quantity ε as *very small but finite*,

$$\varepsilon \approx 0; \tag{3.15b}$$

we moreover assume the $2N$ quantities r_n and w_n, see (3.11), to be bounded (in modulus) above and below as follows:

$$2\,w < |r_n| < r,\quad 2\,w < |v_n| < v. \tag{3.16}$$

It is then clear that the argument of the square root in the right-hand side of (3.10b) is guaranteed not to vanish for $|\tau| \leq \vartheta$ (as required for the applicability of the Theorem [25], namely for the validity of (3.12)), since, in the limit (3.15), it tends to the value $-2\,i\,\omega\,(r_n + w_n)\,(v_n + w_{N+n})$, that certainly does not vanish (see (3.16) and (3.13)). Moreover, in the same limit (3.15) we clearly get (see (3.13) and (3.16))

$$M(\vartheta, w) \leq \varepsilon^2\,(v + w)\,. \tag{3.17}$$

Note that, in order for (3.12) to be applicable [25], we must moreover be sure that the functions $f_n^{(-2\,k-1)}\,(r_m + w_m)$ are *holomorphic* for $|w_m| \leq w,\;\; m = 1, ..., N$; but this is certainly the case for sufficiently small w, see (3.13), provided the initial data $\underline{z}(0)$ – as we of course assume – are assigned where the Newtonian equations of motion (3.1) are not singular (see (3.2), (3.5) and (3.11)).

Insertion of the third relation (3.15a), and of (3.17), in (3.12) implies that, in the limit (3.15b),

$$\rho \geq \varepsilon^{-1}, \tag{3.18}$$

namely that the functions $w_n(\tau)$, hence as well the functions $\zeta_n(\tau)$ (see (3.7)), are *holomorphic* in a circular disk centered at $\tau = 0$, the radius of which can be made arbitrarily large by an appropriate assignment (see (3.11) with (3.15) and (3.16)) of the initial data $\underline{\zeta}(0), \underline{\zeta}'(0)$ (hence as well of the initial data $\underline{z}(0)$, $\underline{\dot{z}}(0)$; see (3.5)) in an appropriate *open* domain. The result we set out to prove is thereby established.

Let us end this section by noting that the proof given herein entails (see (3.11) with (3.15), and (3.5)) that the domain of initial data $\underline{z}(0), \underline{\dot{z}}(0)$ that yield *isochronous* solutions of the Newtonian equations of motion (3.1) with (3.2) is characterized by *very large* (in modulus) values of the initial data $\underline{z}(0), \underline{\dot{z}}(0)$, with

$$\underline{\dot{z}}(0) \approx -i\,\omega\,\underline{z}(0). \tag{3.19}$$

4 Isochronicity of the second class of dynamical systems

In this section we prove that the dynamical system characterized by the Newtonian equations of motion

$$\begin{aligned} &\ddot{z}_n + 2\,i\,\mu_n\,\omega\,\dot{z}_n + (1-\mu_n^2)\,\omega^2\,z_n \\ =\;& -\left[\dot{z}_n + i\,(1+\mu_n)\,\omega\,z_n\right]^{\frac{1}{2}(1-\mu_n)}\left[\dot{z}_n - i\,(1-\mu_n)\,\omega\,z_n\right]^{\frac{1}{2}(1+\mu_n)}\,f_n(\underline{z})\;, \\ n =\;& 1, ..., N, \end{aligned} \tag{4.1a}$$

where the N constants μ_n are *rational* numbers, is *isochronous* (in the sense defined above), provided the functions $f_n(\underline{z})$ are *analytic* in their N arguments and satisfy the scaling property

$$f_n(\underline{z}) = \sum_{k=1}^{K} f_n^{(a_{1k},...,a_{nk}-1,...,a_{Nk})}(\underline{z})\;,\quad n = 1, ..., N, \tag{4.1b}$$

$$\begin{aligned} &f_n^{(a_{1k},...,a_{nk}-1,...,a_{Nk})}(\alpha_1\,z_1, ..., \alpha_N\,z_N) \\ =&\left[\prod_{m=1}^{N}(\alpha_m)^{a_{mk}}\right]^{\frac{1}{N}}\alpha_n^{-1}\,f_n^{(a_{1k},...,a_{nk}-1,...,a_{Nk})}(\underline{z})\;, \\ &n = 1, ..., N,\quad k = 1, ..., K\;, \end{aligned} \tag{4.1c}$$

with the quantities a_{nk} satisfying of course the condition (1.9c). And the $N\,K$ inequalities (1.10) are moreover required to hold.

Clearly these Newtonian equations include as a subcase the (*Hamiltonian*: see Section 2) Newtonian equations of motion (1.8), to which they reduce if

$$f_n(\underline{z}) = \frac{\partial\, F(\underline{z})}{\partial\, z_n}\ , \quad n = 1, ..., N. \tag{4.2}$$

Note the consistency of the scaling properties (4.1b), (4.1c) and (1.9a), (1.9b) with this relation (4.2).

The proof is of course analogous to that given in the preceding section, yet sufficiently different to deserve a separate presentation, which can however be quite terse, see below.

The change of variables we now use reads

$$z_n(t) = \exp\left[-i\ (1+\mu_n)\ \omega\, t\right]\ \zeta_n\,(\tau)\ \ , \quad n = 1, ..., N, \tag{4.3}$$

with the relation among the independent variables t and τ given again by (3.4b). Hence the relation among the initial data reads now as follows:

$$z_n(0) = \zeta_n(0), \quad \dot{z}_n(0) = \zeta_n'(0) - i\ (1+\mu_n)\ \omega\, \zeta_n\,(0)\ \ , \tag{4.4}$$

and the ODEs satisfied by the functions $\zeta_n(\tau)$ read

$$\zeta_n'' = -\sum_{k=1}^{K} (1+2\,i\,\omega\,\tau)^{k-1}\ \left[\zeta_n'\right]^{\frac{1}{2}(1+\mu_n)}\ \cdot$$

$$\cdot \left[-2\,i\,\omega\,\zeta_n + (1+2\,i\,\omega\,\tau)\ \zeta_n'\right]^{\frac{1}{2}(1-\mu_n)}\ f_n^{(a_{nk})}\left(\underline{\zeta}\right), \tag{4.5}$$

where we took of course advantage of the scaling relations (4.1) with (1.9c), and of the relation (3.4b) among t and τ.

We now proceed in close analogy to the treatment of the preceding section by setting

$$\zeta_n(\tau) = \alpha_n\left[r_n + w_n(\tau)\right], \quad \zeta_n'(\tau) = \beta_n\left[v_n + w_{N+n}(\tau)\right], \tag{4.6}$$

where the quantities α_n, β_n are *positive* scaling constants the choice of which remains our privilege, and we utilized again the assignments (3.11). We thereby conclude, via the assignments

$$\alpha_n = \varepsilon^{A_n}, \quad \beta_n = \varepsilon^{B_n}, \quad \vartheta = \varepsilon^{-1}, \quad \varepsilon \approx 0, \tag{4.7}$$

that our result is proven provided a choice of the $2N$ *arbitrary* exponents A_n, B_n can be made such that there hold the following $N\,(K+1)$ inequalities:

$$B_n \geq A_n + 1, \quad n = 1, ..., N, \tag{4.8a}$$

$$\frac{1}{2}\,A_n\ (1-\mu_n) - \frac{1}{2}\,B_n\ (3-\mu_n) + 1 - k + \frac{1}{N}\sum_{m=1}^{N} A_m\,a_{mk}\ \ \geq\ \ 0,$$

$$n = 1, ..., N, \quad k = 1, ..., K. \tag{4.8b}$$

To simplify matters we then replace the inequality (4.8a) with an equality, namely we set

$$B_n = A_n + 1, \quad n = 1, ..., N, \tag{4.8c}$$

and we note that the inequalities (4.8b) become then the $N\,K$ inequalities (1.10).

This concludes our outline of the proof of the *isochronicity* of the Newtonian equations of motion (4.1), confirming the findings reported in Section 1. The period of the *isochronous* motions is of course an integer multiple (coinciding generally with the minimum common multiple of the denominators of the rational numbers μ_n) of the basic period $\frac{T}{2} = \frac{\pi}{\omega}$ (see (4.3) and (3.4b)). And note moreover that (4.8c) with (4.7), via (3.11) and (4.4), imply that the domain of initial data out of which *isochronous* motions emerge is characterized by the condition

$$\dot{z}_n(0) \approx -i\,\left(1 + \mu_n\right)\,\omega\,z_n\left(0\right)\ \ , \tag{4.9}$$

with $z_n\left(0\right)$ *very large* (in modulus) if the corresponding A_n (see (1.10)) is *negative*, *very small* (in modulus) if the corresponding A_n (see (1.10)) is *positive.*

5 Final remarks

In this section we reiterate some considerations analogous to those proffered in the last section of [12].

The findings reported in this paper are mainly based on the simple trick (3.4). Although some of the results reported in this paper might also be proven by standard "Poincaré-Dulac" techniques (see for instance [22] [27] [23] [28] [24], as well as the discussion of this issue in the last section of [16]), the effectiveness of this trick is demonstrated by the generality of the results reported above, as well as by the ease with which they have been proven – including moreover the possibility to obtain *explicit* bounds on the size of the *open* domain of initial data that yield *isochronous* outcomes: see the proof given in the preceding two sections (although we did not insist there on this aspect, being satisfied with demonstrating the *existence* of such an *open* domain of initial data).

The main result of this paper is the demonstration of the *Hamiltonian* character of the *isochronous* Newtonian equations of motion (1.1), (1.3) and (1.8). Let us recall that Hamiltonian systems with N degrees of freedom are, loosely speaking, *completely integrable* if they feature N functionally independent and globally defined constants of motion (including the Hamiltonian itself) that Poisson-commute among themselves; they are *superintegrable* if they feature $N - 1$ additional functionally independent and globally defined constants of motion. All the confined motions of such *superintegrable* Hamiltonian systems (that, loosely speaking, can be likened to one-degree-of-freedom Hamiltonian systems) are *completely periodic*, although not necessarily *isochronous*, since the period may depend on the initial data. Note that, for a many-degree-of-freedom system, the existence of *completely periodic* (rather than *multiply periodic*) motions is quite a nontrivial phenomenon.

It is also useful to introduce the notion of *partially integrable* (and *partially superintegrable*) Hamiltonian systems, to include the possibility that a system feature these properties only in a "part" of its (natural) phase space. Of course for this notion to be reasonable it is required (at least as long as consideration is restricted to *autonomous* Hamiltonian systems, as indeed done above) that this *part* of the phase space remain *invariant* throughout the evolution, namely that *all* motions originating from it *always* remain in it. Hence this distinction among *integrable* (or *superintegrable*) and *partially integrable* (or *partially superintegrable*) Hamiltonian systems is *moot* if one includes in the very definition of a Hamiltonian system the domain in phase space in which it is supposed to live, rather than relying on the *natural* definition of the entire phase space.

As noted above, a *superintegrable Hamiltonian* system, as long as it only produces confined motions, always yields *completely periodic* evolutions, which however need not be *isochronous*. Hence *superintegrability* does not entail *isochronicity*; while the converse, at least loosely speaking, is clearly the case, namely *isochronicity* does entail *superintegrability*. Therefore the Hamiltonian systems considered in this paper, characterized by the Hamiltonian (2.1) with (2.4) respectively with (2.5) and (2.6), and by the Newtonian equations of motion (1.1) respectively (1.8), should be considered *superintegrable*, or at least *partially superintegrable* to the extent these two notions are distinct. This underscores the remarkable character of the findings reported above.

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