Design of State Feedback Controller for Discrete Systems with Time-delay

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***Abstract*- The stability analysis and controller design of discrete linear systems with time-varying delay are addressed. Firstly, the uniformly asymptotical stability criterion with adjustable parameter is derived by Lyapunov-Razumikhin approach. Then, the stabilization approaches for linear systems with time delay by state feedback and observer based-on state feedback are also presented. Sufficient conditions for the existence of state feedback gain and the observer gain are derived through the numerical solution of a set of obtained linear matrix inequalities. Compared with methods in the references, the dynamic performance of systems, such as the overshoot and the convergence rate of the response, can be adjusted by changing the adjustable parameter. Lastly, an illustrative example is given to show the effectiveness of the proposed.**

Keywords -Time delay system; stability; linear matrix inequality; controller

I. INTRODUCTION

It is well known that time delay may be the cause of performance degradations for dynamical systems and even, in some circumstances, the cause of instability if such time delay is not taken into account during the design phase[1]. Time delay may occur in continuous- time or discrete-time systems and may be constant or time-varying. The control of systems with time delay has been a hot topic in research in the last few decades. Stability and stabilizability problems for the continuous- time[2-6] and discrete-time systems[7-10] with time delay have been tackled and interesting results have been reported in the literature.

However, in [2-5,7-10], when system parameters are fixed, the feedback gains will also been fixed, and thus the dynamic performance of closed-loop systems cannot be optimized. However, in real applications, we often want to get better dynamic performance of closed-loop systems although system parameters are fixed. Based on this idea, [6] gave a method to design state feedback controller with adjustable parameter for continuous-time systems, when adjustable parameter takes different values, feedback gains are also different. Different feedback gains lead to different dynamic performance of closed-loop systems, therefore, the dynamic performance of closed-loop systems can be optimized by choosing proper parameter. However, how to design feedback gains with adjustable parameters for discrete-time systems is not mentioned in existing papers.

In this paper, the idea of designing feedback gain with adjustable parameter is extended to discrete systems with time-delay. Firstly, the mathematic model of discrete time-delay systems is described, and then, based on the Lyapunov-Razumikhin stability theorem, the stability conditions are derived in the form of linear matrix inequality. The state feedback controller and observer-based state feedback controller are discussed, too. lastly, an illustrative example is given to show the effectiveness of the proposed.

II. PRELIMINARIES

Consider the following discrete time-delay system:

$$\begin{cases} x(k+1) = Ax(k) + A_d x(k-\tau) + Bu(k) \\ y(k) = Cx(k) + C_d x(k-\tau) \\ x(k) = \phi(k) \quad\quad k \in [-\tau \quad 0] \end{cases} \tag{1}$$

where $x(k) \in R^n$ is the state variable, $u(k) \in R^m$ is the control input, $y(k) \in R^l$ is the output. $\phi(k)$ is the initial condition of state defined on the interval $[-\tau \ 0]$. τ is the bounded delay. And A, A_d, B, C, C_d are some constant matrices of appropriate dimensions, and A_d is nonsingular. Assume $x(k)$ is measurable.

Our goal is to find the state feedback controller and observe-based state feedback controller such that the system (1) is asymptotically stable.

Lemma 1[1]. *Consider the system (1) with* $u(k) = 0$. *Suppose that* $u(s), v(s), w(s)$ *and* $p(s) \in R^+ \mapsto R^+$ *are scalar, continuous and nondecreasing functions,* $u(s), v(s), w(s)$ *are positive for* $s > 0$, $u(0) = 0$, $v(0) = 0$, *if there is a continuous function* $V: R \times \mathbb{S}_{n,z} \mapsto R$ *such that*

1) $u(\|x\|) \le V(t,x) \le v(\|x\|), \quad t \in R, x \in R^n$;

2) $\Delta V(k, x(k)) \le -w(\|x(k)\|)$ *if*

$V(k+\theta, x(k+\theta)) < p(V(k, x(k))) \ \forall \theta \in [-\tau, 0]$,

then the trivial solution of equation (1) with $u(k) = 0$ *is uniformly asymptotically stable.*

Lemma 2[11]. *For given matrices* $S > 0$, $Q \in R^{n \times n}$ *and vectors* $x, y \in R^n$, *the following inequality holds:*

$$2x^T Q y \le x^T Q S^{-1} Q^T x + y^T S y$$

III. MAIN RESULTS

A. Stability analysis

In this section, the stability condition of system (1) with $u(k)=0$ will be discussed, which is described as (2).

$$x(k+1)=Ax(k)+A_d x(k-\tau) \qquad (2)$$

Now we state our first result as follows.

Theorem 1. *The equilibrium of system* (2) *is asymptotic ally stable in large if there exists a common matrix* $P>0$ *such that*

$$\begin{bmatrix} A^T PA+(\gamma-1)P & A^T PA_d \\ A_d^T PA & A_d^T PA_d-\gamma P \end{bmatrix}<0 \qquad (3)$$

where $0<\gamma<1$ *is a constant.*

Proof : Select a Lyapunov function as

$$V(x(k))=x(k)^T Px(k)$$

It is easily to get

$$\lambda_{\min}(P)\|x(k)\|\le V(x(k))\le\lambda_{\max}(P)\|x(k)\|$$

where $\lambda_{\min}(P),\lambda_{\max}(P)$ are the minimum and maxim- um eigenvalues of P, respectively.

The difference of $V(x(k))$ along the trajectories of (2) is

$$\Delta V(x(k))=V(x(k+1))-V(x(k))$$
$$=\begin{bmatrix} x(k) \\ x(k-\tau) \end{bmatrix}^T \begin{bmatrix} A^T PA-P & A^T PA_d \\ * & A_d^T PA_d \end{bmatrix}\begin{bmatrix} x(k) \\ x(k-\tau) \end{bmatrix}$$

For lemma 1, it can be get

$$V(k+\theta,x(k+\theta))<p(V(k,x(k))) \qquad \forall\theta\in[-\tau,0]$$

for some $p>1$, then, for $0<\gamma<1$,

$$\Delta V(x(k))$$
$$<\begin{bmatrix} x(k) \\ x(k-\tau) \end{bmatrix}^T \begin{bmatrix} A^T PA+(\gamma p-1)P & A^T PA_d \\ * & A_d^T PA_d-\gamma P \end{bmatrix}\begin{bmatrix} x(k) \\ x(k-\tau) \end{bmatrix} \qquad (4)$$

The inequality (3) implies that for some sufficient small $\delta>0$, $p=1+\delta$,

$$\begin{bmatrix} A^T PA+(\gamma p-1)P & A^T PA_d \\ A_d^T PA & A_d^T PA_d-\gamma P \end{bmatrix}<0$$

which, according to (4), implies that the Razumikhin difference condition

$$\Delta V(x(k))\le -w\|x(k)\|$$

is satisfied. By lemma 1 we know system (2) is asymptotically stable.

B. State feedback controller design

Consider the following control law for system (1)

$$u(k)=-Fx(k) \qquad (5)$$

By (1) and (5), we get

$$x(k+1)=(A-BF)x(k)+A_d x(k-\tau) \qquad (6)$$

The goal of state feedback controller is to determine the local feedback gain F such that the closed-loop system (6) is asymptotically stable. Similar to the analysis of the last part, the following result can be get easily.

Theorem 2. There exists a state feedback control law (5) such that the equilibrium of the closed-loop system with time-delay described by (6) is asymptotically stable if there exist matrices $P>0$ and Q satisfying the following LMI

$$\begin{bmatrix} (\gamma-1)P & A^T P-Q^T & A^T PA_d-Q^T A_d \\ PA-Q & -P & 0 \\ A_d^T PA-A_d^T Q & 0 & A_d^T PA_d-\gamma P \end{bmatrix}<0 \qquad (7)$$

and the state feedback gain can be constructed as

$$F=(PB)^\dagger Q$$

where $0<\gamma<1$ is the adjustable parameter, $(\cdot)^\dagger$ is the general inverse matrix of $(\cdot)$.

Proof. Similar to the proof of theorem 1, we can get that the sufficient condition for (6) stable is

$$\begin{bmatrix} \begin{bmatrix} (A-BF)^T P(A-BF) \\ +(\gamma p-1)P \end{bmatrix} & (A-BF)^T PA_d \\ A_d^T P(A-BF) & A_d^T PA_d-\gamma P \end{bmatrix}<0 \qquad (8)$$

By Schur compensation lemma we know the inequality (7) implies that for some sufficient small $\delta>0$, $p=1+\delta$,

$$\begin{bmatrix} (A-BF)^T P(A-BF)+(\gamma p-1)P & (A-BF)^T PA_d \\ A_d^T P(A-BF) & A_d^T PA_d-\gamma P \end{bmatrix}<0$$

which, according to (8), implies that the Razumikhin difference condition

$$\Delta V(x(k))\le -w\|x(k)\|$$

is satisfied. According to lemma 1 system (6) is asymptotically stable.

C. Observe-based state feedback controller design

The observer is described as

$$\begin{cases} \tilde{x}(k+1)=A\tilde{x}(k)+A_d\tilde{x}(k-\tau)+Bu(k)+G(y(k)-\tilde{y}(k)) \\ \tilde{y}(k)=C\tilde{x}(k)+C_d\tilde{x}(k-\tau) \end{cases} \qquad (9)$$

With the above observer, the control law should be

$$u(k)=-F\tilde{x}(k) \qquad (10)$$

Let $e(k)=x(k)-\tilde{x}(k)$, then the closed-loop system can be described as

$$\begin{cases} x(k+1)=(A-BF)x(k)+A_d x(k-\tau)+BFe(k) \\ e(k+1)=(A-GC)e(k)+(A_d-GC_d)e(k-\tau) \end{cases} \qquad (11)$$

Let

$$\bar{x}(k)=\begin{bmatrix} x(k) \\ e(k) \end{bmatrix}, M=\begin{bmatrix} A-BF & BF \\ 0 & A-GC \end{bmatrix},$$
$$N=\begin{bmatrix} A_d & 0 \\ 0 & A_d-GC_d \end{bmatrix} \qquad (12)$$

The formula (11) can briefly be described as

$$\bar{x}(k+1)=M\bar{x}(k)+N\bar{x}(k-\tau) \qquad (13)$$

Similar to the proof of theorem 1, we can get that the sufficient condition for (13) stable is

$$\Delta V(x(k)) < \begin{bmatrix} \bar{x}(k) \\ \bar{x}(k-\tau) \end{bmatrix}^T \begin{bmatrix} M^T PM + (\gamma p - 1)P & M^T PN \\ N^T PM & N^T PN - \gamma P \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ \bar{x}(k-\tau) \end{bmatrix} \quad (14)$$

Let

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

and insert (12) into (14) and we get

$$\Delta V(x(k)) < \begin{bmatrix} x(k) \\ e(k) \\ x(k-\tau) \\ e(k-\tau) \end{bmatrix}^T \begin{bmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & 0 \\ * & \Delta_{22} & \Delta_{23} & \Delta_{24} \\ * & * & \Delta_{33} & 0 \\ * & * & * & \Delta_{44} \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \\ x(k-\tau) \\ e(k-\tau) \end{bmatrix}$$

where $\Delta_{11} = (A^T - F^T B^T)P_1(A - BF) + (\gamma - 1)P_1$

$\Delta_{12} = (A^T - F^T B^T)P_1 BF$

$\Delta_{13} = (A^T - F^T B^T)P_1 A_d$

$\Delta_{22} = (A^T - C^T G^T)P_2(A - GC) + (\gamma - 1)P_2$

$\Delta_{23} = F^T B^T P_1 A_d$

$\Delta_{24} = (A^T - C^T G^T)P_2(A_d - GC_d)$

$\Delta_{33} = A_d^T P_1 A_d - \gamma P_1$

$\Delta_{44} = (A_d^T - C_d^T G^T)P_2(A_d - GC_d) - \gamma P_2$

By lemma 2 we get

$$\Delta V(x(k)) < \begin{bmatrix} x(k) \\ e(k) \\ x(k-\tau) \\ e(k-\tau) \end{bmatrix}^T \begin{bmatrix} \Gamma_{11} & 0 & \Gamma_{13} & 0 \\ * & \Gamma_{22} & \Gamma_{23} & 0 \\ * & * & \Gamma_{33} & 0 \\ * & * & * & \Gamma_{44} \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \\ x(k-\tau) \\ e(k-\tau) \end{bmatrix}$$

where $\Gamma_{11} = 2(A^T - F^T B^T)P_1(A - BF) + (\gamma - 1)P_1$

$\Gamma_{13} = (A^T - F^T B^T)P_1 A_d$

$\Gamma_{22} = 2(A^T - C^T G^T)P_2(A - GC) + F^T B^T P_1 BF + (\gamma - 1)P_2$

$\Gamma_{23} = F^T B^T P_1 A_d$

$\Gamma_{33} = A_d^T P_1 A_d - \gamma P_1$

$\Gamma_{44} = 2(A_d^T - C_d^T G^T)P_2(A_d - GC_d) - \gamma P_2$

Similar to the proof of Theorem 1 we know if the following inequality holds

$$\begin{bmatrix} \Gamma_{11} & 0 & \Gamma_{13} & 0 \\ * & \Gamma_{22} & \Gamma_{23} & 0 \\ * & * & \Gamma_{33} & 0 \\ * & * & * & \Gamma_{44} \end{bmatrix} < 0 \quad (15)$$

system (13) is asymptotically stable. By Schur compensation lemma we know a necessary and sufficient condition for (15) is (16) (17) hold. Then we have the following result.

Theorem 3. The equilibrium of the closed-loop time-delay system with observer-based control law (10) described by (13) is asymptotically stable if there exist matrices $P_1 > 0, P_2 > 0, Q_1$ and Q_2 satisfying the following LMIs

$$\begin{bmatrix} -\gamma P_2 & A_d^T P_2 - C_d^T G^T P_2 \\ P_2 A_d - P_2 GC_d & -0.5P_2 \end{bmatrix} < 0 \quad (16)$$

$$\begin{bmatrix} (\gamma-1)P_2 & A^T P_2 - Q_2^T & Q_1^T & Q_1^T A_d & 0 & 0 \\ * & -0.5P_2 & 0 & 0 & 0 & 0 \\ * & * & -P_1 & 0 & 0 & 0 \\ * & * & * & \Gamma_{33} & A_d^T P_1 A - A_d^T Q_1 & 0 \\ * & * & * & * & (\gamma-1)P_1 & A^T P_1 - Q_1^T \\ * & * & * & * & * & -0.5P_1 \end{bmatrix} < 0 \quad (17)$$

The state feedback gain and observer can be constructed as $F = (P_1 B)^{\dagger} Q_1$, $G = P_2^{-1} Q_2$,respectively.

where $0 < \gamma < 1$ is the adjustable parameter.

Remark 1. From (7) (or (16) and (17)) we can find, the feedback gains is related to the adjustable parameter, that is to say, when adjustable parameter take different values, the feedback gains are also different. As we know, different feedback gains lead to different dynamic performance of the closed-loop systems, therefore, we can get better performance by choosing proper adjustable parameter.

Note2. If we only want to know the system is stable or not, we only find a adjustable parameter such that (3) holds. Therefore, Theorem 1 can be rewritten as:

Corollary 1. The equilibrium of the unforced system with time-delay described by (2) is asymptotically stable in large if there exist a scalar $\gamma > 0$ and a common matrix $P > 0$ such that (3) holds.

For there are tow unknown matrices (scalar can be taken as matrix with one string and one row) in the LMI, the feasibility of LMI is much stronger than that only with one unknown matrix, so the conservatism of conclusion is weaken. The case is also hold for theorem 2 and theorem 3.

IV. SIMULATION EXAMPLE

To illustrate the proposed results, we consider the following simulation example from document [7] with $l = 2.8, L = 5.5, v = -1.0, \bar{t} = 2.0$, the constant a defined on the interval $[0,1]$ is the retarded coefficient. The limits 1 and 0 correspond to no delay term and to a completed delay term, respectively. In this example, it is assumed $a = 0.7$.

To illustrate the effectiveness of the approach in this paper, we take state feedback control approach to control the above system. The control law is described as

$$u(k) = -Fx(k)$$

With the MATLAB tools, we get the feedback gains shown in fig 1 with different adjustable parameter.

Table 1. Feedback Gains with Different Adjustable Parameter

Adjustable parameter	Feedback gain

γ=1	F=[-1.2664 -0.1925]
γ=0.5	F=[-1.2428 0.1010]

When the initial condition of the state is chosen as $[0.5\pi \ \ 0.75\pi]^T$ and $\tau = 1$, with different adjustable parameters, we get the state response curves and the output curves shown in fig1-fig3, and the open-loop system as figure 4.

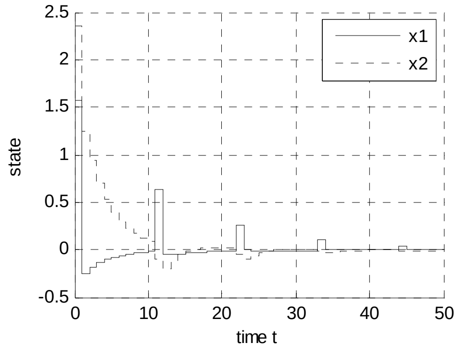

Fig.1. State response (γ=1)

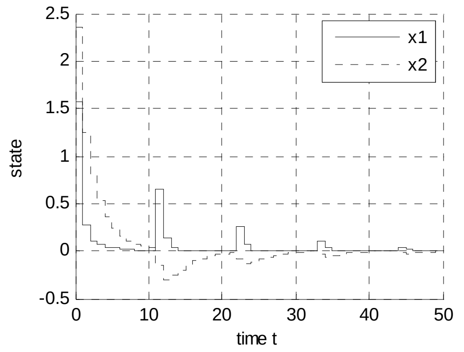

Fig.2. State response (γ=0.5)

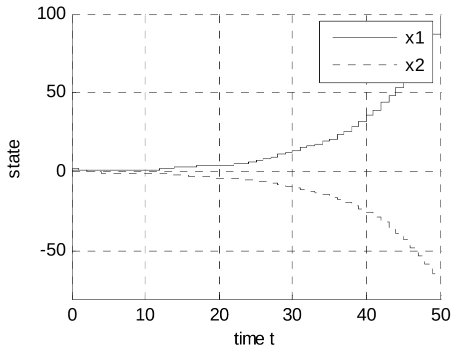

Fig.3. State response of open-loop system

From above simulation we can find : the open-loop system is not stable (fig.3.), but the closed-loop system , which is stabilized by the proposed feedback gains, is stable (fig.1-fig.2), which illustrates the effectiveness of our approach. From Table 1 we can find, when the adjustable parameter takes different value, the feedback gains are also different, different feedback gains lead to different dynamic performance of closed-loop systems(comparing fig 1, fig2), which illustrates Remark 1 . Therefore, in practice, we can get better dynamic performance of the closed-loop systems by choosing proper adjustable parameter.

V. CONCLUSION

In this paper, the stability analysis and controller design of discrete time-delay systems is discussed, and the state feedback gain and observer gain are derived from the solutions of some linear matrix inequalities. Compared with methods in the references, the obtained sufficient conditions have less unknown matrixes and less linear matrix inequalities, and the conservation of the results is weakened. Lastly, an illustrative example is given to show the effectiveness of the proposed. However, how to choose the proper adjustable parameter will be discussed in the future.

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