

Reduction of Order for Systems of Ordinary Differential Equations

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Abstract

The classical reduction of order for scalar ordinary differential equations (ODEs) fails for a system of ODEs. We prove a constructive result for the reduction of order for a system of ODEs that admits a solvable Lie algebra of point symmetries. Applications are given for the case of a system of two second-order ODEs which admits a solvable four-dimensional Lie algebra of point symmetries.

1 Introduction

Assume that one wants to integrate or reduce algorithmically a system of ODEs admitting a solvable Lie algebra of point symmetries. If one tries the classical successive reduction of order as in the case of scalar ODEs [10, 9, 1, 7], there results an ambiguity: a vector field in n variables has a basis of first-order invariants formed by $n + 1$ elements. Thus if $n \geq 1$ there will be ambiguity in the choice of the new variables for reduction of order as there are $n + 1$ possibilities for the choices of the n new variables. As a result, it is not clear which choice of variables does the reduction. This fact highlights the geometrical difference between scalar and systems of ODEs.

The aim of this paper is to provide a method that avoids the above difficulty in an algorithmical fashion, namely we show that for a system of n k th-order ODEs that admits a solvable symmetry Lie algebra of dimension kn , the $(k - 1)$ th prolongations of the generators and the equation vector field are unconnected, is reducible to quadratures.

2 Integrability

There are basically two approaches to integrability using Lie point symmetries. One is that of successive reduction of order using differential invariants for scalar ODEs. The other is that of canonical forms which may apply to scalar as well as systems of ODEs. However, the canonical forms approach only applies in the case when equations are classified for their symmetries, for example Lie's canonical forms for scalar second-order ODEs. Thus

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it is quite restrictive and in the case of systems of two second-order ODEs there are more than 70 canonical forms [13]. This in itself indicates the difficulty in obtaining and using canonical forms. The method of successive reduction of order applies to arbitrary scalar ODEs which admit solvable symmetry Lie algebras. The manner in which this method is presented in the literature does not lend itself to systems of ODEs. The following examples amply illustrate this point.

As an illustration we present an example. Consider for instance the system (see also [13])

$$\ddot{x} = \dot{x}^2 f\left(\frac{\dot{x}}{\dot{y}}\right), \quad \ddot{y} = \dot{x}^2 g\left(\frac{\dot{x}}{\dot{y}}\right) \quad (2.1)$$

the symmetry Lie algebra \mathcal{L}_4 of which is generated by

$$X_1 = -t\frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = \frac{\partial}{\partial x}, \quad X_4 = \frac{\partial}{\partial y}.$$

We form the chain of subalgebras $\langle X_2 \rangle \subset \langle X_2, X_3 \rangle \subset \langle X_2, X_3, X_4 \rangle \subset \mathcal{L}_4$. The first-order differential invariants of $\langle X_2 \rangle$ are x, y, \dot{x}, \dot{y} . Alternatively a basis of invariants of X_2, X_3 and X_4 is \dot{x} and \dot{y} . There arises an ambiguity for the choice of two new dependent variables and one independent variable as invariants: in the first instance there are more choices and in the second there is no choice. However in the second instance (number of invariants equal number of dependant variables), one can sometimes uncouple the system. However, the problem of solving the uncoupled system remains. In the above example this approach leads to full integration of the system.

We point out the failure of the traditional successive reduction of order when it is applied to systems. Next we use a constructive theorem contained in Einsenhart [3] to overcome the ambiguity mentioned above and hence provide a systematic approach to integration when the symmetries satisfy certain properties given below.

Definition 1. A vector field in n variables $X = \xi^i(x)\partial/\partial x^i$ is a symmetry of a first-order linear homogeneous partial differential equation (PDE) $Af \equiv a^i(x)\partial f/\partial x^i = 0$ if there exists $\lambda(x)$ such that $[X, A] = \lambda(x)A$.

Definition 2. Operators $X_l = \xi_l^i(x)\partial/\partial x^i$, $l = 1, \dots, k \leq n$, are said to be unconnected if the rank of the matrix $(\xi_l^i(x))$ is k on an open set.

Theorem 1. If a first-order linear homogeneous PDE in n variables $Af = 0$ admits a solvable symmetry algebra L_{n-1} for which the generators and A are unconnected, the integration of the equation reduces to quadratures.

Proof. Let X_1, X_2, \dots, X_{n-1} be a basis of L_{n-1} , chosen such that the first r symbols for any r from 1 up to at most $n-2$ generate a subalgebra which is an ideal L_r of the Lie algebra L_{r+1} spanned by the first $r+1$ symbols. This choice is possible because L_{n-1} is solvable. Now, if $Af = 0$ is the equation, then

$$Af = 0, \quad X_1 f = 0, \dots, X_{n-2} f = 0 \quad (2.2)$$

form a complete system which admits X_{n-1} . Thus, if u is a solution of (2.2) different from a constant, so is $X_{n-1}u$. If $X_{n-1}u$ were zero, X_1, \dots, X_{n-1} and A would be connected.

Hence $X_{n-1}u = \varphi(u)$, since any two solutions of (2.2) are functions of one another (for, $\text{rank}[A, X_1, \dots, X_{n-2}] = n - 1$). If we put $u_1 = \int du/\varphi(u)$, we find that u_1 is a solution of (2.2) and that $X_{n-1}u_1 = 1$. This last equation and (2.2), in which f is replaced by u_1 , is compatible ($[X_a, X_b]u_1 = 0$, $[X_a, A]u_1 = 0$, $a, b = 1, \dots, n - 1$). Thus it can be solved for the derivatives of u_1 . Therefore u_1 is obtainable by a single quadrature.

If we use the coordinates $x'^1 = u_1$, $x'^2 = x^2, \dots, x'^n = x^n$, then in the new coordinate system, the equations (omitting the primes)

$$Af = 0, \quad X_1f = 0, \dots, X_{n-3}f = 0 \quad (2.3)$$

do not involve derivatives with respect to x^1 . Hence we may treat these equations as a set in $n - 1$ variables, with x^1 entering possibly as a parameter. Since X_1, \dots, X_{n-3} are the symbols of the ideal L_{n-3} of L_{n-2} , we may apply the same process to find u_2 which is independent of u_1 previously found since it involves variables other than x^1 . The x^1 turns out to be u_1 in the present coordinate system. By repeating this process, we obtain by $n - 1$ quadratures $n - 1$ independent solutions of $Af = 0$, that is, the complete solution.

Remark. Note that in the proof of Theorem 1 symmetries are used to reduce the number of independent variables.

Corollary 1. A system of n k th-order ODEs $x_i^{(k)} = f_i(t, x, \dots, x^{(k-1)})$, $i = 1, \dots, n$, which admits a kn -dimensional solvable symmetry algebra L_{kn} for which the $(k - 1)$ th prologation of its symbols as well as $A = \partial/\partial t + \dot{x}_i\partial/\partial x_i + \dots + x_i^{(k-1)}\partial/\partial x_i^{(k-2)} + f_i(t, x, \dots, x^{(k-1)})\partial/\partial x_i^{(k-1)}$ are unconnected, is solvable by quadratures.

Proof. Write the system of n k th-order ODEs as a first-order linear homogeneous PDE and use Theorem 1.

Remarks. In Corollary 1 the unconnectedness is redundant when $n = 1$. Indeed in this case we obtain Lie's classical result. However, for $n \geq 2$ the unconnectedness is vital as we can see from the following example. The system $\ddot{x} = xf(t, \dot{x}/x)$, $\dot{x}\ddot{y} = x\dot{y}f(t, \dot{x}/x)$ admits the four-dimensional solvable Lie algebra spanned by

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial y}, \quad X_4 = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y},$$

but the system itself is not integrable by quadratures [13]. When the hypotheses of Corollary 1 are satisfied, the approach given here is preferable since it does not require the use of canonical variables. Moreover, this approach can be used to integrate canonical forms for which the integration may not be straightforward. From this standpoint the two approaches are complementary. Also, in this approach, the symmetries are used to reduce the number of dependent variables since when the system of ODEs is cast as a first-order linear homogeneous PDE, its dependent variables become independent variables of the resulting PDE (see also the remark after the proof of Theorem 1).

Note that, when the symmetries are connected, linearization is possible [14].

Finally the method presented here enables one to construct first integrals. Recall that systems do not always admit a Lagrangian formulation [2]. The use of Noether's theorem for the construction of first integrals is not always possible.

In the next section we provide two illustrations to systems of two second-order ODEs.

3 Applications

In the following examples, in accordance with Corollary 1, we utilise the first prolongation of the symmetries.

1. Consider the simple Newtonian system with velocity-dependent forces

$$\ddot{x} = \frac{1}{\dot{x}^2 - 2\dot{y}}, \quad \ddot{y} = \frac{1 + \dot{x}}{\dot{x}^2 - 2\dot{y}}. \quad (3.1)$$

It admits the following four symmetries

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = t \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{\partial}{\partial \dot{x}} + \dot{x} \frac{\partial}{\partial \dot{y}}.$$

For equation (3.1) the operator A is

$$A = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + \frac{1}{\dot{x}^2 - 2\dot{y}} \frac{\partial}{\partial \dot{x}} + \frac{1 + \dot{x}}{\dot{x}^2 - 2\dot{y}} \frac{\partial}{\partial \dot{y}}.$$

It can be verified that the operators X_1 to X_4 and A are unconnected. We choose the following sequence of ideals:

$$L_1 = \langle X_1 \rangle \subset L_2 = \langle X_1, X_2 \rangle \subset L_3 = \langle X_1, X_2, X_3 \rangle \subset \langle X_1, X_2, X_3, X_4 \rangle.$$

Solving system (2.2), namely

$$Af = 0, \quad X_1 f = 0, \quad X_2 f = 0, \quad X_3 f = 0,$$

we obtain

$$u = -\dot{y} + \dot{x} + \frac{\dot{x}^2}{2}.$$

Simple calculations show that $X_4 u = 1$. Hence u is already normalised and we can choose it to be u_1 .

In the second step we use the coordinates

$$x^1 = t, \quad x^2 = x, \quad x^3 = y, \quad x^4 = \dot{x}, \quad x^5 = u_1.$$

In these coordinates we have

$$X_1 = \frac{\partial}{\partial x^3}, \quad X_2 = \frac{\partial}{\partial x^2}, \quad X_3 = \frac{\partial}{\partial x^1},$$

and

$$A = \frac{\partial}{\partial x^1} + x^4 \frac{\partial}{\partial x^2} + \left[\frac{(x^4)^2}{2} + x^4 - x^5 \right] \frac{\partial}{\partial x^3} + (2x^5 - 2x^4)^{-1} \frac{\partial}{\partial x^4}.$$

The solution of

$$Af = 0, \quad X_1 f = 0, \quad X_2 f = 0$$

leads to

$$u_2 = x^1 - 2x^4x^5 + (x^4)^2$$

which satisfies $X_3u_2 = 1$.

For the third step we take the coordinates

$$y^1 = t, y^2 = x, y^3 = y, y^4 = u_2, y^5 = u_1.$$

As a result

$$X_1 = \frac{\partial}{\partial y^3}, X_2 = \frac{\partial}{\partial y^2}, A = \frac{\partial}{\partial y^1} + [y^5 \pm \sqrt{(y^5)^2 - y^1 + y^4}] \frac{\partial}{\partial y^2} + () \frac{\partial}{\partial y^3},$$

where the explicit form of the term omitted in the parentheses is not needed. The solution of $Af = 0$ and $X_1f = 0$ yields

$$u_3 = y^2 - y^1y^5 \pm \frac{2}{3} [(y^5)^2 - y^1 + y^4]^{3/2}.$$

It is easy to see that $X_2u_3 = 1$.

In the final step the new variables are

$$z^1 = t, z^2 = y, z^3 = u_3, z^4 = u_2, z^5 = u_1.$$

The operators are

$$X_1 = \frac{\partial}{\partial z^2}, A = \frac{\partial}{\partial z^1} + \left[(z^5)^2 \pm z^5 \sqrt{(z^5)^2 - z^1 + z^4} + \frac{1}{2}z^4 - \frac{1}{2}z^1 \right] \frac{\partial}{\partial z^2}.$$

The solution of $Af = 0$ gives

$$u_4 = z^2 - z^1(z^5)^2 \pm \frac{2}{3}z^5 [(z^5)^2 - z^1 + z^4]^{3/2} - \frac{1}{2}z^1z^4 + \frac{1}{4}(z^1)^2$$

and we verify that $X_1u_4 = 1$.

In summary we have derived the following four first integrals

$$\begin{aligned} u_1 &= -\dot{y} + \frac{\dot{x}^2}{2} + \dot{x}, \\ u_2 &= t + 2\dot{x}\dot{y} - \dot{x}^2 - \dot{x}^3, \\ u_3 &= x - tu_1 \pm \frac{2}{3}(u_1^2 - t + u_2)^{3/2}, \\ u_4 &= y - tu_1^2 \pm \frac{2}{3}(1 + u_1)(u_1^2 - t + u_2)^{3/2} - \frac{1}{2}tu_2 + \frac{1}{4}t^2. \end{aligned}$$

These are indeed first integrals as $du_i/dt = 0$ for $i = 1, \dots, 4$ on the solutions of the system (3.1). One can also write the integrals u_3 and u_4 in terms of the variables t, x, y, \dot{x} and \dot{y} as

$$\begin{aligned} u_3 &= x + t\dot{y} - \frac{1}{2}t\dot{x}^2 - t\dot{x} + \frac{2}{3}(\dot{y} - \frac{1}{2}\dot{x}^2)^3, \\ u_4 &= y - t\dot{y}^2 - \frac{1}{4}t\dot{x}^4 - \frac{1}{2}t\dot{x}^2 + t\dot{y}\dot{x}^2 + t\dot{x}\dot{y} - \frac{1}{2}t\dot{x}^3 - \frac{1}{4}t^2 + \frac{2}{3}(1 - \dot{y} + \frac{\dot{x}^2}{2} + \dot{x})(\dot{y} - \frac{1}{2}\dot{x}^2)^3. \end{aligned}$$

2. Consider the system [13]

$$\dot{y}\ddot{x} = e^{-\dot{x}/\dot{y}}(\dot{y}f(\dot{y}) + \dot{x}g(\dot{y})), \quad \ddot{y} = e^{-\dot{x}/\dot{y}}g(\dot{y}), \quad (3.2)$$

which admits the symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial y}, \quad X_4 = t \frac{\partial}{\partial t} + (x + y) \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \dot{y} \frac{\partial}{\partial \dot{x}}.$$

For equation (3.2) the operator A is

$$A = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} + e^{-\dot{x}/\dot{y}}(f(\dot{y}) + \frac{\dot{x}}{\dot{y}}g(\dot{y})) \frac{\partial}{\partial \dot{x}} + e^{-\dot{x}/\dot{y}}g(\dot{y}) \frac{\partial}{\partial \dot{y}}.$$

As before it can be verified that X_1, \dots, X_4 and A are unconnected and one can choose the same ordering of the ideals as in the previous example. Performing the same steps as in the first example, we obtain the first integrals

$$\begin{aligned} u_1 &= \frac{\dot{x}}{\dot{y}} - \int \frac{f(\dot{y})}{\dot{y}g(\dot{y})} d\dot{y}, \\ u_2 &= y - \int \frac{\dot{y}}{g(\dot{y})} \exp\left(\int \frac{f(\dot{y})}{\dot{y}g(\dot{y})} d\dot{y} + u_1\right) d\dot{y}, \\ u_3 &= x - \int \left[\dot{y} \int \frac{f(\dot{y})}{\dot{y}g(\dot{y})} d\dot{y} + \dot{y}u_1\right] \left[\exp\left(\int \frac{f(\dot{y})}{\dot{y}g(\dot{y})} d\dot{y} + u_1\right)\right] \frac{1}{g(\dot{y})} d\dot{y}, \\ u_4 &= t - \int \frac{\exp\left(\int \frac{f(\dot{y})}{\dot{y}g(\dot{y})} d\dot{y} + u_1\right)}{g(\dot{y})} d\dot{y}. \end{aligned}$$

The integrals u_2 , u_3 and u_4 are further simplified after the substitution of the explicit form of u_1 into them. We have

$$\begin{aligned} u_2 &= y - \int \frac{\dot{y}}{g(\dot{y})} \exp\left(\frac{\dot{x}}{\dot{y}}\right) d\dot{y}, \\ u_3 &= x - \int \frac{\dot{x}}{g(\dot{y})} \exp\left(\frac{\dot{x}}{\dot{y}}\right) d\dot{y}, \\ u_4 &= t - \int \frac{1}{g(\dot{y})} \exp\left(\frac{\dot{x}}{\dot{y}}\right) d\dot{y}. \end{aligned}$$

Remark. Consider the following two-dimensional central force problem, in terms of polar coordinates

$$\ddot{r} - r\dot{\theta}^2 + \mu r^{-3} - \epsilon r = 0, \quad \mu, \epsilon > 0, \quad (3.3)$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.$$

Equation (3.1) has four point symmetries:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial \theta}, \quad X_{3\pm} = \exp(\pm 2t\sqrt{\epsilon}) \left(\frac{1}{\sqrt{\epsilon}} \frac{\partial}{\partial t} \pm r \frac{\partial}{\partial r} \right),$$

where the $+$ and $-$ in \pm refer to the two operators X_{3+} and X_{3-} , respectively. Note that (3.3) is integrable [11], but does not satisfy the unconnectedness condition of Corollary 1. This emphasizes the fact that Corollary 1 provides a sufficient condition for integrability for equations admitting four symmetries. We also refer the reader to, e.g., Leach and Gorringe [4] for other systems.

4 Discussion

Contrary to folklore a system of two second-order ODEs admitting four point symmetries is not necessarily integrable even when the symmetries form a solvable Lie algebra. However, in the case these solvable symmetries together with the equation vector field are unconnected, the system is completely integrable.

In general a system of two second-order ODEs may not admit a Lagrangian [2]. Hence the application of the classical Noether's theorem for the construction of a first integral does not apply. The approach presented here permits the construction of first integrals provided the hypotheses of Corollary 1 are satisfied.

Amongst differential equations, linear ones are generally well understood. Therefore it is of practical interest to know when nonlinear equations are transformable to linear ones by invertible changes of dependent and independent variables. In the case of scalar second-order ODEs, Lie [6] was the first to give necessary and sufficient conditions for linearization. The linearization problem for scalar higher-order ODEs was studied by Mahomed and Leach [8]. The study of linearization of a system of ODEs is recent. The work on the symmetry properties of a system of linear second-order ODEs with constant coefficients by Gorringe and Leach [5] was extended to a system with variable coefficients by Wafo and Mahomed [12]. It was found that the allowable number of point symmetries is 5, 6, 7, 8 or 15. This somewhat surprising symmetry breaking renders the study of linearization nontrivial. Indeed, instead of having a single class (from the symmetry standpoint) of linear ODEs as in the scalar case, we now have five classes. A solution to the linearization problem for a system of two second-order ODEs was given in [14]. In the case when the four symmetries are all connected or three of them are unconnected, linearization can apply and hence complete integrability. Precisely, a system of two second-order ODEs is reducible via a point transformation to the simplest equations $x'' = 0$, $y'' = 0$ if it admits a four-dimensional Lie algebra of symmetries with commutators $[X_i, X_j] = 0$, $[X_i, X_4] = X_i$, $i, j = 1, 2, 3$ such that the symmetries are all connected or three of the symmetries are unconnected [14]. We also can have linearization when the admitted algebra is Abelian. In this case one has that a system of two second-order ODEs is linearizable by means of a point transformation if it admits the four-dimensional Abelian Lie algebra with the condition that two of the symmetries are unconnected [14].

Further work needs to be done on integrability for the other cases of four-dimensional Lie algebras admitted by a system of two second-order ODEs which are not covered by the results of this paper (Corollary 1) and the linearization theorems [14] restated in the preceding paragraph.

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