A New Discrete Hénon-Heiles System

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Abstract

By considering the Darboux transformation for the third order Lax operator of the Sawada-Kotera hierarchy, we obtain a discrete third order linear equation as well as a discrete analogue of the Gambier 5 equation. As an application of this result, we consider the stationary reduction of the fifth order Sawada-Kotera equation, which (by a result of Fordy) is equivalent to a generalization of the integrable case (i) Hénon-Heiles system. Applying the Darboux transformation to the stationary flow, we find a Bäcklund transformation (BT) for this finite-dimensional Hamiltonian system, which is equivalent to an exact discretization of the generalized case (i) Hénon-Heiles system. The Lax pair for the system is 3×3, and the BT satisfies the spectrality property for the associated trigonal spectral curve. We also give an example of how the BT may be used as a numerical integrator for the original continuous Hénon-Heiles system.

1 Introduction

In the last decade or so there has been a huge interest in discrete integrable mappings or correspondences. This started with work on discrete integrable equations with a Lagrangian structure [5, 36]. There are now many discrete integrable analogues of a variety of classical systems including the Lagrange and Euler tops and their generalizations [33, 34]. More recently there has been much interest in discretizations of Hamiltonian systems which preserve all the integrals of the continuous system [18, 19, 20, 21, 22, 11, 12]. In [21] such discrete equations, satisfying an additional *spectrality* property (to be described in Section 5), were referred to as Bäcklund transformations or BTs, by analogy with the corresponding structures for integrable partial differential equations. The purpose of this article is to construct a new BT for a generalized case (i) Hénon-Heiles system, by using the connection with the Darboux transformation (DT) for the Sawada-Kotera Lax operator. DTs have been used in the past to construct integrable maps, both by us [18, 19] and in the context of the dressing chain [31].

The approach to constructing BTs for finite-dimensional systems that was adopted by one of us in [18, 19] made use of the Darboux transformation (DT) for the Schrödinger operator. In particular we found BTs (exact discretizations) for the Garnier and generalized case (ii) Hénon-Heiles system by using the fact that their Lax pairs are obtained by reduction from the KdV hierarchy. In [17] it was explained how this related to the theory of the Ermakov-Pinney equation. This is the second order nonlinear differential equation

$$ff_{xx} - \frac{1}{2}f_x^2 + 2Vf^2 + \frac{\ell^2}{2} = 0, \tag{1.1}$$

where V is an arbitrary function of the independent variable x, the subscript denotes the derivative, and ℓ is a constant. Often this is rewritten in terms of the variable q with $f = -q^2/2$, in which case it is known as Pinney's equation,

$$q_{xx} + Vq + \frac{\ell^2}{q^3} = 0. \tag{1.2}$$

In the classical work of Ermakov [9] and Pinney [25] it was shown that (1.1), or equivalently (1.2), is connected to the Schrödinger equation

$$\mathcal{L}\psi := (\partial_x^2 + V)\psi = 0; \tag{1.3}$$

the general solution of (1.1) is given by $f = \psi_1\psi_2$, a product of two (zero energy) eigenfunctions of the operator $\mathcal{L}$ with Wronskian ℓ. As a member of the class 22 of Gambier [14, 15], the equation (1.1) is linearisable by differentiation with respect to x into the third order equation

$$f_{xxx} + 4Vf_x + 2V_xf = 0 . \tag{1.4}$$

Various authors [6, 27, 30, 37] have been interested in obtaining discrete versions of (1.1) or (1.2) which preserve the properties of the continuous equation. Making use of the connection with (1.3) and the Darboux transformation [8, 7] (DT) for the Schrödinger operator, one of us [17] obtained a BT for (1.1), leading to an exact discretization of the Ermakov-Pinney equation which may be written in terms of q as

$$\sqrt{\ell^2 + \kappa^2 q^2 \overline{q}^2} + \sqrt{\ell^2 + \underline{\kappa}^2 q^2 \underline{q}^2} = \Omega q^2. \tag{1.5}$$

The quantity Ω is a function of Bäcklund parameters κ, $\underline{\kappa}$ and a sequence of potentials $\underline{V}$, V, $\overline{V}$ related by the Darboux transformation. (Note that we have taken $p \to f = -q^2/2$, $\ell \to \ell/2$ compared with the formulae in reference [17].) The discrete Pinney equations in [6, 17] have also been related to a discrete Schwarzian equation of form

$$\frac{(\overline{\varphi} - \underline{\varphi})\,(\overline{\overline{\varphi}} - \varphi)}{(\varphi - \underline{\varphi})\,(\overline{\overline{\varphi}} - \overline{\varphi})} - \frac{\Omega\overline{\Omega}}{\kappa^2} = 0 , \tag{1.6}$$

first given by Faddeev and Takhtajan [10]. In the next Section we apply the DT approach to a third order operator, leading to a discrete Gambier 5 equation, before using these results in the context of the case (i) Hénon-Heiles system.

2 Discrete Gambier 5 equation

In the Gambier classification [15], another second order nonlinear equation belonging to the class 5,

$$Y_{xx} + 3YY_x + Y^3 + rY + q = 0 , \tag{2.1}$$

where r and q are two functions of x, is linearizable by the transformation $Y = \partial_x \log \psi$ into the third order linear equation

$$\psi_{xxx} + r\psi_x + q\psi = 0 .$$

Setting $Y_1 = \psi_x/\psi$, $Y_2 = \psi_{xx}/\psi$, the equation (2.1) is equivalent to the projective Riccati system [1]

$$Y_{1,x} = Y_2 - Y_1^2 , \tag{2.2}$$
$$Y_{2,x} = -Y_1Y_2 - rY_1 - q . \tag{2.3}$$

Several attempts to discretize the system (2.2)–(2.3) and therefore the equation (2.1) have been investigated by Lie group consideration and discrete Painlevé approach [26, 16], in order to recover (2.2) and (2.3) in the continuous limit.

In order to derive a discrete version of the equation (2.1) we here extend the approach explained in the previous section for the discrete EP equation by considering the DT associated with the third order linear equation

$$\psi_{xxx} + V\psi_x = \lambda\psi \tag{2.4}$$

where V is a function of x and λ is a constant. This equation corresponds to the scattering problem associated with the Sawada-Kotera [29] nonlinear partial differential equation. The DT for (2.4) has been previously considered in [3, 23]. Under the DT a new potential $\overline{V}$ is obtained from the eigenfunction ϕ of an associated third order scattering problem

$$\phi_{xxx} + V\phi_x = \mu\phi , \tag{2.5}$$

according to the formula

$$\overline{V} = V + 6\partial_x^2 \log \phi . \tag{2.6}$$

Then the transformation law for a new function $\overline{\psi}$ satisfying

$$\overline{\psi}_{xxx} + \overline{V}\overline{\psi}_x = \lambda\overline{\psi} \tag{2.7}$$

follows from the DT. In matrix form, this becomes

$$\begin{pmatrix} \overline{\psi} \\ \overline{\psi}_x \\ \overline{\psi}_{xx} \end{pmatrix} = g \begin{pmatrix} \mu - \lambda & -2Y_2 & 2Y_1 \\ 2\lambda Y_1 & 2Y_1Y_2 - \lambda - \mu & -2Y_1^2 \\ 2\lambda(Y_2 - 2Y_1^2) & \begin{matrix} 2Y_2(Y_2 - 2Y_1^2) \\ +2Y_1(\lambda + \mu) \end{matrix} & \begin{matrix} -2Y_1(Y_2 - 2Y_1^2) \\ -\lambda - \mu \end{matrix} \end{pmatrix} \begin{pmatrix} \psi \\ \psi_x \\ \psi_{xx} \end{pmatrix} \tag{2.8}$$

with $g = \left[(\mu - \lambda)(\mu + \lambda)^2 \right]^{-1/3}$.

From (2.8) and its inverse

$$
\begin{pmatrix} \psi \\ \psi_x \\ \psi_{xx} \end{pmatrix} = \hat{g}
\begin{pmatrix}
\mu + \lambda & 2(2Y_1^2 - Y_2) & 2Y_1 \\
2\lambda Y_1 & \begin{array}{c} -2Y_1(Y_2 - 2Y_1^2) \\ -\mu + \lambda \end{array} & 2Y_1^2 \\
2\lambda Y_2 & \begin{array}{c} -2Y_2(Y_2 - 2Y_1^2) \\ -2Y_1(\mu - \lambda) \end{array} & 2Y_1 Y_2 + \lambda - \mu
\end{pmatrix}
\begin{pmatrix} \overline{\psi} \\ \overline{\psi}_x \\ \overline{\psi}_{xx} \end{pmatrix}, \tag{2.9}
$$

with $\hat{g} = g^{-1}(\mu^2 - \lambda^2)^{-1}$, one may obtain the following three equations:

$$
\overline{\psi} - g(\mu - \lambda)\psi = g(-2Y_2\psi_x + 2Y_1\psi_{xx}) \tag{2.10}
$$

$$
g(\underline{\mu}^2 - \lambda^2)\underline{\psi} - (\underline{\mu} + \lambda)\psi = 2(2\underline{Y}_1^2 - \underline{Y}_2)\psi_x + 2\underline{Y}_1\psi_{xx} \tag{2.11}
$$

$$
\overline{\overline{\psi}} - \overline{g}(\overline{\mu} - \lambda)\overline{\psi} + 4g\overline{g}\lambda(\overline{Y}_2 Y_1 - \overline{Y}_1 Y_2 + 2\overline{Y}_1 Y_1^2)\psi
$$
$$
= 2g\overline{g}\left((\lambda + \mu)(\overline{Y}_2 + 2Y_1\overline{Y}_1) - 2\overline{Y}_2 Y_1 Y_2 + 2\overline{Y}_1 Y_2(Y_2 - 2Y_1^2)\right)\psi_x
$$
$$
+ 2g\overline{g}\left(-(\lambda + \mu)\overline{Y}_1 + 2Y_1(\overline{Y}_2 Y_1 + 2\overline{Y}_1 Y_1^2 - \overline{Y}_1 Y_2)\right)\psi_{xx} \tag{2.12}
$$

The elimination of ψ_x, ψ_{xx} between (2.10), (2.11) and (2.12) yields the discrete third order equation

$$
A\,\overline{\overline{\psi}} + \overline{g}\left((\lambda + \mu)\,B + (\lambda - \overline{\mu})\,A + 2A\overline{A}\right)\overline{\psi}
$$
$$
+ g\overline{g}(\lambda + \mu)\left((\lambda - \mu)\,B + (\lambda + \underline{\mu})\overline{A} - 2A\overline{A}\right)\psi
$$
$$
+ g\overline{g}g(\lambda^2 - \underline{\mu}^2)(\lambda + \mu)\overline{A}\underline{\psi} = 0, \tag{2.13}
$$

where

$$
A = Y_1\underline{Y}_2 - \underline{Y}_1 Y_2 - 2\underline{Y}_1^2 Y_1, \quad B = \overline{Y}_1\underline{Y}_2 - \underline{Y}_1\overline{Y}_2 - 2\underline{Y}_1^2\overline{Y}_1 - 2\underline{Y}_1 Y_1\overline{Y}_1.
$$

In the continuous limit ($h \to 0$),

$$
\overline{\overline{\psi}} \equiv \psi(x + 2h) = \psi + 2h\psi_x + 2h^2\psi_{xx} + \frac{4}{3}h^3\psi_{xxx} + \mathcal{O}(h^4), \tag{2.14}
$$

$$
\overline{\psi} \equiv \psi(x + h) = \psi + h\psi_x + \frac{h^2}{2}\psi_{xx} + \frac{h^3}{6}\psi_{xxx} + \mathcal{O}(h^4), \tag{2.15}
$$

$$
\underline{\psi} \equiv \psi(x - h) = \psi - h\psi_x + \frac{h^2}{2}\psi_{xx} - \frac{h^3}{6}\psi_{xxx} + \mathcal{O}(h^4), \tag{2.16}
$$

with

$$
\mu = -\frac{8}{h^3} + \mathcal{O}(1), \qquad Y_1 = -\frac{2}{h} + \frac{h}{6}V + \mathcal{O}(h^2), \tag{2.17}
$$

the discrete equation (2.13) tends to the third order linear equation (2.4) and the relation (2.17) for Y_1 and the one for $\overline{Y}_1$ are compatible with the transformation (2.6). Therefore defining

$$
\overline{\psi}/\psi = 1 + hY \;; \overline{\overline{\psi}}/\overline{\psi} = 1 + h\overline{Y} \;; \psi/\underline{\psi} = 1 + h\underline{Y} \tag{2.18}
$$

and substituting in (2.13) we obtain the discrete Gambier 5 equation which in the continuous limit tends to (2.1).

3   Generalized Hénon-Heiles case (i)

A generalization of the integrable case (i) of the Hénon-Heiles system has the Hamiltonian

$$h_1 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{6}q_1^3 + \frac{1}{2}q_1 q_2^2 - \frac{\ell^2}{2q_2^2}. \tag{3.1}$$

The positions q_j and p_j are canonically conjugate variables, and ℓ is a constant parameter. Hamilton's equations are

$$q_1' = p_1, \qquad q_2' = p_2, \qquad p_1' = -\frac{1}{2}q_1^2 - \frac{1}{2}q_2^2, \qquad p_2' = -q_1 q_2 - \frac{\ell^2}{q_2^3}, \tag{3.2}$$

giving the Newton equations

$$q_1'' = -\frac{1}{2}\left(q_1^2 + q_2^2\right), \qquad q_2'' = -q_1 q_2 - \frac{\ell^2}{q_2^3}. \tag{3.3}$$

Observe that the second Newton equation for q_2 has the same form as Pinney's equation (1.2). The equations of motion (3.2) can be written in the Lax form

$$\mathbf{L}' = [\mathbf{N}, \mathbf{L}], \tag{3.4}$$

with

$$\mathbf{L}(\lambda) = \begin{pmatrix} 6\lambda q_1 & -\frac{3}{2}q_1^2 - \frac{1}{2}q_2^2 & 9\lambda - 3p_1 \\ 9\lambda^2 + 3\lambda p_1 & -3\lambda q_1 - q_2 p_2 & q_2^2 \\ -\lambda\left(\frac{3}{2}q_1^2 + \frac{1}{2}q_2^2\right) & 9\lambda^2 + \frac{\ell^2}{q_2^2} - p_2^2 & -3\lambda q_1 + q_2 p_2 \end{pmatrix},$$

and

$$\mathbf{N}(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & -q_1 & 0 \end{pmatrix}, \tag{3.5}$$

where λ is the spectral parameter, and the prime in (3.4) denotes the derivation corresponding to the flow generated by the Hamiltonian h_1 (3.1). The Lax equation (3.4) is just the compatibility condition for the matrix linear system

$$\mathbf{L}\,\Psi = \eta\,\Psi, \tag{3.6}$$

$$\Psi' = \mathbf{N}\Psi, \tag{3.7}$$

which was derived from the Lax pair for the Sawada-Kotera equation in [13].

The Hamiltonian flow preserves the spectral curve $\Gamma : \mathcal{P}(\lambda, \eta) = 0$, with

$$\mathcal{P}(\lambda, \eta) \equiv \det\left(\eta\mathbf{1} - \mathbf{L}(\lambda)\right) = \eta^3 - \ell^2\eta - 729\lambda^5 + 162h_1\lambda^3 - 9h_2\lambda. \tag{3.8}$$

The two independent conserved quantities for the generalized case (i) Hénon-Heiles system are the coefficients h_1, h_2 appearing in the formula (3.8) for the spectral curve, given by (3.1) and

$$h_2 = \left(p_1 p_2 + \frac{1}{2}q_1^2 q_2 + \frac{1}{6}q_2^3\right)^2 - \ell^2\left(\frac{p_1^2}{q_2^2} + \frac{2}{3}q_1\right). \tag{3.9}$$

The algebraic variety Γ in $\mathbb{C}^2$ defined by the vanishing of the polynomial (3.8) is a Riemann surface of genus 4. Observe from (3.8) that Γ has the involution

$$(\lambda, \eta) \in \Gamma \iff (-\lambda, -\eta) \in \Gamma. \tag{3.10}$$
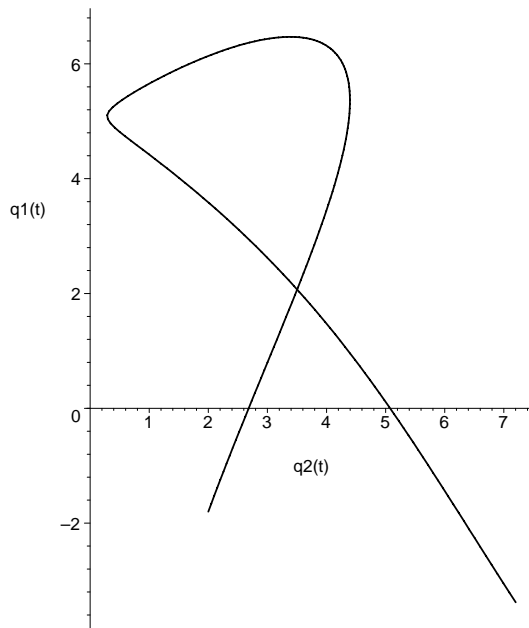
Figure 1. Plot of q_1 versus q_2 from numerical integration of the case (i) generalized Hénon-Heiles system with parameter $\ell^2 = -10$ and initial data $q_1(0) = -1.8$, $p_1(0) = 14$, $q_2(0) = 2$, $p_2(0) = 5$.

4   Bäcklund transformation

To construct a Bäcklund transformation for the generalized integrable case (i) Hénon-Heiles system, we use the fact that the Hamiltonian flow generated by (3.1) is a stationary flow in the Sawada-Kotera hierarchy, as first observed in [13]. Thus if Ψ in (3.6) and (3.7) is written as a column vector, $\Psi = (\psi, \psi', \psi'')^T$, then with the form of the matrix $\mathbf{N}$ as in (3.5), the first component just satisfies the spatial part of the Sawada-Kotera Lax pair, i.e.

$$\psi''' + q_1\psi' = \lambda\psi. \tag{4.1}$$

From our previous formula (2.8) in Section 2, we know that the Darboux transformation for the linear problem (4.1) can be written in matrix form as

$$\Psi \longrightarrow \tilde{\Psi} = \mathbf{M}(\lambda, \mu)\Psi, \tag{4.2}$$

where (up to rescaling by a constant prefactor, which can depend on λ, μ)

$$\mathbf{M}(\lambda, \mu) = \begin{pmatrix} \mu - \lambda & -2Y_2 & 2Y_1 \\ 2Y_1\lambda & 2Y_1Y_2 - \lambda - \mu & -2Y_1^2 \\ 2(Y_2 - 2Y_1^2)\lambda & \begin{matrix} 2Y_2(Y_2 - 2Y_1^2) \\ +2Y_1(\lambda + \mu) \end{matrix} & \begin{matrix} -2Y_1(Y_2 - 2Y_1^2) \\ -\lambda - \mu \end{matrix} \end{pmatrix}, \tag{4.3}$$

and

$$Y_1 = \frac{\phi'}{\phi}, \qquad Y_2 = \frac{\phi''}{\phi}, \qquad \phi''' + q_1\phi' = \mu\phi. \tag{4.4}$$

The Darboux transformation (4.2) acts on both parts of the Lax pair, so that $\tilde{\Psi}$ satisfies the same linear system (3.6) and (3.7), but with $\mathbf{L}$, $\mathbf{N}$ replaced by $\tilde{\mathbf{L}}$, $\tilde{\mathbf{N}}$, which are the same matrices but with q_j, p_j replaced by new positions and momenta $\tilde{q}_j$, $\tilde{p}_j$. In particular, the first component $\tilde{\psi}$ of the vector $\tilde{\Psi}$ is an eigenfunction of the Darboux-transformed Sawada-Kotera Lax operator, i.e.

$$\tilde{\psi}''' + \tilde{q}_1 \tilde{\psi}' = \lambda \tilde{\psi}.$$

Thus the Darboux transformation acting on the linear system induces a Bäcklund transformation (BT) on the finite-dimensional Hamiltonian system, and applying (4.2) to the first equation (3.6) yields the discrete Lax equation

$$\mathbf{M}(\lambda, \mu)\, \mathbf{L}(\lambda) = \tilde{\mathbf{L}}(\lambda)\, \mathbf{M}(\lambda, \mu). \tag{4.5}$$

However, a priori the quantities Y_1 and Y_2 are defined in terms of derivatives of the eigenfunction ϕ as in (4.4). In order for (4.5) to define a discrete dynamical system, these quantities must be obtained as suitable functions of the dynamical variables q_j, p_j and/or $\tilde{q}_j$, $\tilde{p}_j$. As it stands, (4.5) provides the correct form of the BT, with Y_1 and Y_2 to be determined.

To find the precise form of the mapping from q_j, p_j to $\tilde{q}_j$, $\tilde{p}_j$, it is necessary to eliminate Y_1 and Y_2 by comparing coefficients of λ on both sides of (4.5), with μ being regarded as a parameter in the BT (the Bäcklund parameter). Initially (to prove that the BT is a canonical transformation) it will be most convenient to consider the BT in *implicit* form, with both the Y_j and the momenta given as functions of the coordinates, i.e. $Y_j = Y_j(q_k, \tilde{q}_k)$, $p_j = p_j(q_k, \tilde{q}_k)$, $\tilde{p}_j = \tilde{p}_j(q_k, \tilde{q}_k)$ for $j = 1, 2$. Comparison of both sides of (4.5) yields the following relations:

$$\begin{aligned}
p_1 &= 3Y_1^3 + (2q_1 + \tilde{q}_1)Y_1 - \tfrac{1}{12Y_1}\left(q_2^2 - \tilde{q}_2^2\right) - 3\mu, \\
\tilde{p}_1 &= -3Y_1^3 - (q_1 + 2\tilde{q}_1)Y_1 - \tfrac{1}{12Y_1}\left(q_2^2 - \tilde{q}_2^2\right) + 3\mu,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
p_2 q_2 &= Z + q_2^2 \left(Y_1 - \tfrac{1}{6Y_1}(q_1 - \tilde{q}_1) - \tfrac{\mu}{2Y_1^2}\right), \\
\tilde{p}_2 \tilde{q}_2 &= -Z + \tilde{q}_2^2 \left(-Y_1 - \tfrac{1}{6Y_1}(q_1 - \tilde{q}_1) + \tfrac{\mu}{2Y_1^2}\right),
\end{aligned} \tag{4.7}$$

where

$$Z = 9Y_1^5 + \frac{9}{2}(q_1 + \tilde{q}_1)Y_1^3 - 9\mu Y_1^2 + \frac{1}{2}\left(q_1^2 + q_1 \tilde{q}_1 + \tilde{q}_1^2 + \frac{1}{2}q_2^2 + \frac{1}{2}\tilde{q}_2^2\right)Y_1$$

$$+ \frac{1}{24Y_1}(q_1 - \tilde{q}_1)(q_2^2 - \tilde{q}_2^2) + \frac{\mu}{4Y_1^2}(q_2^2 + \tilde{q}_2^2), \tag{4.8}$$

with

$$Z^2 = \ell^2 + \frac{\mu^2 q_2^2 \tilde{q}_2^2}{4Y_1^4}, \tag{4.9}$$

and also

$$Y_2 = Y_1^2 + \frac{1}{6}(\tilde{q}_1 - q_1). \tag{4.10}$$

The above formulae deserve some further explanation. Observe that from (4.6) and (4.7) the momenta are given in terms of the coordinates as well as Y_1 and the quantity Z. In turn, Z is defined in terms of the coordinates and Y_1 according to the expression (4.8). Finally, substituting for Z from (4.8) into (4.9), we see that Y_1 satisfies a polynomial of degree 14 with coefficients depending on the coordinates q_1, q_2, $\tilde{q}_1$, $\tilde{q}_2$ and μ. The discrete Lax equation (4.5) means that the spectral curve given by (3.8) is preserved, and hence the two quantities h_1, h_2 are conserved under the BT. So as it stands the Bäcklund transformation is defined only implicitly; the equation for Y_1 introduces multivaluedness, so we have an integrable correspondence of the kind first detailed in [36]. In order to have an *explicit* BT, we make use of the spectrality property in the next Section. However, first it is necessary to derive the exact generating function $S(q_k, \tilde{q}_k)$ which proves that the transformation is canonical. From the expressions (4.6), (4.7) for the momenta we find that

$$dS = \sum_{j=1,2} p_j \, dq_j - \tilde{p}_j \, d\tilde{q}_j$$

with

$$S = \frac{63}{5} Y_1^5 + \frac{15}{2}(q_1 + \tilde{q}_1) Y_1^3 - 18\mu Y_1^2 + \frac{3}{2}\left(q_1^2 + q_1\tilde{q}_1 + \tilde{q}_1^2 + \frac{1}{2}q_2^2 + \frac{1}{2}\tilde{q}_2^2 \right) Y_1$$

$$-3\mu(q_1 + \tilde{q}_1) - \frac{1}{24Y_1}(q_1 - \tilde{q}_1)(q_2^2 - \tilde{q}_2^2) + \frac{\ell}{2}\log\left(\frac{Z - \ell}{Z + \ell}\right). \tag{4.11}$$

It is interesting to compare this with the formula for the generating function of the case (ii) Hénon-Heiles system, which was presented in [18, 19]. In that case, S has exactly the same functional dependence on an analogous quantity Z, and is a quintic expression in another quantity y (the analogue of Y_1) which for case (ii) is a function of the coordinates given by the root of a quadratic (rather than a degree 14 equation as here for case (i)).

5   Spectrality and the explicit mapping

The spectrality property is the key to writing the BT for generalized case (i) Hénon-Heiles in an explicit form. Essentially it means that given an eigenvalue ξ of the Lax matrix $\mathbf{L}(\mu)$, so that $\mathcal{P}(\mu, \xi) = 0$ and (μ, ξ) is a point on the spectral curve Γ, this eigenvalue is canonically conjugate to the Bäcklund parameter μ. To see this we first note that when $\lambda = \mu$ the Darboux matrix $\mathbf{M}(\mu, \mu)$ as in (4.3) becomes degenerate, with a one-dimensional kernel spanned by a vector Φ:

$$\mathbf{M}(\mu, \mu)\, \Phi = 0, \qquad \Phi = (1, Y_1, Y_2)^T. \tag{5.1}$$

By applying both sides of the discrete Lax equation (4.5) to the vector Φ and setting $\lambda = \mu$ we see that $\mathbf{L}(\mu)\Phi \in \ker \mathbf{M}(\mu, \mu)$, which implies that Φ is an eigenvector of $\mathbf{L}(\mu)$, i.e.

$$\mathbf{L}(\mu)\, \Phi = \xi\, \Phi \tag{5.2}$$

and so $\mathcal{P}(\mu, \xi) \equiv \det\left(\xi\mathbf{1} - \mathbf{L}(\mu)\right) = 0$.

A careful calculation (taking into account the derivatives $\frac{\partial Z}{\partial \mu}$, $\frac{\partial Y_1}{\partial \mu}$) shows that

$$\frac{\partial S}{\partial \mu} = \frac{Z}{\mu} - 9Y_1^2 - \frac{1}{4Y_1^2}(q_2^2 + \tilde{q}_2^2) - 3(q_1 + \tilde{q}_1).$$

If we consider the first row of the equation (5.2) and substitute for Y_2 from (4.10), then it is easy to see that

$$\xi = -\mu \frac{\partial S}{\partial \mu}, \tag{5.3}$$

which shows that μ, ξ are canonically conjugate variables. This is the required spectrality property, as introduced in [21].

Given the conjugate pair (μ, ξ) it is now possible to describe the BT explicitly as an iterative map, rather than just an integrable correspondence. The exact formulae for the new variables $\tilde{q}_j$, $\tilde{p}_j$ in terms of q_j, p_j and the Bäcklund parameter μ are very unwieldy if written out in full, so instead we prefer to present each iterative step of the BT in the form of an algorithm, which is easily implemented on a computer:

Step 1. Pick a point on the spectral curve: Given the initial data q_j, p_j, the values of the conserved quantities h_1 and h_2 are fixed; these are preserved under iteration of the BT. Choosing a value for the Bäcklund parameter μ, the conjugate variable ξ is then found from the formula (3.8) for the spectral curve $\Gamma : \mathcal{P}(\mu, \xi) = 0$, i.e. as a root of a cubic equation. This gives a point (μ, ξ) on the spectral curve Γ.
Step 2. Find Y_1 and Y_2: By picking any two rows of the eigenvector equation (5.2), we can solve for Y_1 and Y_2 in terms of q_j, p_j and the spectral data $(\mu, \xi) \in \Gamma$. For instance, taking the first two rows we have

$$\begin{pmatrix} -\frac{3}{2}q_1^2 - \frac{1}{2}q_2^2 & 9\mu - 3p_1 \\ -3\mu q_1 - q_2 p_2 - \xi & q_2^2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \xi - 6\mu q_1 \\ -9\mu^2 - 3\mu p_1 \end{pmatrix}, \tag{5.4}$$

which is easily solved to obtain explicit expressions for Y_1 and Y_2 (which we omit for reasons of length).
Step 3. Find $\tilde{q}_1$: Given the explicit form of Y_1 and Y_2 from (5.4), the new coordinate $\tilde{q}_1$ is found in terms of the initial and spectral data by solving (4.10) to yield

$$\tilde{q}_1 = q_1 + 6(Y_2 - Y_1^2). \tag{5.5}$$

Step 4. Find $\tilde{p}_1$: Subtracting the two equations (4.6) and solving for $\tilde{p}_1$ gives

$$\tilde{p}_1 = p_1 - 6Y_1^3 - 3(q_1 + \tilde{q}_1)Y_1 + 6\mu, \tag{5.6}$$

and this is found entirely in terms of q_j, p_j and μ, ξ by substituting for Y_1 from Step 2 and for $\tilde{q}_1$ from (5.5) as in Step 3.
Step 5. Find $\tilde{q}_2$: Given $\tilde{p}_1$ as in (5.6) and the other quantities obtained in Steps 1-4, $\tilde{q}_2$ is found by solving the second equation of (4.6) to give

$$\tilde{q}_2^2 = q_2^2 + 12Y_1\left(\tilde{p}_1 + 3Y_1^2 + (q_1 + 2\tilde{q}_1)Y_1 - 3\mu\right). \tag{5.7}$$

Step 6. Find $\tilde{p}_2$: Adding the pair of equations (4.7) and solving for $\tilde{p}_2$ gives

$$\tilde{p}_2 = \frac{1}{\tilde{q}_2}\left(-p_2 q_2 + (q_2^2 - \tilde{q}_2^2)Y_1 - \frac{1}{6Y_1}(q_1 - \tilde{q}_1)(q_2^2 + \tilde{q}_2^2) - \frac{\mu}{2Y_1^2}(q_2^2 - \tilde{q}_2^2)\right). \tag{5.8}$$

From (5.7) and Steps 1-4, the quantities Y_j and $\tilde{q}_j$ for $j = 1, 2$ are all found in terms of the initial data q_j, p_j and spectral data (μ, ξ), and thus from (5.8) it is clear that $\tilde{p}_2$ is found in these terms in the final Step.

6   Continuum limit

The discrete equations (4.6), (4.7) have a continuum limit to the Hamilton's equations (3.2) with continuous time t by taking $q_j = q_j(t)$, $p_j = p_j(t)$ with

$$\tilde{q}_j = q_j(t + h) \sim q_j + q_j'h + \dots, \qquad \tilde{p}_j = p_j(t + h) \sim p_j + p_j'h + \dots,$$

for $j = 1, 2$, and

$$Y_1 \sim -\frac{2}{h} + \frac{q_1}{6}h + \mathcal{O}(h^2), \qquad \mu \sim -\frac{8}{h^3} + \mathcal{O}(h^0), \tag{6.1}$$

by taking the limit $h \to 0$. Thus it is apparent that the Bäcklund parameter μ plays the role of an inverse time step. With the limiting behaviour as in (6.1) is clear from (4.10) that Y_2 has the leading order behaviour

$$Y_2 \sim \frac{4}{h^2} - \frac{2}{3}q_1 + \mathcal{O}(h). \tag{6.2}$$

An easier way to check the continuum limits, rather than substituting the small h expansions into (4.6), (4.7), is to consider the discrete Lax equation (4.5). From the expressions (6.1) and (6.2) it is straighforward to see that the matrix $\mathbf{M}$ has the expansion

$$\mathbf{M}(\lambda, \mu) \sim -\frac{8}{h^3}\left(\mathbf{1} + h\,\mathbf{N}(\lambda) + \mathcal{O}(h^2)\right), \tag{6.3}$$

If the shifted Lax matrix is expanded as $\tilde{\mathbf{L}} \sim \mathbf{L} + h\,\mathbf{L}' + \mathcal{O}(h^2)$, and this is substituted with (6.3) into (4.5), then the continuous Lax equation (3.4) arises in the limit $h \to 0$.

It is also instructive to consider the discrete Newton equations

$$\tilde{p}_j(\underline{q}_k, q_k) = p_j(q_k, \tilde{q}_k) \tag{6.4}$$

(with $\underline{q}_j$ denoting q_j evaluated at a reverse time step) arising from the BT. Using the formula (4.6) we see that the first discrete Newton equation, $j = 1$ in (6.4), has the form

$$3(Y_1^3 + \underline{Y}_1^3) + (2q_1 + \tilde{q}_1)Y_1 + (\underline{q}_1 + 2q_1)\underline{Y}_1 - \frac{1}{12}\left((q_2^2 - \tilde{q}_2^2)/Y_1 - (\underline{q}_2^2 - q_2^2)/\underline{Y}_1\right) - 3(\mu + \underline{\mu}) = 0,$$

and this has a continuum limit to the first Newton equation in (3.3). Taking $j = 2$ in (6.4) yields the second discrete Newton equation, which can be rearranged as

$$\sqrt{\ell^2 + \frac{\mu^2\,q_2^2\,\tilde{q}_2^2}{4Y_1^4}} + \sqrt{\ell^2 + \frac{\underline{\mu}^2\,q_2^2\,\underline{q}_2^2}{4\underline{Y}_1^4}} = \Omega q_2^2, \tag{6.5}$$

with

$$\Omega = -Y_1 - \underline{Y}_1 + \frac{1}{6}\left((q_1 - \tilde{q}_1)/Y_1 - (\underline{q}_1 - q_1)/\underline{Y}_1\right) + \frac{1}{2}\left(\frac{\mu}{Y_1^2} + \underline{\mu}\,\underline{Y}_1^2\right).$$

In order to obtain the expression (6.5), we have taken a square root in the equation (4.9) for Z and substituted into (4.7), and similarly for $\underline{Z}$ with all variables shifted back by one lattice step. This second discrete Newton equation has a continuum limit to the second Newton equation for q_2 in (3.3). It is interesting to observe that (6.5) is almost identical in form to the discrete Pinney equation (1.5) found in [17] using the Darboux transformation for the (second order) Schrödinger equation (instead of the Darboux transformation for the third order Sawada-Kotera Lax operator, as in this case).

7 Numerical experiments

Since the BT we have constructed has a continuum limit to the Hamilton's equations (3.2), and preserves the value both integrals h_1, h_2, we expect that it should be a good symplectic integrator for this generalized Hénon-Heiles system if the Bäcklund parameter μ is large. As an example we consider solving the initial value problem for (3.2) with parameter $\ell^2 = -10$ and initial data

$$q_1(0) = -1.8, \qquad p_1(0) = 14, \qquad q_2(0) = 2, \qquad p_2(0) = 5. \tag{7.1}$$

Using its default numerical integration package (RFK45), the computer algebra package MAPLE produced the graphical plot Figure 1 of q_1 versus q_2 for an integration from $t = 0$ to $t = 2$ with timestep $h = 0.01$.
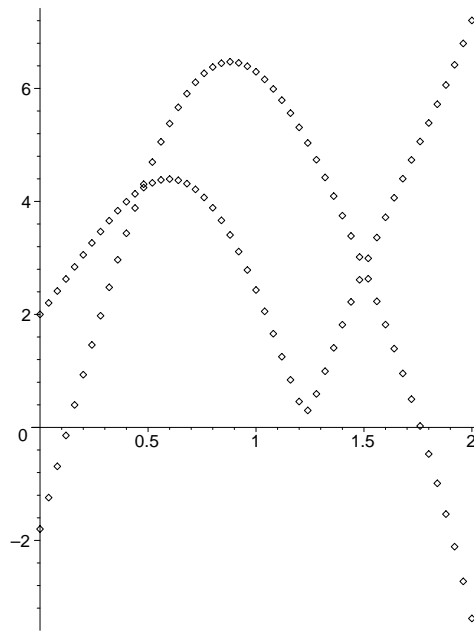

Figure 2. Successive iterates of the BT plotted as $q_1(j/25)$ (lower curve) and $q_2(j/25)$ (upper curve). The discrete mapping is iterated 50 times ($j = 0, \ldots, 50$) with parameter $\ell^2 = -10$, initial data $q_1(0) = -1.8$, $p_1(0) = 14$, $q_2(0) = 2$, $p_2(0) = 5$ and Bäcklund parameter $\mu = -125000$.

To make a comparison with some standard numerical integrators, we have applied 50 iterations of the BT, starting from the initial values (7.1), and choosing $\ell^2 = -10$ and

Bäcklund parameter $\mu = -125000$ corresponding to a time step $h = 0.04$ (equivalent to integrating from $t = 0$ to $t = 2$). This corresponds to values of the Hamiltonians of

$$h_1 = 107.178000, \qquad h_2 = 6039.18204 \tag{7.2}$$

to 9 significant figures. The chosen value of the other spectral parameter is roughly $\xi = 2.8 \times 10^9$, corresponding to a real root of the cubic $\mathcal{P}(\mu, \xi) = 0$. The iteration of the algorithm in Section 5 is straightforward to perform with MAPLE, and produces final values (to 9 significant figures)

$$q_{1,BT}(2) = -3.38140912, \qquad p_{1,BT}(2) = -17.0082640,$$

$$q_{2,BT}(2) = 7.20251308, \qquad p_{2,BT}(2) = 10.6388167. \tag{7.3}$$

We have plotted the iteration of the BT in Figure 2. Using MAPLE's default RFK45 integrator with the same timestep $h = 0.04$ instead gives

$$q_{1,RFK}(2) = -3.38140943, \qquad p_{1,RFK}(2) = -17.0082654,$$

$$q_{2,RFK}(2) = 7.20251377, \qquad p_{2,RFK}(2) = 10.6388180. \tag{7.4}$$

At the end of 50 steps of the BT, the values (7.2) of the Hamiltonians are not preserved exactly due to rounding errors, but differ only in the tenth significant figure. For the RFK45 method we find instead

$$h_{1,RFK} = 107.178010, \qquad h_{2,RFK} = 6039.18310,$$

which clearly differ from (7.2). It is also interesting to note that when we integrated from $t = 0$ to $t = 2$ with the initial conditions (7.1) using a second-order Runge-Kutta method (RK2) with the same timestep $h = 0.04$, the integration blew up and gave values for the Hamiltonians of approximately

$$h_{1,RK2} = 1.0 \times 10^4, \qquad h_{2,RK2} = 6.5 \times 10^7.$$

Thus the BT seems to give the most accurate results with this timestep.

8   Conclusions

We have constructed an exact discrete analogue of the generalized case (i) Hénon-Heiles system, which is associated with a trigonal spectral curve (3.8) of genus 4. From general considerations (see e.g. [11, 12, 22]) we know that this discrete mapping corresponds to a discrete linear flow on the (four-dimensional) Jacobian of this spectral curve. However, the Hénon-Heiles system has two degrees of freedom, and with two other authors one of us [35] constructed the separation of variables for this system in terms of a genus two hyperelliptic spectral curve of the form

$$r^2 = P_6(s), \tag{8.5}$$

with P_6 a sextic polynomial in s. From the Liouville-Arnold theorem [2] it is clear that the level set of the integrals h_1, h_2 is a two-dimensional torus. Thus the (discrete or continuous) Hénon-Heiles flow must correspond to a linear flow on a two-dimensional subvariety of the trigonal Jacobian. However, we conjecture that this can be understood directly on the level of the spectral curves in the sense that the genus four curve (3.8) should be a double cover of the hyperelliptic curve (8.5). We will describe this explicitly in future work, together with the discretization of the case (iii) Hénon-Heiles system which is related to case (i) by a canonical transformation [4, 28].

Note added in proof: Since this paper was originally written, one of our colleagues, John Merriman, has explicitly constructed the genus two hyperelliptic curve (8.5) as a quotient of (3.8) by the involution (3.10). This will be presented in a forthcoming article.

References

[1] R. Anderson, J. Harnad and P. Winternitz, Physica D **4** (1982) 164–182.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd edition, Springer-Verlag (1989).

[3] C. Athorne and J.J.C. Nimmo, Inverse Problems 7 (1991) 645-654.

[4] M. Blaszak and S. Rauch-Wojciechowski, J. Math. Phys. **35** (1994) 1693–1709.

[5] M. Bruschi, O. Ragnisco, P.M. Santini and T. Gui-Zhang, Physica D 49 (1991) 273-294.

[6] A.K. Common and M. Musette, Phys. Lett. A 235 (1997) 574-580.

[7] M.M. Crum, Quart. J. Math. Oxford 6 (1955) 121.

[8] G. Darboux, C. R. Acad. Sc. (Paris) **94** (1882) 1456–1459.

[9] V.P. Ermakov, Univ. Izv. Kiev Series III, 9 (1880) 1-25.

[10] L.D. Faddeev and L.A. Takhtajan, Springer Lect. Notes Phys. 246 (1986) 166-179.

[11] Y. Fedorov, *Discrete Versions of Some Algebraic Integrable Systems Related to Generalized Jacobians*, in 'Proceedings of the SIDE III meeting, Sabaudia 1998' (eds. D. Levi and O. Ragnisco), CRM Proceedings and Lecture Notes Series 25, American Mathematical Society (2000) 147-160.

[12] Y. Fedorov, J. Nonlin. Math. Phys. 9, Supplement 1 (2002) 29-46.

[13] A.P. Fordy, Physica D 52 (1991) 204-210.

[14] B. Gambier, Comptes Rendus 142 (1906) 1403-1406.

[15] B. Gambier, Thèse, Paris (1909); Acta Math. **33** (1910) 1–55.

[16] B. Grammaticos, A. Ramani and P. Winternitz, Phys. Lett. A **245** (1998) 382–388.

[17] A.N.W. Hone, Physics Letters A263 (1999) 347-354.

[18] A.N.W. Hone, V.B. Kuznetsov and O. Ragnisco, J. Phys. A 32 (1999) L299-L306.

[19] A.N.W. Hone, V.B. Kuznetsov and O. Ragnisco, *Bäcklund transformations for the Hénon-Heiles and Garnier systems*, in 'Proceedings of the SIDE III meeting, Sabaudia 1998' (eds. D. Levi and O. Ragnisco), CRM Proceedings and Lecture Notes Series 25, American Mathematical Society (2000) 231-235.

[20] A.N.W. Hone, V.B. Kuznetsov and O. Ragnisco, J. Phys. A 34 (2001) 2477-2490.

[21] V.B. Kuznetsov and E.K. Sklyanin, J. Phys. A 31 (1998) 2241-2251.

[22] V.B. Kuznetsov and P. Vanhaecke, J. Geom. Phys. 44 (2002) 1-40.

[23] D. Levi and O. Ragnisco, Inverse Problems 4 (1988) 815-828.

[24] M. Musette and R. Conte, J. Math. Phys. 39 (1998) 5617-5630.

[25] E. Pinney, Proc. Am. Math. Soc. 1 (1950) 681.

[26] A. Ramani, B. Grammaticos and G. Karra, Physica A **180** (1992) 115–127.

[27] C. Rogers, W.K. Schief and P. Winternitz, J. Math. Anal. Appl. 216 (1997) 246-264.

[28] M. Salerno. V.Z. Enolskii and D.V. Leykin, Phys. Rev. E 49 (1994) 5897-5899.

[29] K. Sawada and T. Kotera, Prog. Theor. Phys. **51** (1974) 1355–1367.

[30] W.K. Schief, Appl. Math. Lett. 10 (1997) 13-15.

[31] A.B. Shabat and A.P. Veselov, *Funct. Anal. Appl.* **27:2** (1993) 1-21.

[32] E.K. Sklyanin, Prog. Theor. Phys. Suppl. 118 (1995) 35-60.

[33] Y.B. Suris, Int. Math. Res. Notices 12 (2000) 643-663.

[34] Y.B. Suris, J. Nonlin. Math. Phys. 8 (2001) 534-560.

[35] C. Verhoeven, M. Musette and R. Conte, J. Math. Phys. 43 (2002) 1906-1915.

[36] A.P. Veselov, Russian Math. Surveys 46 (1991) 1-51.

[37] P. Winternitz, *Nonlinear difference equations with superposition formulae*, in 'Symmetries and Integrability of Difference Equations' (eds. P.A. Clarkson and F.W. Nijhoff), LMS Lecture Note Series 255 (1998) 275-283.