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Lax Matrices for Yang-Baxter Maps

Yuri B SURIS [1] *and Alexander P VESELOV* [2]

[1] *Institut für Mathematik, Technische Universität Berlin, Str. des 17. Juni 136, 10623 Berlin, Germany*
E-mail: suris@sfb288.math.tu-berlin.de

[2] *Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, UK and Landau Institute for Theoretical Physics, Moscow, Russia*
E-mail: A.P.Veselov@lboro.ac.uk

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Abstract

It is shown that for a certain class of Yang-Baxter maps (or set-theoretical solutions to the quantum Yang-Baxter equation) the Lax representation can be derived straight from the map itself. A similar phenomenon for 3D consistent equations on quad-graphs has been recently discovered by A. Bobenko and one of the authors, and by F. Nijhoff.

1 Introduction

In 1990 V.G. Drinfeld suggested the problem of studying the solutions of the quantum Yang-Baxter equation in the case when the vector space V is replaced by an arbitrary set X and tensor product by the direct product of the sets ("set-theoretical solutions to the quantum Yang-Baxter equation") [1]. In the paper [2] one of the authors investigated the dynamical aspects of this problem and suggested a shorter term "Yang-Baxter map" for such solutions.

For each Yang-Baxter map one can introduce the hierarchy of commuting transfer-maps which are believed to be integrable (see [2]). In this note we explain how to find Lax representations for a certain class of Yang-Baxter maps thus giving another justification for this conjecture. We were motivated by the explicit examples of the Yang-Baxter maps from [2] and recent results on the equations on quad-graphs, satisfying the so-called "3D consistency condition" [3, 4].

2 Yang-Baxter maps and their Lax representations

Let X be any set and R be a map:

$$R : X \times X \to X \times X.$$

Let $R_{ij} : X^n \to X^n$, $X^n = X \times X \times ..... \times X$ be the map which acts as R on i-th and j-th factors and identically on the others. Let $R_{21} = PRP$, where $P : X^2 \to X^2$ is the permutation: $P(x, y) = (y, x)$.

Following [2], we call R the *Yang-Baxter map* if it satisfies the Yang-Baxter relation

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}, \tag{2.1}$$

considered as the equality of the maps of $X \times X \times X$ into itself. If additionally R satisfies the relation

$$R_{21}R = Id, \tag{2.2}$$

it is called *reversible Yang-Baxter map*. Reversibility condition will not play an essential role in this note but it is satisfied in all the examples we present.

The standard way to represent the Yang-Baxter relation is given by the diagram in Fig. 1. However we would like to use here also an alternative (dual) way to visualize

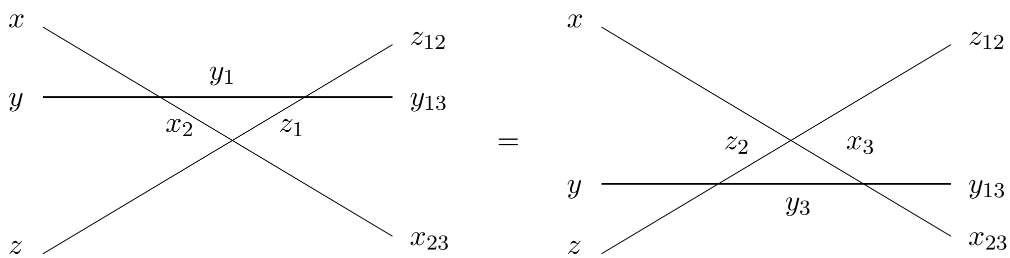

Figure 1. Standard representation of the Yang–Baxter relation

it, which emphasizes the relation with 3D consistency condition for discrete equations on quad-graphs (see [3, 5]). In this representation the fields (elements of X) are assigned to the edges of elementary quadrilaterals, so that Fig.2 encodes the map $R : (x, y) \mapsto (\tilde{x}, \tilde{y})$. Then the Yang-Baxter relation is illustrated as in Fig. 3. It encodes the 3D consistency

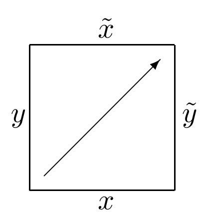

Figure 2. A map associated to an elementary quadrilateral; fields are assigned to edges

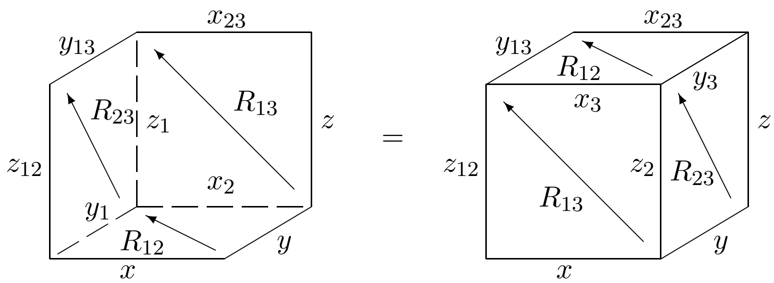

Figure 3. "Cubic" representation of the Yang–Baxter relation

of the maps R attached to all facets of an elementary cube. The left–hand side of (2.1) corresponds to the chain of maps along the three rear faces of the cube on Fig. 3:

$$R_{12}:(x,y)\mapsto(x_2,y_1),\quad R_{13}:(x_2,z)\mapsto(x_{23},z_1),\quad R_{23}:(y_1,z_1)\mapsto(y_{13},z_{12}),$$

while its right–hand side corresponds to the chain of the maps along the three front faces of the cube:

$$R_{23}:(y,z)\mapsto(y_3,z_2),\quad R_{13}:(x,z_2)\mapsto(x_3,z_{12}),\quad R_{12}:(x_3,y_3)\mapsto(x_{23},y_{13}).$$

So, (2.1) assures that two ways of obtaining (x_{23},y_{13},z_{12}) from the initial data (x,y,z) lead to the same results.

One can consider also *parameter-dependent Yang-Baxter maps* $R(\lambda,\mu)$, with $\lambda,\mu\in\mathbf{C}$, satisfying the corresponding version of Yang-Baxter relation

$$R_{23}(\mu,\nu)R_{13}(\lambda,\nu)R_{12}(\lambda,\mu)=R_{12}(\lambda,\mu)R_{13}(\lambda,\nu)R_{23}(\mu,\nu). \tag{2.3}$$

The reversibility condition in this situation reads

$$R_{21}(\mu,\lambda)R(\lambda,\mu)=Id. \tag{2.4}$$

One thinks of the parameters λ,μ as assigned to the same edges of the quadrilateral in Fig. 2 as the fields x,y are. Moreover, opposite edges are thought of as carrying the same parameters. Thus, in Fig. 3 all edges parallel to the x (resp. y,z) axis, carry the parameter λ (resp. μ,ν). Although this can be considered as a particular case of the general notion, by introducing $\tilde{X}=X\times\mathbf{C}$ and $\tilde{R}(x,\lambda;y,\mu)=R(\lambda,\mu)(x,y)$, it is convenient for us to keep the parameter separately.

By the *Lax matrix* (or Lax representation) for such a map we will mean the matrix $A(x,\lambda;\zeta)$ depending on the point $x\in X$, parameter λ and additional ("spectral") parameter $\zeta\in\mathbf{C}$, which satisfies the following relation:

$$A(x,\lambda;\zeta)A(y,\mu;\zeta)=A(\tilde{y},\mu;\zeta)A(\tilde{x},\lambda;\zeta), \tag{2.5}$$

whenever $(\tilde{x},\tilde{y})=R(\lambda,\mu)(x,y)$. As it was shown in [2], such a matrix allows one to produce integrals for the dynamics of the related transfer-maps.

Our main result is the following observation.

Suppose that on the set X we have an action of the linear group $G = GL_N$, and that the Yang-Baxter map $R(\lambda, \mu)$ has the following special form:

$$\tilde{x} = B(y, \mu, \lambda)[x], \quad \tilde{y} = A(x, \lambda, \mu)[y], \tag{2.6}$$

where $A, B : X \times \mathbf{C} \times \mathbf{C} \to GL_N$ are some matrix valued functions on X depending on parameters λ and μ and $A[x]$ denotes the action of $A \in G$ on $x \in X$. Suppose for the beginning that the action of G on X is effective, i.e. A acts identically on X only if $A = I$. Then we claim that both $A(x, \lambda, \zeta)$ and $B^{\mathrm{T}}(x, \lambda, \zeta)$ are Lax matrices for R. The claim about B is equivalent to saying that $B(x, \lambda, \zeta)$ is a Lax matrix for R_{21}.

The following argument is illustrated by either the standard or the "cubic" diagram for the Yang-Baxter relation (Figs. 1,3). Look at the values of z_{12} produced by the both parts of the Yang-Baxter relation (2.3): the left-hand side gives $z_{12} = A(y_1, \mu, \nu)A(x_2, \lambda, \nu)[z]$, while the right-hand side gives $z_{12} = A(x, \lambda, \nu)A(y, \mu, \nu)[z]$. Now since we assume that the action of G is effective, we immediately arrive at the relation

$$A(x, \lambda, \nu)A(y, \mu, \nu) = A(y_1, \mu, \nu)A(x_2, \lambda, \nu),$$

which holds whenever $(x_2, y_1) = R(\lambda, \mu)(x, y)$. This coincides with (2.5), an arbitrary parameter ν playing the role of the spectral parameter ζ.

Similarly, one could look at the values of x_{23} produced by the both parts of (2.3): the left-hand side gives $x_{23} = B(z, \nu, \lambda)B(y, \mu, \lambda)[x]$, while the right-hand side gives $x_{23} = B(y_3, \mu, \lambda)B(z_2, \nu, \lambda)[x]$. Effectiveness of the action of G again implies:

$$B(z, \nu, \lambda)B(y, \mu, \lambda) = B(y_3, \mu, \lambda)B(z_2, \nu, \lambda),$$

whenever $(y_3, z_2) = R(\mu, \nu)(y, z)$. This turns into (2.5) for the transposed matrices B^{T} (or for the inverse matrices B^{-1}); the role of spectral parameter is here played by an arbitrary parameter λ.

It should be mentioned that this kind of arguments was first used to derive Lax representations for 3D consistent discrete equations on quad-graphs with fields on vertices in [3, 4]. In fact the 3D consistency condition is the exact analog of the Yang-Baxter relation for the problems with fields on vertices (see [5]).

In order to cover all the known examples we have to extend the proposed scheme in the following way. Let us say that $A(x, \lambda, \zeta)$ gives a *projective Lax representation* for the Yang-Baxter map R if the relation (2.5) holds up to multiplication by a scalar matrix cI, where c may depend on all the variables in the relation. One can easily modify the arguments from [2] to produce the integrals for the transfer-maps using the projective Lax matrix: all the ratios of the eigenvalues of the monodromy matrix are obviously preserved by these maps.

Assume now that the action of $G = GL_N$ on X is projective, i.e. scalar matrices are acting trivially and moreover if the action of A on X is trivial then A is a scalar. Then our previous considerations show that the matrices $A(x, \lambda, \zeta)$ and $B^{\mathrm{T}}(x, \lambda, \zeta)$ give projective Lax representations for the corresponding Yang-Baxter maps (2.6). In practice for a natural choice of matrices A, B in (2.6) we have actually proper Lax representations, as the following examples show.

3 Example 1: Adler's map

Here $X = \mathbf{CP}^1$ and the map has the form

$$\tilde{x} = y - \frac{\lambda - \mu}{x + y}, \qquad \tilde{y} = x - \frac{\mu - \lambda}{x + y}. \tag{3.1}$$

This map (modulo additional permutation) first appeared in Adler's paper [6] as a symmetry of the periodic dressing chain [7]. The Lax pair for this map was known from the very beginning since it comes from re-factorization problem for the matrix

$$A(x, \lambda, \zeta) = \begin{pmatrix} x & x^2 + \lambda - \zeta \\ 1 & x \end{pmatrix}.$$

Our point is that we can actually see this matrix directly in the map:

$$\tilde{y} = x - \frac{\mu - \lambda}{x + y} = \frac{x^2 + xy - (\mu - \lambda)}{x + y} = A(x, \lambda, \mu)[y],$$

where the group $G = GL_2$ is acting on $\mathbf{CP}^1$ by Möbius transformations. In this example $B(x, \lambda, \zeta) = A(x, \lambda, \zeta)$, which reflects the symmetry of the map: $R_{21} = R$.

4 Example 2: Interaction of matrix solitons

One-soliton solutions of the matrix KdV equation

$$U_t + 3UU_x + 3U_xU + U_{xxx} = 0$$

have the form [8]

$$U = 2\lambda^2 P \operatorname{sech}^2(\lambda x - 4\lambda^3 t),$$

where the matrix amplitude P must be a projector: $P^2 = P$, and λ is the parameter measuring the soliton velocity. If we assume that P has rank 1 then $P = \dfrac{\xi \otimes \eta}{\langle \xi, \eta \rangle}$. Here ξ is a vector in a vector space V of dimension N, η is a (co)vector from the dual space V^*, and bracket $\langle \xi, \eta \rangle$ means the canonical pairing between V and V^*.

The change of the matrix amplitudes P of two solitons with the velocities λ_1 and λ_2 after their interaction is described by the following Yang-Baxter map [8, 9]:

$$R(\lambda_1, \lambda_2) : (\xi_1, \eta_1; \xi_2, \eta_2) \to (\tilde{\xi}_1, \tilde{\eta}_1; \tilde{\xi}_2, \tilde{\eta}_2),$$

$$\tilde{\xi}_1 = \xi_1 + \frac{2\lambda_2 \langle \xi_1, \eta_2 \rangle}{(\lambda_1 - \lambda_2)\langle \xi_2, \eta_2 \rangle} \xi_2, \qquad \tilde{\eta}_1 = \eta_1 + \frac{2\lambda_2 \langle \xi_2, \eta_1 \rangle}{(\lambda_1 - \lambda_2)\langle \xi_2, \eta_2 \rangle} \eta_2, \tag{4.1}$$

$$\tilde{\xi}_2 = \xi_2 + \frac{2\lambda_1 \langle \xi_2, \eta_1 \rangle}{(\lambda_2 - \lambda_1)\langle \xi_1, \eta_1 \rangle} \xi_1, \qquad \tilde{\eta}_2 = \eta_2 + \frac{2\lambda_1 \langle \xi_1, \eta_2 \rangle}{(\lambda_2 - \lambda_1)\langle \xi_1, \eta_1 \rangle} \eta_1. \tag{4.2}$$

In this example X is the set of projectors P of rank 1 which is the variety $\mathbf{CP}^{N-1} \times \mathbf{CP}^{N-1}$, and the group $G = GL_N$ is acting on the projectors by conjugation (which corresponds to the natural action of $GL(V)$ on $V \otimes V^*$).

It is easy to see that the formulas (4.1), (4.2) are of the form (2.6) with the matrices

$$A(P,\lambda,\zeta) = B(P,\lambda,\zeta) = I + \frac{2\lambda}{\zeta-\lambda}P = I + \frac{2\lambda}{\zeta-\lambda}\cdot\frac{\xi\otimes\eta}{\langle\xi,\eta\rangle}$$

(note that again $R_{21} = R$). Our results show that the matrix $A(P,\lambda,\zeta)$ gives a projective Lax representation for the interaction map. In [9] it is shown that this is actually a genuine Lax representation. One can explain in the same way the Lax matrices for more general Yang-Baxter maps on Grassmannians from [9].

5 Example 3: Yang-Baxter maps arising from geometric crystals

Let $X = \mathbf{C}^n$, and define $R : X \times X \to X \times X$ by the formulas [10],[11]

$$\tilde{x}_j = x_j\,\frac{P_j}{P_{j-1}}, \qquad \tilde{y}_j = y_j\,\frac{P_{j-1}}{P_j}, \qquad j = 1,\ldots,n, \tag{5.1}$$

where

$$P_j = \sum_{a=1}^{n}\left(\prod_{k=1}^{a-1} x_{j+k}\prod_{k=a+1}^{n} y_{j+k}\right) \tag{5.2}$$

(in this formula subscripts $j+k$ are taken (mod n)). Clearly, the map (5.1) keeps the following subsets invariant: $X_\lambda \times X_\mu \subset X \times X$, where $X_\lambda = \{(x_1,\ldots,x_n) \in X : \prod_{k=1}^n x_k = \lambda\}$. It can be shown that the restriction of R to $X_\lambda \times X_\mu$ may be written in the form (2.6). For this, the following trick is used. Embed this set into $\mathbf{CP}^{n-1} \times \mathbf{CP}^{n-1}$:

$$J(x,y) = (z(x), w(y)), \quad z(x) = (1 : z_1 : \ldots : z_{n-1}), \quad w(y) = (w_1 : \ldots : w_{n-1} : 1),$$

$$z_j = \prod_{k=1}^{j} x_k\,, \qquad w_j = \prod_{k=j+1}^{n} y_k\,.$$

Then it is easy to see that in coordinates (z, w) the map R is written as

$$\tilde{z} = B(y,\mu,\lambda)[z]\,, \qquad \tilde{w} = A(x,\lambda,\mu)[w]\,,$$

with certain matrices B, A from $G = GL_n$, where the standard projective action of GL_n on $\mathbf{CP}^{n-1}$ is used. Moreover, a simple calculation shows that the inverse matrices are cyclic two-diagonal:

$$B^{-1}(y,\mu,\lambda) = \begin{pmatrix} y_1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & y_2 & -1 & \ldots & 0 & 0 \\ 0 & 0 & y_3 & \ldots & 0 & 0 \\ & \ldots & & & \ldots & \\ 0 & 0 & 0 & \ldots & y_{n-1} & -1 \\ -\lambda & 0 & 0 & \ldots & 0 & y_n \end{pmatrix}, \tag{5.3}$$

$$A^{-1}(x,\lambda,\mu)=\begin{pmatrix} x_1 & 0 & 0 & \dots & 0 & -\mu \\ -1 & x_2 & 0 & \dots & 0 & 0 \\ 0 & -1 & x_3 & \dots & 0 & 0 \\ & \dots & & & \dots & \\ 0 & 0 & 0 & \dots & x_{n-1} & 0 \\ 0 & 0 & 0 & \dots & -1 & x_n \end{pmatrix}. \tag{5.4}$$

To be more precise the matrices A, B are defined only up to multiplication by scalar matrices. These scalar matrices are chosen in (5.3), (5.4) in such a way that the dependence of the matrices B^{-1}, A^{-1} on their "own" parameters (μ and λ, resp.) drops out, so that the only parameter remaining in the Lax representation is the spectral one. In other words, the Lax representation does not depend on the subset $X_\lambda \times X_\mu$ to which we restricted the map. Note also that we get this time only *one* Lax representation for R, since the matrices B^{T} coincide with A. It can be checked that this is actually a genuine (not only projective) Lax representation.

As the last remark we would like to mention that our Lax representation is closely related to the notion of the *structure group* G_R of the Yang-Baxter map R [10]. It was shown by Etingof in [10] that for the map (5.1) the so-called reduced structure groups G_R^+ and G_R^- can be realized as the subgroups of the loop group $PGL_n(C(\lambda))$ generated by the matrix functions $A^{-1}(x,\cdot,\lambda)$ with $x \in X$, resp. by $B^{-1}(x,\cdot,\lambda)$ with $x \in X$.

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