Journal of Nonlinear Mathematical Physics Volume 10, Supplement 2 (2003), 133–142 SIDE V

On a Certain Fractional q-Difference and its Eigen Function

Atsushi NAGAI

Department of Mathematical Sciences, Graduate School of Engineering Science, Osaka University, Toyonaka 560-8531, Japan
E-mail: a-nagai@sigmath.es.osaka-u.ac.jp

This paper is part of the Proceedings of **SIDE V***;*
Giens, June 21-26, 2002

Abstract

A fractional q-difference operator is presented and its properties are investigated. Especially, it is shown that this operator possesses an eigen function, which is regarded as a q-discrete analogue of the Mittag-Leffler function. An integrable nonlinear mapping with fractional q-difference is also presented.

1 Introduction

Fractional derivative goes back to the Leipniz's note in his list to L'Hospital in 1695 and we now have many definitions of fractional derivatives [9]. In the last few decades, many authors pointed out that derivatives and integrals of fractional order, especially 1/2-derivative, are very suitable for the description of physical phenomena.

We first define a fractional integral operator I^a as follows.

Definition 1. *Let a be a nonnegative real number. For a given function $u(t)(t > 0)$, its integral of order a is defined as follows.*

$$I^a u(t) = \int_0^t K(a; t-s) u(s) \mathrm{d}s \tag{1.1}$$

$$I^0 u(t) = u(t) \tag{1.2}$$

where $K(a;t)$ is a monomial given by

$$K(a;t) \equiv \frac{t^{a-1}}{\Gamma(a)} \quad (t > 0,\ a > 0). \tag{1.3}$$

Fractional derivatives of order $a > 0$ are defined by a combination of normal derivative and fractional integral in the following two manners.

Definition 2. *Let m be a positive integer such as $m-1 < a \le m$. Then for a given m times continuously differentiable function $u(t)$, its derivative of order a is defined by*

$$D^a u(t) \equiv (I^{m-a} D^m u)(t) = \int_0^t K(m-a; t-s) u^{(m)}(s) \mathrm{d}s \tag{1.4}$$

Definition 3. *For the same $a, m, u(t)$ in the previous definition, a derivative of order a is defined by*

$$D^a u(t) \equiv (D^m I^{m-a} u)(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_0^t K(m-a; t-s) u(s) \mathrm{d}s. \tag{1.5}$$

These two definitions are called Caputo and Riemann-Liouville fractional derivatives, respectively. We here adopt Caputo's definition 3.

The Mittag-Leffler function,

$$E_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+1)} \quad (a > 0, z \in \mathbb{C}) \tag{1.6}$$

was proposed by Mittag-Leffler [6] in 1903 as an entire function whose order can be calculated exactly. Afterwards, it was clarified that the Mittag-Leffler function also plays an important role in fractional calculus (See refs. [5, 8] for example). In other words, the Mittag-Leffer function,

$$u(t) = \sum_{j=0}^{\infty} \lambda^j K(aj+1; t) = E_a(\lambda t^a) \tag{1.7}$$

is an eigen function of Caputo's fractional derivative [5],

$$D^a u(t) = \lambda u(t) \quad (t > 0). \tag{1.8}$$

In a present paper [7], a discrete analogue of Mittag-Leffler function is presented, together with its relation with a certain fractional difference and a nonlinear integrable mapping with fractional difference has been proposed. The main purpose of this paper is a q-discretization of the above result. In section 2, we present a certain fractional q-difference operator, which is a slight modification of Al-Salam's fractional q-difference operator [1], and investigate its properties. Section 3 is devoted to q-discretization of the Mittag-Leffler function. We also show that q-Mittag-Leffler function serves as an eigen function of the fractional q-difference operator. Finally in section 4, a new type of nonlinear integrable mapping equipped with fractional q-difference is presented.

2 Fractional q-difference

In this section, we present fractional q-addition and q-difference operators and investigate their properties.

Before getting onto the main subject, we first give definitions of q-number, q-binomial coefficient and q-difference operator, together with their properties, which are required in this paper.[1] Let q be a given complex number. Throughout this paper, we impose the assumption,

$$|q| > 1. \tag{2.1}$$

We introduce q-number $[a]_q$ defined by

$$[a]_q = \frac{q^a - 1}{q - 1}, \tag{2.2}$$

we here rewrite $[a]_q$ as $[a]$ for the sake of simplicity. By making use of the q-number, q-binomial coefficient is given as follows.

$$\begin{bmatrix} x \\ n \end{bmatrix} = \frac{[x][x-1]\cdots[x-n+1]}{[n]!} = \frac{[x][x-1]\cdots[x-n+1]}{[n][n-1]\cdots[1]} \tag{2.3}$$

We here list some important properties of q-number and q-binomial coefficient used in future.

$$[-x] = -q^{-x}[x] \tag{2.4}$$

$$\begin{bmatrix} -x \\ n \end{bmatrix} = (-1)^n q^{-nx-\frac{1}{2}n(n-1)} \begin{bmatrix} x+n-1 \\ n \end{bmatrix} \tag{2.5}$$

$$\begin{bmatrix} x \\ n \end{bmatrix} - \begin{bmatrix} x-1 \\ n \end{bmatrix} = q^{x-n} \begin{bmatrix} x-1 \\ n-1 \end{bmatrix} \tag{2.6}$$

$$\begin{bmatrix} x \\ n \end{bmatrix} - \begin{bmatrix} x-1 \\ n-1 \end{bmatrix} = q^{n} \begin{bmatrix} x-1 \\ n \end{bmatrix} \tag{2.7}$$

$$\sum_{k=0}^{n} \begin{bmatrix} x \\ n-k \end{bmatrix} \begin{bmatrix} y \\ k \end{bmatrix} q^{k^2-nk+kx} = \begin{bmatrix} x+y \\ n \end{bmatrix} \tag{2.8}$$

We here adopt backward q-difference operator Δ_q defined by

$$\Delta_q f(x) = \frac{f(x) - f(q^{-1}x)}{(1-q^{-1})x} \tag{2.9}$$

Through dependent and independent variable transformations

$$x = q^n, f(x) = f(q^n) = f_n, \tag{2.10}$$

the q-difference operator in eq. (2.9) is rewritten equivalently as

$$\Delta_q f_n = \frac{f_n - f_{n-1}}{q^n - q^{n-1}}. \tag{2.11}$$

We next introduce a fractional q-addition operator I_q^α defined as follows.

Definition 4. *Let α be a non-negative real number and $\{f_n\}$ is a given complex sequence. Then a q-addition operator of fractional order α for $\{f_n\}$ is defined by*

$$I_q^\alpha f_n = q^{(n-1)\alpha}(q-1)^\alpha \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} f_{n-k} \qquad (\alpha > 0, n \geq 1) \tag{2.12}$$

$$I_q^0 f_n = f_n \qquad (n \geq 1) \tag{2.13}$$

[1] For details of q-analysis, see ref. [2] for example.

Substitution of $\alpha = 1$ into eq. (2.12) gives

$$
\begin{aligned}
I_q f_n &= q^{n-1}(q-1)\sum_{k=0}^{n-1}\begin{bmatrix}-1\\ k\end{bmatrix} q^{\frac{1}{2}k(k-1)} f_{n-k} \\
&= q^{n-1}(q-1)\sum_{k=0}^{n-1}(-1)^k(-1)^k q^{-\frac{1}{2}k(k+1)} q^{\frac{1}{2}k(k-1)} f_{n-k} \\
&= (q-1)\sum_{k=0}^{n-1} q^{n-1-k} f_{n-k} \\
&= (q-1)\sum_{k=1}^{n} q^{k-1} f_k,
\end{aligned}
$$

which is a finite version of Jackson integral. This fractional q-addition operator satisfies the following lemma.

Lemma 1. *Let α, β be non-negative real numbers, a, b be complex numbers and $\{f_n\}, \{g_n\}$ be given complex sequences. Then q-addition operators satisfy the following linearity and commutation rules.*

$$
I_q^{\alpha}(af_n + bg_n) = a(I_q^{\alpha} f_n) + b(I_q^{\alpha} g_n) \tag{2.14}
$$

$$
I_q^{\alpha} I_q^{\beta} f_n = I_q^{\beta} I_q^{\alpha} f_n = I_q^{\alpha+\beta} f_n \tag{2.15}
$$

Proof of Lemma 1. Equation (2.14) is obvious. We prove a commutation rule (2.15) by employing some properties of a q-binomial coefficient.

$$
\begin{aligned}
&I_q^{\alpha} I_q^{\beta} f_n \\
=&q^{(n-1)\alpha}(q-1)^{\alpha}\sum_{k=0}^{n-1}(-1)^k \begin{bmatrix}-\alpha\\ k\end{bmatrix} q^{k(k-1)/2} q^{(n-k-1)\beta}(q-1)^{\beta} \\
&\times \sum_{j=0}^{n-k-1}(-1)^j \begin{bmatrix}-\beta\\ j\end{bmatrix} q^{j(j-1)/2} f_{n-k-j} \\
=&q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{k=0}^{n-1}(-1)^k \begin{bmatrix}-\alpha\\ k\end{bmatrix} q^{k(k-1)/2} q^{-\beta k} \\
&\times \sum_{j=0}^{n-k-1}(-1)^j \begin{bmatrix}-\beta\\ j\end{bmatrix} q^{j(j-1)/2} f_{n-k-j} \\
=&q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{k=0}^{n-1}(-1)^k \begin{bmatrix}-\alpha\\ k\end{bmatrix} q^{k(k-1)/2} q^{-\beta k} \\
&\times \sum_{j=0}^{n-k-1}(-1)^{n-k-1-j} \begin{bmatrix}-\beta\\ n-j-1-k\end{bmatrix} q^{(n-k-1-j)(n-k-2-j)/2} f_{j+1}
\end{aligned}
$$

$$
\begin{aligned}
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{n-j-1}f_{j+1} \\
&\quad\times\sum_{k=0}^{n-j-1}\begin{bmatrix}-\alpha\\ k\end{bmatrix}\begin{bmatrix}-\beta\\ n-j-1-k\end{bmatrix}q^{k(k-1)/2+(n-k-1-j)(n-k-2-j)/2}q^{-\beta k} \\
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{n-j-1}q^{(n-j-1)(n-j-2)/2}f_{j+1} \\
&\quad\times\sum_{k=0}^{n-j-1}\begin{bmatrix}-\alpha\\ k\end{bmatrix}\begin{bmatrix}-\beta\\ n-j-1-k\end{bmatrix}q^{k^2-k(n-j-1)-\beta k} \\
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{n-j-1}q^{(n-j-1)(n-j-2)/2}f_{j+1}\begin{bmatrix}-\alpha-\beta\\ n-j-1\end{bmatrix} \\
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{j}q^{j(j-1)/2}f_{n-j}\begin{bmatrix}-\alpha-\beta\\ j\end{bmatrix} \\
&= I^{\alpha+\beta}f_n,
\end{aligned}
$$

which completes the proof. ■

Next we present a fractional q-difference operator Δ_q^α, which can be regarded as a q-discrete version of Caputo's fractional derivative operator.

Definition 5. *Let α be a positive real number and m be a positive integer which satisfies $m-1<\alpha\le m$. Then a fractional q-difference operator of order $\alpha>0$ is given by*

$$
\begin{aligned}
\Delta_q^\alpha f_n &= I_q^{m-\alpha}\Delta_q^m f_n \\
&= q^{-(n-1)(\alpha-m)}(q-1)^{-(\alpha-m)}\sum_{k=0}^{n-1}(-1)^k\begin{bmatrix}\alpha-m\\ k\end{bmatrix}q^{\frac{1}{2}k(k-1)}\Delta_q^m f_{n-k}
\end{aligned}
\tag{2.16}
$$

Remark 1. *Fractional q-difference operator was first proposed by Al-Salam [1] in 1966. Let $f(x)$ be a given function and $\alpha\in\mathbb{R}\backslash\{1,2,3,\cdots\}$. Then a q-difference operator K_q^α is given by*

$$
K_q^\alpha f(x)=x^{-\alpha}(1-q)^{-\alpha}\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}\alpha\\ k\end{bmatrix}q^{k(k-1)/2-\alpha(\alpha-1)/2}f(xq^{\alpha-k}) \tag{2.17}
$$

Fractional q-difference operator Δ_q^α presented here is a slight modification of Al-Salam's operator K_q^α. The operator K_q^α satisfies the commutative rule,

$$
K_q^\alpha K_q^\beta = K_q^\beta K_q^\alpha = K_q^{\alpha+\beta} \tag{2.18}
$$

for any α,β, whereas the commutation rule for Δ_q^α does not always hold. However, as is mentioned in the next section, the operator Δ_q^α possesses an eigen function, which is regarded as a q-discrete analogue of the Mittag-Leffler function.

3 q-Mittag-Leffler function

This section provides a q-discrete analogue of the Mittag-Leffler function and its relation with the fractional q-difference operator Δ_q^α. We first introduce a fundamental function $M_q(a;n)$ defined by

$$M_q(a;n) = (q-1)^{a-1} \begin{bmatrix} n+a-2 \\ n-1 \end{bmatrix} \quad (a>0, n \in \mathbb{Z}_{\geq 1}). \tag{3.1}$$

$$M_q(a;0) = \begin{cases} 1 & (a=1) \\ 0 & (a \neq 1) \end{cases} \tag{3.2}$$

Remark 2. *In the limit $q \to 1$ and $n \to \infty$ with $t = (q-1)n > 0$ fixed, the above function converges to a monomial,*

$$M_q(a;n) \to K(a;t) = \frac{t^{a-1}}{\Gamma(a)}. \tag{3.3}$$

It is a well-known fact that this function $K(a;t)$ plays an essential role in the theory of fractional derivatives.

The above fundamental function $M_q(a;n)$ satisfies the following two lemmas which states the relation between $M_q(a;n)$ and q-difference (or fractional q-addition) operator.

Lemma 2. *If $a > 0$ and $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\Delta_q M_q(a+1;n) = M_q(a;n). \tag{3.4}$$

Lemma 3. *If $\alpha \geq 0$, $a > 0$ and $n \in \mathbb{Z}_{\geq 1}$, we have*

$$I_q^\alpha M_q(a;n) = M_q(a+\alpha;n). \tag{3.5}$$

Proof of Lemma 2. This is proved essentially by using an addition rule of q-binomial coefficient given by eq. (2.7).

$$\begin{aligned}
\Delta_q M_q(a+1;n) &= \frac{M_q(a+1;n) - M_q(a+1;n-1)}{q^n - q^{n-1}} \\
&= (q-1)^a \left(\begin{bmatrix} n+a-1 \\ n-1 \end{bmatrix} - \begin{bmatrix} n+a-2 \\ n-2 \end{bmatrix} \right) \frac{1}{q^{n-1}(q-1)} \\
&= (q-1)^a q^{n-1} \begin{bmatrix} n+a-2 \\ n-1 \end{bmatrix} \frac{1}{q^{n-1}(q-1)} \\
&= (q-1)^{a-1} \begin{bmatrix} n+a-2 \\ n-1 \end{bmatrix} \\
&= M_q(a;n)
\end{aligned}$$

which completes the proof. ■

Proof of Lemma 3. If $\alpha = 0$, it is obvious. We suppose $\alpha > 0$.

$$\begin{aligned}
I_q^\alpha M_q(a;n) &= q^{(n-1)\alpha}(q-1)^\alpha \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} M_q(a;n-k) \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \begin{bmatrix} n-k+a-2 \\ n-k-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} \cdot \\
&\quad \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)}(-1)^{n-1-k} q^{(n-k-1)a+\frac{1}{2}(n-k-1)(n-k-2)} \begin{bmatrix} -a \\ n-k-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} q^{(n-1)a+\frac{1}{2}(n-1)(n-2)}(-1)^{n-1} \cdot \\
&\quad \sum_{k=0}^{n-1} \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{k^2-(n-1)k+k(-a)} \begin{bmatrix} -a \\ n-k-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} q^{(n-1)a+\frac{1}{2}(n-1)(n-2)}(-1)^{n-1} \begin{bmatrix} -(a+\alpha) \\ n-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} q^{(n-1)a+\frac{1}{2}(n-1)(n-2)} q^{-(n-1)(a+\alpha)-\frac{1}{2}(n-1)(n-2)} \begin{bmatrix} n+a+\alpha-2 \\ n-1 \end{bmatrix} \\
&= (q-1)^{a+\alpha-1} \begin{bmatrix} n+a+\alpha-2 \\ n-1 \end{bmatrix} = M_q(a+\alpha;n),
\end{aligned}$$

where we have employed an upper negation rule (2.5) twice and a Vandermonde convolution rule (2.8). This completes the proof. ■

We next introduce a q-analogue of the Mittag-Leffler function.

Definition 6. *Let a be a positive real number. Then q-Mittag-Leffler function $F_{a,q}(\lambda;n)$ is given by*

$$F_{a,q}(\lambda;n) = \sum_{j=0}^{\infty} \lambda^j M_q(aj+1;n) = \sum_{j=0}^{\infty} \lambda^j (q-1)^{aj} \begin{bmatrix} n+aj-1 \\ n-1 \end{bmatrix} \tag{3.6}$$

It can be verified easily from eq. (3.3) that the above function $F_{a,q}(\lambda;n)$ converges to the Mittag-Leffler function $E_a(\lambda t^a)$ in the limit $q \to 1$ and $n \to \infty$ with $t = (q-1)n$ fixed. The following main theorem states that q-Mittag-Leffler function serves as an eigen function of the fractional q-difference operator Δ_q^a.

Theorem 1. *If $a > 0$ and $n \in \mathbb{Z}_{\geq 1}$, we have*

$$\Delta_q^a F_{a,q}(\lambda;n) = \lambda F_{a,q}(\lambda;n) \tag{3.7}$$

Proof of Theorem 1. Let m be a positive integer such as $m-1 < a \leq m$. Operating

Δ_q^m on $F_{a,q}(\lambda;n)$ and noticing $\Delta_q M_q(1;n) = \Delta_q 1 = 0$, we have from Lemma 2

$$\begin{aligned}\Delta_q^m F_{a,q}(\lambda;n) &= \sum_{j=0}^{\infty} \lambda^j \Delta_q^m M_q(aj+1;n) \\ &= \sum_{j=1}^{\infty} \lambda^j M_q(aj-m+1;n). \end{aligned} \tag{3.8}$$

Operating fractional q-addition operator I_q^{m-a} on both sides of the above equation and employing Lemma 3, we finally obtain

$$\begin{aligned}\Delta_q^a F_{a,q}(\lambda;n) &= I_q^{m-a} \Delta_q^m F_{a,q}(\lambda;n) \\ &= \sum_{j=1}^{\infty} \lambda^j I^{m-a} M_q(aj-m+1;n) \\ &= \sum_{j=1}^{\infty} \lambda^j M_q(aj-a+1;n) \\ &= \sum_{j=0}^{\infty} \lambda^{j+1} M_q(aj+1;n) = \lambda F_{a,q}(\lambda;n), \end{aligned} \tag{3.9}$$

which completes the proof. ■

4 An integrable nonlinear mapping with fractional q-difference

We here give a new type of integrable nonlinear mapping which is equipped with fractional q-difference. We start with a linear mapping,

$$\Delta_q^p g_n = -a g_n, \quad 0 < p \le 1, 0 < a. \tag{4.1}$$

The above equation is rewritten equivalently as

$$(1 + a(q^n - q^{n-1})^p) g_n = g_{n-1} + \sum_{k=1}^{n-1} (-1)^{k-1} \begin{bmatrix} p-1 \\ k \end{bmatrix} q^{k(k+1)/2} (g_{n-k} - g_{n-1-k}) \tag{4.2}$$

Through dependent variable transformation,

$$u_n = \frac{1}{g_n + 1}, \tag{4.3}$$

we obtain the following nonlinear mapping with fractional q-difference.

$$u_n = \frac{1 + a(q^n - q^{n-1})^p}{u_{n-1}^{-1} + a(q^n - q^{n-1})^p + \sum_{k=1}^{n-1} (-1)^{k-1} \begin{bmatrix} p-1 \\ k \end{bmatrix} q^{k(k+1)/2} (u_{n-k}^{-1} - u_{n-1-k}^{-1})}. \tag{4.4}$$

The solution for eq. (4.4) is written as

$$u_n = \frac{u_0}{u_0 + (1 - u_0)F_{p,q}(-a; n)}. \tag{4.5}$$

Putting $p = 1$ in eq. (4.4), we have

$$\frac{u_n - u_{n-1}}{q^n - q^{n-1}} = a u_{n-1}(1 - u_n)$$

which converges to the Riccati equation,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = au(1 - u) \tag{4.6}$$

in the continuum limit $t = q^n, u(t) = u_n$ and $q \to 1$.

The following Figure 1 illustrates the time evolution of the fractional mapping with parameter $p = n/4 (n = 1, 2, 3, 4)$ and $u_0 = 0.2, a = 4, q = 2^{1/10}$.

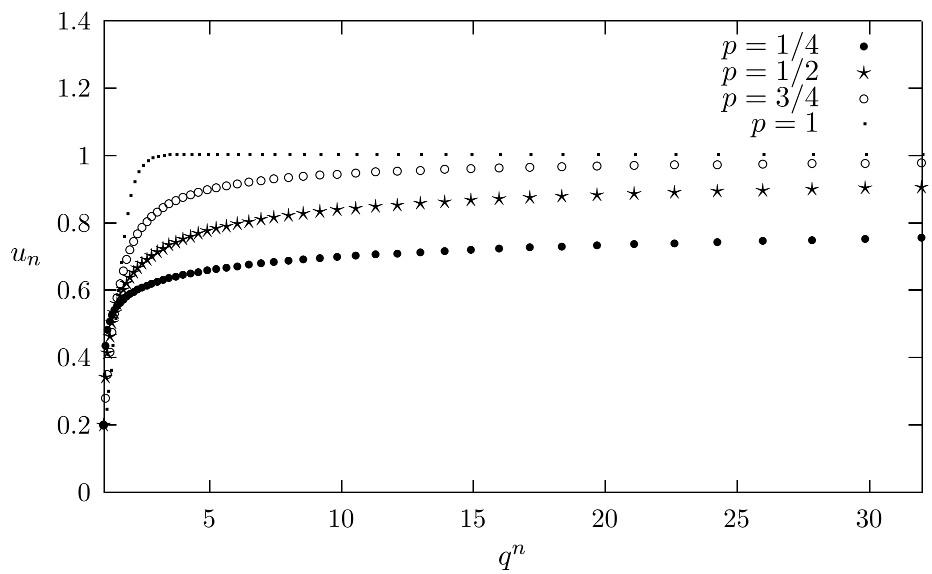

Figure 1. Time evolutions of the fractional mapping (4.4)

5 Concluding Remarks

We have presented one definition of fractional q-difference operator. We have also shown that a q-discrete version of Mittag-Leffer function preserves the property that Mittag-Leffler function is an eigen function of a fractional derivative. It should be noted, however, that the Mittag-Leffler function possesses more abundant properties such as complex-integral expression, asymptotic behavior [10]. It is unknown whether its q-discrete version preserves such properties as well.

It is also an interesting problem to construct nonlinear integrable equations equipped with fractional derivative, difference or q-difference. Although it contains many difficult problems, it is no doubt that the Mittag-Leffler function and its discrete analogues hold the key to this problem.

The author is supported by J.S.P.S. Grant-in-Aid for Scientific Research (C) No. 13640212.

References

[1] W. A. Al-Salam, *Some Fractional q-Integrals and q-Derivatives*, Proc. Edinburgh Math. Soc., **15** (1966) pp. 135–140.

[2] G. E. Andrews, *q-Series : Their Development and Application in Analysis, Number Theory, Combinatorics, Physics and Computer Algebra*, CBMS Regional Conference Lacture **66**, Amer. Math. Soc., Providence, R. I., 1986.

[3] J. B. Diaz and T. J. Osler, *Differences of Fractional Order*, Math. Comp., **28**(1974) pp. 185–202.

[4] R. Hirota, *Sabunhouteishiki Kougi* (Lectures on difference equations) (Science-sha, 2000) p.132 [in Japanese].

[5] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., 1993.

[6] G. M. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$, C. R. Acad. Sci. Paris, **137** (1903) pp. 554–558.

[7] A. Nagai, *An integrable mapping with fractional difference*, J. Phys. Soc. Jpn., **72** (2003) pp. 2181–2183.

[8] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[9] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Science Publishers, 1993.

[10] G. Sansonne and J. Gerretsen, *Lectures on the Theory of Functions of a Complex Variable*, P. Noordhoff-Groningen, 1960.