Hierarchies of Difference Equations and Bäcklund Transformations

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Abstract

In this paper we present a method for deriving infinite sequences of difference equations containing well known discrete Painlevé equations by using the Bäcklund transformations for the equations in the second Painlevé equation hierarchy.

1   Introduction

The six Painlevé equations (P_I–P_{VI}) were discovered by Painlevé, Gambier and their colleagues whilst studying second order ordinary differential equations of the form

$$w'' = F\left(z, w, w'\right), \qquad ' \equiv \mathrm{d}/\mathrm{d}z \tag{1.1}$$

where F is rational in w' and w and analytic in z. The general solutions of the Painlevé equations, called the *Painlevé transcendents*, are transcendental in the sense that they cannot be expressed in terms of known elementary functions and can be thought of as nonlinear analogues of the classical special functions. However, P_{II}–P_{VI} also possess rational solutions and solutions expressible in terms of special functions for certain values of the parameters. Further P_{II}–P_{VI} possess Bäcklund transformations which relate one solution to another such solution either of the same equation, with different values of the parameters, or another such equation (cf. [1, 3, 4, 14, 19, 21, 38, 40, 42] and the references therein).

The discrete Painlevé equations (dP_I–dP_{VI}), which have the form

$$x_{n+1} = \frac{f_1(x_n; n) + x_{n-1} f_2(x_n; n)}{f_3(x_n; n) + x_{n-1} f_4(x_n; n)}, \tag{1.2}$$

where the $f_j(x_n; n)$ are polynomials of degree at most four in x_n, have stimulated interest due to their role as integrable mappings (cf. [17]). In the continuum limit ($nh = \mathcal{O}(1)$

as $h \to 0$), the discrete Painlevé equations yield a Painlevé equation, though some of the discrete equations have limits to more than one Painlevé equation. The discrete Painlevé equations share a variety of other properties in common with the Painlevé equations including Lax pairs, bilinear representations, Bäcklund transformations and exact solutions for certain parameter values, expressible in terms of rational functions or discrete special functions (cf. [9, 17, 18, 28, 35, 41, 43, 46, 50] and the references therein).

The discrete Painlevé equations are in some respects much richer than the Painlevé equations. For example, they have several solutions which have no continuum limit and other properties which are lost in the continuum limit. Perhaps the most fundamental difference is that there is a canonical form for each Painlevé equation which are unique up to a Möbius transformation. However, for each Painlevé equation there is more than one inequivalent discrete equation which has the Painlevé equation as its continuum limit.

There have been several studies of the derivation of discrete Painlevé equations from Bäcklund transformations of the Painlevé equations. Fokas, Grammaticos and Ramani [15] (see also [17, 18, 41]) used an approach which is based on the Schlesinger transformations related to the associated isomonodromy problem for the Painlevé equation. Gromak and Tsegel'nik [22, 48] also derived such difference equations, though these were not identified as discrete Painlevé equations; recent studies include [7, 8, 44].

In this paper we are primarily interested in the derivation of an infinite sequence of systems of discrete equations from Bäcklund transformations of the Painlevé equations and the equations in the P_{II} hierarchy. For completeness, we review the procedure for the derivation of discrete Painlevé equations from Bäcklund transformations of the Painlevé equations in §2. In §3 we generalize this procedure and derive systems of discrete equations from Bäcklund transformations of the equations in the P_{II} hierarchy. In §4 we derive the isomonodromy problems (Lax pairs) for these systems of discrete difference equations, from the isomonodromy problems for the continuous Painlevé equations and in §5 we discuss our results.

2 Deriving discrete equations from Bäcklund transformations for the continuous Painlevé equations

The general procedure for deriving discrete Painlevé equations from Bäcklund transformations for the Painlevé equations is as follows. Suppose there are two Bäcklund transformations for a Painlevé equation in the form

$$w^{\pm}(z; \boldsymbol{\alpha}^{\pm}) = \mathcal{T}^{\pm}(w(z; \boldsymbol{\alpha})) = F^{\pm}(w(z; \boldsymbol{\alpha}), w'(z; \boldsymbol{\alpha}), z, \boldsymbol{\alpha}), \tag{2.1}$$

where $w(z; \boldsymbol{\alpha})$ and $w^{\pm}(z; \boldsymbol{\alpha}^{\pm})$ are solutions of the associated Painlevé equation corresponding to the parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $\boldsymbol{\alpha}^{\pm} = (\alpha_1^{\pm}, \alpha_2^{\pm}, \ldots, \alpha_m^{\pm})$, respectively. Eliminating $w'(z; \boldsymbol{\alpha})$ in (2.1) yields an algebraic relation, which is a recurrence relation, between three solutions $w^+(z; \boldsymbol{\alpha}^+)$, $w(z; \boldsymbol{\alpha})$ and $w^-(z; \boldsymbol{\alpha}^-)$ of the Painlevé equation. This algebraic relation can be thought of as a nonlinear superposition principle for solutions of the Painlevé equation, as an alternate form of the Bäcklund transformations or as a discrete equation of Painlevé type. The continuous variable z enters as a parameter and the independent variable of the discrete equation comes from the parameters $\boldsymbol{\alpha}$.

In order to do this consistently for a given Painlevé equation, we require two Bäcklund transformations $\mathcal{T}^\pm$ such that $\mathcal{T}^+ = (\mathcal{T}^-)^{-1}$, i.e. they are inverse transformations, and so the solutions and parameters are linked as follows

$$\{w^-(z;\boldsymbol{\alpha}^-),\boldsymbol{\alpha}^-\} \underset{\mathcal{T}^-}{\overset{\mathcal{T}^+}{\rightleftarrows}} \{w(z;\boldsymbol{\alpha}),\boldsymbol{\alpha}\} \underset{\mathcal{T}^-}{\overset{\mathcal{T}^+}{\rightleftarrows}} \{w^+(z;\boldsymbol{\alpha}^+),\boldsymbol{\alpha}^+\}$$

Hence setting $w(z;\boldsymbol{\alpha}) = x_n$, $w^\pm(z;\boldsymbol{\alpha}^\pm) = x_{n\pm1}$, with $\boldsymbol{\alpha} = \boldsymbol{a}_n$ and $\boldsymbol{\alpha}^\pm = \boldsymbol{a}_{n\pm1}$, then we obtain the chain

$$\{x_{n-1},\boldsymbol{a}_{n-1}\} \underset{\mathcal{T}^-}{\overset{\mathcal{T}^+}{\rightleftarrows}} \{x_n,\boldsymbol{a}_n\} \underset{\mathcal{T}^-}{\overset{\mathcal{T}^+}{\rightleftarrows}} \{x_{n+1},\boldsymbol{a}_{n+1}\}.$$

We illustrate this procedure using the second Painlevé equation ($\mathrm{P_{II}}$) as an example (see also [15]).

Example 2.1. $\mathrm{P_{II}}$

Suppose that $w(z;\alpha)$ is a solution of $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha, \tag{2.2}$$

where α is a constant. It is well known that (cf. [14, 37])

$$w(z;\alpha+1) = \mathcal{T}^+(w(z;\alpha)) = -w(z;\alpha) - \frac{2\alpha+1}{2w^2(z;\alpha) + 2w'(z;\alpha) + z}, \tag{2.3}$$

$$w(z;\alpha-1) = \mathcal{T}^-(w(z;\alpha)) = -w(z;\alpha) - \frac{2\alpha-1}{2w^2(z;\alpha) - 2w'(z;\alpha) + z}, \tag{2.4}$$

provided that $\alpha \neq -\frac{1}{2}$ in (2.3) and $\alpha \neq \frac{1}{2}$ in (2.4), are also solutions of $\mathrm{P_{II}}$ (2.2) for the parameters $\alpha+1$ and $\alpha-1$, respectively. In the cases $\alpha = j + \frac{1}{2}$, with $j \in \mathbb{Z}$, $\mathrm{P_{II}}$ admits special solutions expressed in terms of Airy functions. Eliminating $w'(z;\alpha)$ from (2.3) and (2.4) yields

$$\frac{2\alpha+1}{w(z;\alpha+1) + w(z;\alpha)} + \frac{2\alpha-1}{w(z;\alpha) + w(z;\alpha-1)} + 4w^2(z;\alpha) + 2z = 0. \tag{2.5}$$

Now if we set

$$a_{n+1} = \alpha + 1, \qquad a_n = \alpha, \qquad a_{n-1} = \alpha - 1, \tag{2.6}$$

then solving these difference equations yields $\alpha = a_n = n + \kappa - \frac{1}{2}$, where κ is an arbitrary constant (the factor $\frac{1}{2}$ is for convenience); since $\mathcal{T}^+$ and $\mathcal{T}^-$ are inverse transformations then (2.6) have a common solution. Next setting $w_m = w(z;a_m)$, in (2.5) gives the equation

$$\frac{n+\kappa}{w_{n+1} + w_n} + \frac{n-1+\kappa}{w_n + w_{n-1}} + 2w_n^2 + z = 0, \tag{2.7}$$

where z is a parameter, which is called alt-$\mathrm{dP_I}$, an alternative form of $\mathrm{dP_I}$ [15]. We note that the difference equation (2.7) appears in [26] — see equation (5.3).

Thus solutions of P_{II} (2.2) satisfy both a differential equation, namely P_{II} itself, and a difference equation, namely alt-dP_I (2.7), which is completely analogous to the situation for Bessel functions (cf. [7, 8]). We remark that in P_{II} (2.2), α is a fixed parameter and z varies whilst in alt-dP_I (2.7), z is fixed and $\alpha = n + \kappa - \frac{1}{2}$ varies.

Example 2.2. P_{II} and P_{34}

In this example we derive a system of discrete equations. Suppose $w(z;\alpha)$ and $q(z;\alpha)$ are respective solutions of P_{II} (2.2) and P_{34}

$$qq'' = \tfrac{1}{2}(q')^2 - 2q^3 - zq^2 - \tfrac{1}{8}(2\alpha + 1)^2, \tag{2.8}$$

with α a parameter. The equation P_{34} (2.8) is so-called since it is equivalent to equation XXXIV of Chapter 14 in Ince [25]. It is well known (cf. [14]) that solutions of P_{II} (2.2) and P_{34} (2.8) are related through the following invertible transformation

$$w(z;\alpha) = \mathcal{B}(q(z;\alpha)) = [2q'(z;\alpha) + 2\alpha + 1]/[4q(z;\alpha)], \tag{2.9}$$

$$q(z;\alpha) = \mathcal{B}^{-1}(w(z;\alpha)) = -w^2(z;\alpha) - w'(z;\alpha) - \tfrac{1}{2}z, \tag{2.10}$$

since eliminating $q(z;\alpha)$ in (2.9,2.10) yields (2.2) and eliminating $w(z;\alpha)$ yields (2.8). The Bäcklund transformation for P_{II} is (2.3) and for P_{34} is

$$q(z;\alpha + 1) = \mathcal{Q}^+(q(z;\alpha)) = -q(z;\alpha) - z - \tfrac{1}{2}\left[(q'(z;\alpha) - \alpha - \tfrac{1}{2})/q(z;\alpha)\right]^2. \tag{2.11}$$

Eliminating $w'(z;\alpha)$ between (2.10) and (2.3), yields

$$w(z;\alpha + 1) = -w(z;\alpha) + (\alpha + \tfrac{1}{2})/q(z;\alpha), \tag{2.12}$$

whilst eliminating $q'(z;\alpha)$ between (2.9) and (2.11) yields

$$q(z;\alpha + 1) = -q(z;\alpha) - z - 2\left[w(z;\alpha) - (\alpha + \tfrac{1}{2})/q(z;\alpha)\right]^2. \tag{2.13}$$

Note that although it appears as though we have only used one Bäcklund transformation, i.e. (2.3), to obtain equation (2.12), in fact, it arises from the composition of two transformations, namely (2.3) and the transformation (2.10), which plays the role of a second Bäcklund transformation. Analogously the transformation (2.9), plays the role of the second Bäcklund transformation in the derivation of (2.13). As in the previous example, solving the difference equations for the parameter relations yields $\alpha = a_n = n + \kappa - \frac{1}{2}$. Thus if we set $w_m = w(z;a_m)$ and $q_m = q(z;a_m)$ in (2.12) and (2.13) then we obtain the discrete system

$$w_{n+1} = -w_n + (n + \kappa)/q_n, \tag{2.14a}$$

$$q_{n+1} = -q_n - z - 2\left[w_n - (n + \kappa)/q_n\right]^2, \tag{2.14b}$$

with z a parameter.

Thus we have generalized the procedure outlined above and obtained the discrete system (2.14). This system includes the discrete equations arising from the Bäcklund transformations of P_{II} and P_{34}, i.e. (2.7) and (2.15), respectively.

Now consider the discrete system (2.14). Eliminating q_n yields alt-dP$_\text{I}$ (2.7), whilst eliminating w_n yields the second-order second-degree discrete equation

$$(q_{n+1} - q_{n-1})^2 q_n^4 + 4(n + \kappa)^2 (q_{n+1} + 2q_n + q_{n-1} + 2z) q_n^2 + 4(n + \kappa)^4 = 0. \qquad (2.15)$$

It is straightforward to generate this discrete equation from the Bäcklund transformation (2.11) of P$_{34}$ together with the associated inverse transformation

$$\mathcal{Q}^-(q(z;\alpha)) = q(z;\alpha - 1) = -q(z;\alpha) - \tfrac{1}{2}\left\{[q'(z;\alpha) + \alpha + \tfrac{1}{2}]/q(z;\alpha)\right\}^2.$$

which is easily obtained from the Bäcklund transformation (2.11) and the discrete symmetry $q(z; -\alpha - \tfrac{1}{2}) = q(z; \alpha + \tfrac{1}{2})$. Note that the second-order second-degree discrete equation (2.15) appears not to be integrable, since the latest point, q_{n+1}, on an orbit is multivalued as a function of the earlier points q_n and q_{n-1}. However, it is a consequence of equations (2.14) which do provide single-valued definitions of the pair (w_{n+1}, q_{n+1}). Equations (2.14) form an integrable system because their Lax pairs can be deduced from Schlesinger transformations of the Lax pairs for the corresponding ordinary differential equations (see §4.)

3 Discrete systems associated with the P$_\text{II}$ hierarchy

3.1 The P$_\text{II}$ hierarchy

The Korteweg-de Vries (KdV) hierarchy can be written as

$$u_{t_{2n+1}} + \partial_x \mathcal{L}_{n+1}[u] = 0, \qquad n = 0, 1, 2, \ldots, \qquad (3.1)$$

where $\partial_x = \partial/\partial x$, and the sequence $\mathcal{L}_n$ satisfies the Lenard recursion relation [34]

$$\partial_x \mathcal{L}_{n+1} = \left(\partial_x^3 + 4u\partial_x + 2u_x\right)\mathcal{L}_n.$$

Beginning with $\mathcal{L}_0[u] = \tfrac{1}{2}$, this then gives

$$\mathcal{L}_1[u] = u, \qquad \mathcal{L}_2[u] = u_{xx} + 3u^2, \qquad \mathcal{L}_3[u] = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3,$$

and so on. The modified Korteweg-de Vries (mKdV) hierarchy is obtained from the KdV hierarchy (3.1) via the Miura map $u = v_x - v^2$, and can be written as

$$v_{t_{2n+1}} + \partial_x \left(\partial_x + 2v\right)\mathcal{L}_n\left[v_x - v^2\right] = 0, \qquad n = 0, 1, 2, \ldots \qquad (3.2)$$

A P$_\text{II}$ hierarchy [3, 13] is obtained from this equation via the similarity reduction

$$v(x, t_{2n+1}) = \frac{w(z)}{[(2n + 1)t_{2n+1}]^{1/(2n+1)}}, \qquad z = \frac{x}{[(2n + 1)t_{2n+1}]^{1/(2n+1)}},$$

which gives the $2n^\text{th}$ order equation P$_\text{II}^{[n]}$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} + 2w\right)\mathcal{L}_n\left[w' - w^2\right] - wz - \alpha = 0, \qquad n = 1, 2, 3, \ldots \qquad (3.3)$$

where we have excluded the trivial case $n = 0$ from consideration. We note that the P$_\text{II}$ hierarchy has the discrete symmetry $(w, \alpha) \to (-w, -\alpha)$, inherited from the associated discrete symmetry $v \to -v$ of the mKdV hierarchy.

- Since $\mathcal{L}_1[u] = u$ the first member of the P_{II} hierarchy is P_{II} (2.2)

- Since $\mathcal{L}_2[u] = u_{xx} + 3u^2$, the second member of the P_{II} hierarchy is the fourth order equation $(P_{II}^{[2]})$

$$w'''' = 10w^2 w'' + 10w \left(w'\right)^2 - 6w^5 + zw + \alpha. \tag{3.4}$$

- Since $\mathcal{L}_3[u] = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3$, the third member of the P_{II} hierarchy is the sixth order equation $(P_{II}^{[3]})$

$$w'''''' = 14w^2 w'''' + 56ww'w''' + 42w \left(w''\right)^2 - 70 \left[w^4 - \left(w'\right)^2\right] w''$$
$$-140w^3 \left(w'\right)^2 + 20w^7 + zw + \alpha. \tag{3.5}$$

We remark that there is much current interest in the properties of solutions of the P_{II} hierarchy – see, for example, [5, 27, 29, 30, 31, 32, 36, 39].

3.2   Bäcklund transformations for the P_{II} hierarchy

The Bäcklund transformation for the P_{II} hierarchy is

$$\mathcal{T}_n^\pm(w_\alpha) = w_{\alpha\pm1} = -w_\alpha + \frac{2\alpha \pm 1}{2\mathcal{L}_n\left[\mp w_\alpha' - w_\alpha^2\right] - z}, \tag{3.6}$$

where $w_\alpha \equiv w(z; \alpha)$ and $w_{\alpha\pm1} \equiv w(z; w_{\alpha\pm1})$ (see, for example, [3, 6] for derivation and further details).

- For $n = 1$, (3.6) gives the Bäcklund transformations of P_{II} given by (2.3) and (2.4)

- For $n = 2$, (3.6) gives the Bäcklund transformations of $P_{II}^{[2]}$

$$\mathcal{T}_2^\pm(w_\alpha) = w_{\alpha\pm1} = -w_\alpha \mp \frac{2\alpha \pm 1}{2w_\alpha''' \pm 4w_\alpha w_\alpha'' \mp 2(w_\alpha')^2 - 12w_\alpha^2 w_\alpha' \mp 6w_\alpha^4 \pm z}, \tag{3.7}$$

which appears in [19, 24].

- For $n = 3$, (3.6) gives the Bäcklund transformations of $P_{II}^{[3]}$

$$\mathcal{T}_3^\pm(w_\alpha) = w_{\alpha\pm1} = -w_\alpha \mp (2\alpha \pm 1)/F_3^\pm, \tag{3.8}$$

where

$$F_3^\pm \equiv 2w_\alpha''''' \mp 4w_\alpha w_\alpha'''' - (20w_\alpha^2 \mp 4w_\alpha')w_\alpha''' \pm 2(w_\alpha'')^2 - 40w_\alpha(2w_\alpha' \pm w_\alpha^3)w_\alpha''$$
$$- 20(w_\alpha')^3 \mp 20w_\alpha^2(w_\alpha')^2 + 60w_\alpha^4 w_\alpha' \pm 20w_\alpha^6 \pm z.$$

3.3   Derivation of a hierarchy of discrete systems

Here we derive a hierarchy of discrete systems associated with the P_{II} hierarchy (3.3) from the Bäcklund transformation (3.6). The basic idea is to let $w_\alpha = w(z; \alpha)$, $p_\alpha = w'_\alpha$, $q_\alpha = w''_\alpha$, $r_\alpha = w'''_\alpha$ and so on, in (3.6) so that the Bäcklund transformation $\mathcal{T}_n^+$ has the form

$$w_{\alpha+1} + w_\alpha = \frac{2\alpha + 1}{2\mathcal{L}_n\left[-p_\alpha - w_\alpha^2\right] - z} \equiv \Phi_n(\alpha, w_\alpha, p_\alpha, q_\alpha, r_\alpha, \dots), \tag{3.9}$$

where $w_{\alpha+1} \equiv w(z; \alpha + 1)$. Then successively differentiating this yields

$$p_{\alpha+1} + p_\alpha = \left.\frac{\mathrm{d}\Phi_n}{\mathrm{d}z}\right|_{P_{II}^{[n]}}, \qquad q_{\alpha+1} + q_\alpha = \left.\frac{\mathrm{d}^2\Phi_n}{\mathrm{d}z^2}\right|_{P_{II}^{[n]}}, \qquad r_{\alpha+1} + r_\alpha = \left.\frac{\mathrm{d}^3\Phi_n}{\mathrm{d}z^3}\right|_{P_{II}^{[n]}},$$

and so on. Thus we obtain a discrete system of the form

$$\boldsymbol{w}_{\alpha+1} + \boldsymbol{w}_\alpha = \boldsymbol{f}(\boldsymbol{w}_\alpha), \tag{3.10}$$

where $\boldsymbol{w}_\alpha = (w_\alpha, p_\alpha, q_\alpha, r_\alpha, \dots)$ and $\boldsymbol{f} = (f_1, f_2, \dots, f_n)$, with $f_m = \left.\mathrm{d}^{m-1}\Phi_n/\mathrm{d}z^{m-1}\right|_{P_{II}^{[n]}}$.

   We now illustrate this procedure using the first three members of the P_{II} hierarchy.

Example 3.1. Suppose that $w \equiv w(z; \alpha)$ is a solution of P_{II}. Then $w(z; \alpha + 1)$ defined by (2.3) is also a solution of P_{II}. Letting $p(z; \alpha) = w'(z; \alpha)$ yields

$$w(z; \alpha + 1) = -w(z; \alpha) - \frac{2\alpha + 1}{2w^2(z; \alpha) + 2p(z; \alpha) + z}, \tag{3.11}$$

and so differentiating this gives

$$p(z; \alpha+1) = -p(z; \alpha) + \frac{2(2\alpha + 1)w(z; \alpha)}{2w^2(z; \alpha) + 2p(z; \alpha) + z} + \left(\frac{2\alpha + 1}{2w^2(z; \alpha) + 2p(z; \alpha) + z}\right)^2. \tag{3.12}$$

Thus setting $w_\beta = w(z; \beta)$ and $p_\beta = p(z; \beta)$ yields the discrete system

$$w_{\alpha+1} + w_\alpha = \Phi_1, \tag{3.13a}$$
$$p_{\alpha+1} + p_\alpha = \Phi_1^2 - 2w_\alpha \Phi_1, \tag{3.13b}$$

where

$$\Phi_1(w_\alpha, p_\alpha, z) = -\frac{2\alpha + 1}{2w_\alpha^2 + 2p_\alpha + z}. \tag{3.14}$$

Example 3.2. The second member of the P_{II} hierarchy, i.e. $P_{II}^{[2]}$, is the fourth order equation (3.4). Thus letting $w_\alpha = w(z; \alpha)$, $p_\alpha = w'_\alpha$, $q_\alpha = w''_\alpha$ and $r_\alpha = w'''_\alpha$, then the Bäcklund transformation (3.7) becomes

$$w_{\alpha+1} + w_\alpha = \frac{2\alpha + 1}{6w_\alpha^4 + 12w_\alpha^2 p_\alpha - 4w_\alpha q_\alpha + 2p_\alpha^2 - 2r_\alpha - z}. \tag{3.15}$$

Differentiating this successively gives the fourth-order discrete system

$$w_{\alpha+1} + w_\alpha = \Phi_2, \tag{3.16a}$$

$$p_{\alpha+1} + p_\alpha = \Phi_2^2 - 2w_\alpha \Phi_2, \tag{3.16b}$$

$$q_{\alpha+1} + q_\alpha = 2\Phi_2^3 - 6w_\alpha \Phi_2^2 + (2w_\alpha^2 - p_\alpha)\Phi_2, \tag{3.16c}$$

$$r_{\alpha+1} + r_\alpha = 6\Phi_2^4 - 24w_\alpha \Phi_2^3 + 4(7w_\alpha^2 - 2p_\alpha)\Phi_2^2 - 2(4w_\alpha^3 - 6w_\alpha p_\alpha + q_\alpha)\Phi_2, \tag{3.16d}$$

where

$$\Phi_2(w_\alpha, p_\alpha, q_\alpha, r_\alpha, z) \equiv \frac{2\alpha + 1}{6w_\alpha^4 + 12w_\alpha^2 p_\alpha - 4w_\alpha q_\alpha + 2p_\alpha^2 - 2r_\alpha - z}. \tag{3.17}$$

Example 3.3. The third member of the P_{II} hierarchy, i.e. $P_{II}^{[3]}$, is the sixth order equation (3.5) associated Bäcklund transformation (3.8). Thus letting $w_\alpha = w(z; \alpha)$, $p_\alpha = w_\alpha'$, $q_\alpha = w_\alpha''$, $r_\alpha = w_\alpha'''$, $s_\alpha = w_\alpha''''$ and $t_\alpha = w_\alpha'''''$ then the Bäcklund transformation (3.8) becomes

$$w_{\alpha+1} + w_\alpha = \Phi_3(w_\alpha, p_\alpha, q_\alpha, r_\alpha, s_\alpha, t_\alpha, z) \equiv -(2\alpha + 1)/F_3, \tag{3.18a}$$

where

$$F_3 \equiv 20w_\alpha^6 + 60p_\alpha w_\alpha^4 - 40w_\alpha^3 q_\alpha - 20(p_\alpha^2 + r_\alpha)w_\alpha^2 + 4(s_\alpha - 20p_\alpha q_\alpha)w_\alpha$$

$$-20p_\alpha^3 - 4p_\alpha r_\alpha + 2q_\alpha^2 + 2t_\alpha + z. \tag{3.18b}$$

Differentiating this successively gives the sixth-order discrete system

$$w_{\alpha+1} + w_\alpha = \Phi_3 \tag{3.19a}$$

$$p_{\alpha+1} + p_\alpha = \Phi_3^2 - 2w_\alpha \Phi_3 \tag{3.19b}$$

$$q_{\alpha+1} + q_\alpha = 2\Phi_3^3 - 6w_\alpha \Phi_3^2 + (2w_\alpha^2 - p_\alpha)\Phi_3 \tag{3.19c}$$

$$r_{\alpha+1} + r_\alpha = 6\Phi_3^4 - 24w_\alpha \Phi_3^3 + 4(7w_\alpha^2 - 2p_\alpha)\Phi_3^2 - 2(4w_\alpha^3 - 6w_\alpha p_\alpha + q_\alpha)\Phi_3 \tag{3.19d}$$

$$s_{\alpha+1} + s_\alpha = 24\Phi_3^5 - 120w_\alpha \Phi_3^4 + 40(5w_\alpha^2 - p_\alpha)\Phi_3^3 - 10(12w_\alpha^3 - 10w_\alpha p_\alpha + q_\alpha)\Phi_3^2$$

$$+2(8w_\alpha^4 - 24w_\alpha^2 p_\alpha + 8w_\alpha q_\alpha + 6p_\alpha^2 - r_\alpha)\Phi_3 \tag{3.19e}$$

$$t_{\alpha+1} + t_\alpha = 120\Phi_3^6 - 720w_\alpha \Phi_3^5 + 120(13w_\alpha^2 - 2p_\alpha)\Phi_3^4 - 60(24w_\alpha^3 - 14w_\alpha p_\alpha + q_\alpha)\Phi_3^3$$

$$+4(124w_\alpha^4 - 202w_\alpha^2 p_\alpha + 39w_\alpha q_\alpha + 28p_\alpha^2 - 3r_\alpha)\Phi_3^2$$

$$-2\left[16w_\alpha^5 - 80p_\alpha w_\alpha^3 + 40q_\alpha w_\alpha^2 + 10(6p_\alpha^2 - r_\alpha)w_\alpha - 20p_\alpha q_\alpha + s_\alpha\right]\Phi_3. \tag{3.19f}$$

In these examples we note that the similarity in the expressions for w_{a+1}, $p_{\alpha+1}$, $q_{\alpha+1}$, etc. in terms of the quantity Φ_n. This is clarified through the following theorem.

Theorem 3.4. *When restricted to solutions of $P_{II}^{[n]}$, Φ_n as defined by (3.9), satisfies the Riccati equation*

$$\Phi_n' = \Phi_n^2 - 2w_\alpha \Phi_n. \tag{3.20}$$

where $w_\alpha \equiv w(z; \alpha)$ is a solution of $P_{II}^{[n]}$ (3.3).

Proof. From the definition of Φ_n through equation (3.9) and using the fact that $w'_\alpha = p_\alpha$

$$
\begin{aligned}
\Phi'_n &= -\frac{2\alpha + 1}{\{2\mathcal{L}_n\left[-w'_\alpha - w^2_\alpha\right] - z\}^2}\left\{2\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{L}_n\left[-w'_\alpha - w^2_\alpha\right]\right) - 1\right\} \\
&= -\frac{2\alpha + 1}{\{2\mathcal{L}_n\left[-w'_\alpha - w^2_\alpha\right] - z\}^2}\left\{2w_\alpha\left(2\mathcal{L}_n\left[-w'_\alpha - w^2_\alpha\right] - z\right) - (2\alpha + 1)\right\} \\
&= \Phi^2_n - 2w_\alpha\Phi_n,
\end{aligned}
$$

as required, since from equation (3.3) and the discrete symmetry $(w_\alpha, \alpha) \to (-w_{-\alpha}, -\alpha)$

$$
\frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{L}_n\left[-w'_\alpha - w^2_\alpha\right]\right) = 2w_\alpha\mathcal{L}_n\left[-w'_\alpha - w^2_\alpha\right] - zw_\alpha - \alpha.
$$

Successively differentiating (3.20) gives

$$
\Phi''_n = 2\Phi^3_n - 6w_\alpha\Phi^2_n + (4w^2_\alpha - p_\alpha)\Phi_n, \tag{3.21}
$$

$$
\Phi'''_n = 6\Phi^4_n - 24w_\alpha\Phi^3_n + (28w^2_\alpha - 8p_\alpha)\Phi^2_n - (8w^3_\alpha - 12w_\alpha p_\alpha + 2q_\alpha)\Phi_n, \tag{3.22}
$$

and so on, where $p_\alpha = w'_\alpha$ and $q_\alpha = w''_\alpha$. We remark that setting $\Phi_n(z) = -\varphi'_n(z)/\varphi_n(z)$ in (3.20) yields the linear homogeneous equation

$$
\varphi''_n + 2w_\alpha\varphi'_n = 0, \tag{3.23}
$$

where $w_\alpha = w(z; \alpha)$ is a solution of $\mathrm{P}^{[n]}_{\mathrm{II}}$ (3.3). Also setting $\Phi_n(z) = w_\alpha - \psi'_n(z)/\psi_n(z)$ in (3.20) yields the linear homogeneous Schrödinger equation

$$
\psi''_n = [w'_\alpha + w^2_\alpha]\psi_n = -V_\alpha\psi_n, \tag{3.24}
$$

where V_α satisfies (4.1) below, which is the nth equation in the P_{34} hierarchy. Further setting $\Phi_n(z) = 1/\Psi_n(z)$ in (3.20) yields the Bernoulli-type equation

$$
\Psi'_n = 2w_\alpha\Psi_n - 1. \tag{3.25}
$$

3.4 Confinement

Consider the difference equation

$$
\boldsymbol{x}_{n+1} = \boldsymbol{H}\left(\boldsymbol{x}_n\right) \tag{3.26}
$$

with $\boldsymbol{x}_n \in \mathbb{C}^{k+1}$, where $\boldsymbol{H}$ is rational in its arguments. The following definition of singularity confinement is the one given in [10, 11].

Definition 3.5. Equation (3.26) is called *admissible* if $\boldsymbol{H}$ has some movable singular point $\boldsymbol{x}_n \in \mathbb{C}^{k+1}$. An admissible equation is said to have the *singularity confinement property* if the following conditions hold.

1. The maps $\boldsymbol{x}_n \mapsto \boldsymbol{x}_{n+1}$ and $\boldsymbol{x}_n \mapsto \boldsymbol{x}_{n-1}$ induced by equation (3.26) are both well posed, i.e. unique and a continuous function of the pre-image, at the ordinary points of $\boldsymbol{H}$ in $\mathbb{C}^{k+1}$.

2. If the movable singular point of $\boldsymbol{H}$ is isolated.

It is proved in [10, 11] that if a difference equation arises as a consistent composition of Bäcklund transformations of an ordinary differential equation with the Painlevé property, then it must have the singularity confinement property. This leads to the following result.

Theorem 3.6. *For each integer $n \geq 1$, the system (3.10) has the singularity confinement property.*

4   Isomonodromy problems

Here we obtain the isomonodromy problems for the discrete difference equations, from the isomonodromy problems for the (continuous) Painlevé equations. Making the similarity reduction

$$u(x, t_{2n+1}) = \frac{V(z)}{[(2n+1)t_{2n+1}]^{2/(2n+1)}}, \qquad z = \frac{x}{[(2n+1)t_{2n+1}]^{1/(2n+1)}},$$

in the KdV hierarchy (3.1), and integrating once yields

$$(2\mathcal{L}_n[V_\alpha] - z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\mathcal{L}_n[V_\alpha]) - \left[\frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}_n[V_\alpha])\right]^2 + \frac{\mathrm{d}}{\mathrm{d}z}(\mathcal{L}_n[V_\alpha])$$

$$+ (2\mathcal{L}_n[V_\alpha] - z)^2 V_\alpha - (\alpha + \tfrac{1}{2})^2 = 0. \tag{4.1}$$

which is known as the P_{34} hierarchy (see [6, 24] for further details).

Using the method of Flaschka and Newell [13] it is straightforward to derive the isomonodromy problems for each member of the P_{34} hierarchy (4.1), through a similarity reduction of the Lax pair for the nth member in the KdV hierarchy. The nth term in the sequence of isomonodromy problems has the form

$$\frac{\partial \boldsymbol{\Psi}_\alpha}{\partial z} = \begin{pmatrix} 0 & 1 \\ \zeta - V_\alpha & 0 \end{pmatrix} \boldsymbol{\Psi}_\alpha, \qquad \zeta\frac{\partial \boldsymbol{\Psi}_\alpha}{\partial \zeta} = \mathbf{G}_n[V_\alpha; \zeta]\,\boldsymbol{\Psi}_\alpha. \tag{4.2}$$

For each n, the matrix $\mathbf{G}_n$ is a polynomial of degree $n + 1$ in the spectral parameter ζ whose coefficients are polynomials in V_α and its derivatives. Written as a scalar equation, the z part of (4.2) becomes a Schrödinger equation with potential V_α and eigenvalue ζ.

Similarly, applying the Flaschka-Newell method to the Lax pair of each equation in the mKdV hierarchy the isomonodromy problems for the P_{II} hierarchy (3.3) given by

$$\frac{\partial \boldsymbol{\Xi}_\alpha}{\partial z} = \begin{pmatrix} -\lambda & w_\alpha \\ w_\alpha & \lambda \end{pmatrix} \boldsymbol{\Xi}_\alpha, \qquad \lambda\frac{\partial \boldsymbol{\Xi}_\alpha}{\partial \lambda} = \mathbf{J}_n[w_\alpha, \alpha; \lambda]\,\boldsymbol{\Xi}_\alpha. \tag{4.3}$$

The matrix $\mathbf{J}_n$ is a polynomial in w_α and its derivatives, and a polynomial in λ of degree $2n + 1$ [23]; details for the cases $n = 1$ and $n = 2$ are given below. We note that for each n, the z evolution is an isomonodromic deformation of the second equation in (4.3), which has a regular singular point at $\lambda = 0$ and an irregular singular point at $\lambda = \infty$ whose order increases with n.

The Lax pairs (4.2) and (4.3) are related by the gauge transformation

$$\boldsymbol{\Xi}_\alpha = \mathbf{g}(w_\alpha; \lambda)\, \boldsymbol{\Psi}_\alpha, \qquad \mathbf{g}(w_\alpha; \lambda) = \frac{1}{\sqrt{2\lambda}} \begin{pmatrix} -w_\alpha - \lambda & 1 \\ w_\alpha - \lambda & -1 \end{pmatrix}, \tag{4.4}$$

with $\zeta = \lambda^2$, which is the analogue of the Miura map that relates the mKdV and KdV linear problems. The dependent variables are related by the correspondence

$$V_\alpha = -w'_\alpha - w_\alpha^2, \qquad w_\alpha = \frac{1}{2\mathcal{L}_n\,[V_\alpha] - z} \left\{ \frac{\mathrm{d}}{\mathrm{d}z}\left(\mathcal{L}_n\,[V_\alpha] - \tfrac{1}{2}z\right) + \alpha + \tfrac{1}{2} \right\}. \tag{4.5}$$

In the framework of the first linear problem (4.2), the Bäcklund transformation for the P_{II} hierarchy (3.6) is equivalent to a Darboux transformation for the corresponding Schrödinger spectral problem. More precisely, V_α is given in terms of two solutions at adjacent parameter values by the formula

$$V_\alpha = -w'_\alpha - w_\alpha^2 = w'_{\alpha+1} - w_{\alpha+1}^2. \tag{4.6}$$

Applying the Bäcklund transformation to the solution V_α of the P_{34} hierarchy is equivalent to the Darboux transformation that produces a new potential

$$V_{\alpha+1} = -w'_{\alpha+1} - w_{\alpha+1}^2. \tag{4.7}$$

At the level of the z part of the linear problem (4.2) $\boldsymbol{\Psi}_\alpha$ is mapped as

$$\boldsymbol{\Psi}_{\alpha+1} = \mathbf{M}\,\boldsymbol{\Psi}_\alpha, \qquad \mathbf{M} = \kappa(\zeta) \begin{pmatrix} w_{\alpha+1} & 1 \\ w_{\alpha+1}^2 + \zeta & w_{\alpha+1} \end{pmatrix}. \tag{4.8}$$

The Darboux matrix $\mathbf{M}$ is a central object in the theory of the dressing chain for Schrödinger operators [2, 45], and κ must be independent of z. Under this transformation, the new eigenfunction satisfies

$$\frac{\partial \boldsymbol{\Psi}_{\alpha+1}}{\partial z} = \begin{pmatrix} 0 & 1 \\ \zeta - V_{\alpha+1} & 0 \end{pmatrix} \boldsymbol{\Psi}_{\alpha+1}, \qquad \zeta \frac{\partial \boldsymbol{\Psi}_{\alpha+1}}{\partial \zeta} = \mathbf{G}_n[V_{\alpha+1}, \alpha + 1; \zeta]\, \boldsymbol{\Psi}_{\alpha+1}. \tag{4.9}$$

To obtain the correct transformation of the ζ part of the Lax pair (4.2) it turns out that the determinant of the Darboux matrix $\mathbf{M}$ must also be independent of ζ, so that $\kappa(\zeta) = \pm i/\sqrt{\zeta}$ for $\mathbf{M} \in SL(2)$.

It is now straightforward to obtain the discrete Lax pair for the sequence of difference equations derived from the Bäcklund transformation of each member in the P_{II} hierarchy. The gauge transformation

$$\boldsymbol{\Xi}_{\alpha+1} = \mathbf{g}(w_{\alpha+1}; \lambda)\, \boldsymbol{\Psi}_{\alpha+1}, \tag{4.10}$$

relates the eigenfunction in (4.9) to the eigenfunction $\boldsymbol{\Xi}_{\alpha+1}$ in the Lax pair (4.3) for P_{II} with w_α replaced by $w_{\alpha+1}$. Combining (4.4) and (4.8) with the gauge (4.10) gives the transformation between the eigenfunctions in the P_{II} Lax pair which takes the same form for any n, i.e.

$$\boldsymbol{\Xi}_{\alpha+1} = \mathbf{N}\,\boldsymbol{\Xi}_\alpha, \tag{4.11a}$$

where

$$\mathbf{N} \equiv \mathbf{g}(w_{\alpha+1}; \lambda)\, \mathbf{M}\, \mathbf{g}^{-1}(w_\alpha; \lambda) = \frac{\kappa}{2} \begin{pmatrix} w_{\alpha+1} + w_\alpha - 2\lambda & w_{\alpha+1} + w_\alpha \\ w_{\alpha+1} + w_\alpha & w_{\alpha+1} + w_\alpha + 2\lambda \end{pmatrix}. \quad (4.11b)$$

Then the discrete isomonodromic Lax pair for the nth system of difference equations is just

$$\Xi_{\alpha+1} = \mathbf{N}_n\, \Xi_\alpha, \qquad \lambda \frac{\partial \Xi_\alpha}{\partial \lambda} = \mathbf{J}_n[w_\alpha, \alpha; \lambda]\, \Xi_\alpha, \quad (4.12a)$$

where

$$\mathbf{N}_n = \frac{\mathrm{i}}{2\lambda} \begin{pmatrix} \Phi_n - 2\lambda & \Phi_n \\ \Phi_n & \Phi_n + 2\lambda \end{pmatrix} \quad (4.12b)$$

and $\mathbf{J}_n[w_\alpha, \alpha; \lambda]$ is as in (4.3) above, with $\kappa = \mathrm{i}/\lambda$ so that $\mathbf{N}_n \in \mathrm{SL}(2)$.

For each n, the system of discrete difference equations in the variables $w = w_\alpha, p = p_\alpha, q = q_\alpha, r = r_\alpha, \ldots$ is then equivalent to the compatibility condition

$$\frac{\partial \mathbf{N}_n}{\partial \lambda} + \mathbf{N}_n\, \mathbf{J}_n[w_\alpha, \alpha; \lambda] - \mathbf{J}_n[w_{\alpha+1}, \alpha + 1; \lambda]\, \mathbf{N}_n = 0 \quad (4.13)$$

for the discrete Lax pair (4.12a). In the case $n = 1$, the matrix $\mathbf{J}_1$ is

$$\mathbf{J}_1[w_\alpha, \alpha; \lambda] = \begin{pmatrix} 4\lambda^2 - 2w_\alpha^2 - z & -4\lambda w_\alpha + 2p_\alpha - \alpha/\lambda \\ -4\lambda w_\alpha - 2p_\alpha - \alpha/\lambda & -4\lambda^2 + 2w_\alpha^2 + z \end{pmatrix} \quad (4.14)$$

In the case $n = 2$, the matrix $\mathbf{J}_2$ is

$$\mathbf{J}_2[w_\alpha, \alpha; \lambda] = \begin{pmatrix} A_2 & B_2 - C_2 \\ B_2 + C_2 & -A_2 \end{pmatrix}, \quad (4.15)$$

with

$$A_2 = 16\lambda^4 - 8w_\alpha^2\lambda^2 - 4w_\alpha q_\alpha + 2p_\alpha^2 + 6w_\alpha^4 - z,$$
$$B_2 = -16w_\alpha\lambda^3 - 4(q_\alpha - 2w_\alpha^3)\lambda - \alpha/\lambda,$$
$$C_2 = -8p_\alpha\lambda^2 - 2r + 12w_\alpha^2 p_\alpha.$$

The matrix elements of $\mathbf{N}_2$ in (4.12a) are given in terms of the quantity Φ_2 in equation (3.17).

5   Discussion

We have illustrated that there is a close relationship between the Bäcklund transformations and the solution hierarchies for the Painlevé and discrete Painlevé equations. Hierarchies of solutions of Painlevé equations satisfy both a differential equation and a difference equation. Using Bäcklund transformations of Painlevé equations one can derive various difference equations, including discrete Painlevé equations, second-degree difference equations and systems of difference equations. Gromak [20], see also [12, 33], derived a

fourth-order difference equation relating five solutions of $P_{II}^{[2]}$ (3.4), with parameter values $w_{\alpha-1}$, w_α, $w_{\alpha+1}$, $w_{\alpha+2}$ and $w_{\alpha+3}$, which is equivalent to the system (3.16).

Hierarchies of rational and one-parameter families of exact solutions of P_{II}, P_{III} and P_{IV} are well-known, as remarked above. Since there is an explicit relationship between P_{II}, P_{III}, P_{IV} and some discrete Painlevé equations, then these solution hierarchies also satisfy difference equations as well as ordinary differential equations. This is entirely analogous to the situation for the classical special functions. This is further evidence that the Painlevé equations may be thought of as nonlinear special functions and that there is a fundamental relationship between special functions, Painlevé equations and discrete Painlevé equations (see also [46]).

Acknowledgements

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