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On Negatons of the Toda Lattice

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Abstract

Negatons are a solution class with the following characteristic properties: They consist of solitons which are organized in groups. Solitons belonging to the same group are coupled in the sense that they drift apart from each other only logarithmically. The groups themselves rather behave like particles. Moving with constant velocity, they collide elastically with the only effect of a phase-shift. The main result of this article is the rigorous proof of this characterization (including an explicit formula for the phase-shift) in terms of the asymptotic behaviour. To illustrate our result, we also discuss prototypical examples.

1 Introduction

The topic of the present article is a general study of solutions of the Toda lattice with weakly bounded groups of solitary waves. To our knowledge, the first systematic approach to solutions of this kind goes back to Wadati/Ohkuma [13], Tsuru/Wadati [12], who considered the modified Korteweg-de Vries and the sine-Gordon equation by means of the inverse scattering method. By his formalism of Darboux transformations, Matveev was able to study degenerate solutions of the Korteweg-de Vries equation and discovered the new class of positons [5]. Based on the same method Rasinariu et al. [7] derived the negatons of the Korteweg-de Vries equation, a solution class corresponding to the multiple pole solutions in [13], [12].

Our aim is to treat for the first time negatons for a lattice equation. The background of our work is an operator theoretic method developed by Aden, Carl and the author [1], [2], which was inspired by pioneering work of Marchenko [4]. It provides a general scheme to derive solution formulas in terms of determinants of operators. In the case at hand its application to finite matrices leads to transparent solution formulas which shall allow us a complete and rigorous treatment of the asymptotic behavior of negatons.

After reviewing some basic material in Section 2, we present the precise statement of the main theorem followed by a qualitative discussion in Section 3. Illustrating examples are sampled in Section 4. The rest of the paper is devoted to the proof of the main theorem.

2 Basic material

First we derive the negatons of the Toda lattice

$$\frac{\partial^2}{\partial t^2}\Big(\log(1+v_m(t))\Big) = \frac{1+v_{m+1}(t)}{1+v_m(t)} - \frac{1+v_m(t)}{1+v_{m-1}(t)}, \qquad m \in \mathbb{Z}. \tag{2.1}$$

In our context they are given by (see [9]):

Theorem 2.1. Let V be a matrix of dimension n ($V \in \mathcal{M}_{n,n}(\mathbb{C})$) which is already in Jordan form with N Jordan blocks V_j of dimension n_j and eigenvalues k_j,

$$V = \begin{pmatrix} V_1 & 0 & \cdots & 0 \\ 0 & V_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & V_N \end{pmatrix}, \qquad V_j = \begin{pmatrix} k_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & k_j \end{pmatrix}, \tag{2.2}$$

where all k_j are real, $\neq 0$, pairwise different and $k_i k_j - 1 \neq 0$ ($\forall i, j = 1, \ldots, N$). In particular $n = \sum_{j=1}^{N} n_j$.

a) The elementary operator $\Phi_V : \mathcal{M}_{n,n}(\mathbb{C}) \longrightarrow \mathcal{M}_{n,n}(\mathbb{C})$ given by $\Phi_V(X) = VXV - X$ is invertible.

b) If we define $p_m(t) := \det\big(I + L_m(t)\big)$ with $L_m(t) = V^{2m} \exp\big((V - V^{-1})t\big)\Phi_V^{-1}(a \otimes c)$ for arbitrary real non-zero n-dimensional vectors a, c (Recall that $a \otimes c$ is defined by $a \otimes c(x) =< x, a > c$. It is one-dimensional since $\mathrm{rank}(a \otimes c) = \mathrm{span}\{\mathrm{c}\}$), then

$$v_m(t) = \frac{p_{m+1}(t)}{p_m(t)} - 1 \tag{2.3}$$

is a solution of the Toda lattice (2.1).

Theorem 2.1 is a special case of more general results from [9]. For the sake of completeness, we give the proof in the appendix.

Remark 2.2. We want to point out that Theorem 2.1 holds for arbitrary matrices V with $0 \notin \mathrm{spec}(V)\mathrm{spec}(V) - 1$. However, it is not difficult to show that the restriction to matrices in Jordan form can be done without loss of generality ([9], Lemma 5.1).

In the sequel we shall concentrate on the equation

$$\frac{\partial^2}{\partial t^2}\Big(\log(1+f_m(t))\Big) = f_{m+1}(t) - 2f_m(t) + f_{m-1}(t), \tag{2.4}$$

the form of the Toda lattice which is usually used for drawing pictures (see Toda [11]). Note that Theorem 2.1 yields solutions of (2.4) by the transformation

$$[Tp]_m(t) = \frac{p_{m+1}(t)p_{m-1}(t)}{p_m(t)^2} - 1. \tag{2.5}$$

(For relations to the original Toda lattice, the Toda lattice in Flaschka's variables see [9]).

3 Statement and discussion of the main theorem

Since negatons may have poles, we first provide an appropriate notion of convergence.

First note that we can quite naturally interprete the discrete variable $m \in \mathbb{Z}$ as a continuous one. Thus in the sequel we always assume $m \in \mathbb{R}$ identifying $f_m(t) = f(m,t)$. Then the discrete results are obtained by again restricting m to the lattice.

Next, for any fixed t, $f_t(m) := f(m,t)$ can be viewed as a mapping to the Riemann number sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Now we equip $\hat{\mathbb{C}}$ with some metric, say

$$\mathrm{d}^\infty(w,z) = |\pi^{-1}(w) - \pi^{-1}(z)|, \qquad \pi^{-1} : \hat{\mathbb{C}} \to S^2 \subseteq \mathbb{R}^3, \tag{3.1}$$

where π denotes the stereographic projection $\pi : S^2 \to \hat{\mathbb{C}}$.

With these conventions, we say that two functions $f(m,t)$ and $g(m,t)$ have the same asymptotic behaviour for $t \to \pm\infty$, briefly

$$f(m,t) \approx g(m,t) \qquad \text{for } t \approx \pm\infty, \tag{3.2}$$

if the family $(f-g)_t : \mathbb{R} \to \hat{\mathbb{C}}$ converges to zero as $t \to \pm\infty$ uniformly with respect to the metric d^∞.

Now we are in the position to formulate our main result.

As the physics is not changed if we replace k_j by $1/k_j$, we shall always assume $|k_j| > 1 \forall j$.

Theorem 3.1. For the N-negatons as given in Theorem 2.1, the following asymptotic behaviour holds

$$f(m,t) \approx \sum_{j=1}^{N} \sum_{j'=0}^{n_j-1} f^{\pm}_{jj'}(m,t) \qquad \text{for } t \approx \pm\infty \tag{3.3}$$

with solitons $f^{\pm}_{jj'}(m,t) = (k_j^2 - k_j^{-2}) \frac{\ell^{\pm}_{jj'}}{(1+\ell^{\pm}_{jj'})^2}$, where $\ell^{\pm}_{jj'}(m,t) = (-1)^{j'} \epsilon_j \exp\big(\Gamma^{\pm}_{jj'}(m,t)\big)$, moving along the curves

$$\Gamma^{\pm}_{jj'}(m,t) = (\log k_j^2)\big(m - v_j t\big) \mp J' \log|t| + \Big(\varphi_j + \varphi_j^{\pm} + \varphi^{\pm}_{jj'}\Big), \tag{3.4}$$

where we have associated the velocity $v_j = -(k_j - k_j^{-1})/\log k_j^2$ to the j-th negaton and set $J' = -(n_j - 1) + 2j'$. The phase shifts $\varphi_j^{\pm}$ due to external collisions of negatons with different velocities are given by

$$\exp(\varphi_j^-) = \prod_{i:v_i<v_j} \left[\frac{k_j - k_i}{k_j k_i - 1}\right]^{2n_i}, \qquad \exp(\varphi_j^+) = \prod_{i:v_i>v_j} \left[\frac{k_j - k_i}{k_j k_i - 1}\right]^{2n_i}, \tag{3.5}$$

and the internal collisions between the solitons belonging to the j-th negaton result in phase shifts $\varphi^{\pm}_{jj'}$, where

$$\exp(\varphi^{\pm}_{jj'}) = \Big(\frac{j'!}{(j'-J')!} d_j^{-J'}\Big)^{\pm 1}, \qquad d_j = (k_j^2 - k_j^{-2}) - \frac{(k_j - k_j^{-1})^2}{\log|k_j|}. \tag{3.6}$$

Here we suppose that the vector a (and c accordingly) is decomposed as $a = (a_1, \ldots, a_N)^t$ with $a_j = (a_j^{(1)}, \ldots, a_j^{(n_j)})^t$, corresponding to the Jordan canonical form of V, and that $\varphi_j \in \mathbb{R}$ and $\epsilon_j = \pm 1$ are defined by $a_j^{(1)} c_j^{(n_j)} / (k_j^2 - 1)^{n_j} = \epsilon_j \exp(\varphi_j)$.

In addition, the following conservation law holds.

Corollary 3.2. The sum of the phase shifts vanishes: $\sum_{j=1}^{N} n_j \big(\varphi_j^+ - \varphi_j^-\big) = 0.$

Qualitatively our main result can be summarized as follows:

Discussion a) First consider a single eigenvalue k of algebraic multiplicity n. Then the solution is a wave packet consisting of n solitons. We call such a solution a (single) negaton of order n. The main observation is that the geometric center of the wave packet propagates with constant velocity, but its members drift away from each other at most logarithmically.

Hence, for large negative times we can imagine each soliton to be located on one side of the center, approaching the center logarithmically. Sometime it changes sides, and for large positive times it is located on the other side of the center, moving away from the center again logarithmically. Hence the solitons appear in reversed order for $\pm\infty$. Moreover, regular and singular solitons always alternate. Finally we stress that the path of the geometrical center is not affected by the internal collisions.

b) In the general case of N eigenvalues $k_1, \ldots, k_N$ of algebraic multiplicities $n_1, \ldots, n_N$, the solution is a superposition of N wave packets as in a). Their behaviour under collision is a natural generalization of what is known for N-solitons. But now every wave packet interacts and suffers a phase shift as a whole.

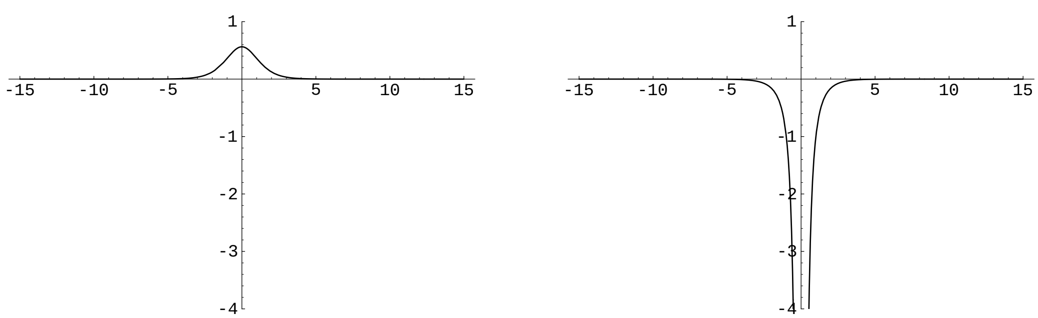
Remark 3.3. N-solitons correspond to diagonal matrices V of dimension N.

4 Examples

Example 4.1. (Regular and singular solitons) For $N = 1$, $V = k$ with $k^2 \neq 1$. If we set $\ell(m,t) = \dfrac{ac}{k^2-1} k^{2m} \exp\big((k-k^{-1})t\big) = \epsilon \exp\big((\log k^2)m + (k-k^{-1})t + \varphi\big)$, for $(ac)/(k^2-1) = \epsilon \exp(\varphi)$ with $\epsilon = \pm 1$ and φ real, we get

$$
\begin{aligned}
f(m,t) &= (k-k^{-1})^2 \ell(m,t)/\big(1+\ell(m,t)\big)^2 \\
&= \begin{cases} \frac{1}{4}(k-k^{-1})^2 \cosh^{-2}\left(\frac{1}{2}\left[(\log k^2)m + (k-k^{-1})t + \varphi\right]\right), & \epsilon = 1 \\ -\frac{1}{4}(k-k^{-1})^2 \sinh^{-2}\left(\frac{1}{2}\left[(\log k^2)m + (k-k^{-1})t + \varphi\right]\right), & \epsilon = -1 \end{cases},
\end{aligned}
\tag{4.1}
$$

the usual regular soliton for $\epsilon = 1$, and its singular counterpart for $\epsilon = -1$.



Typical shape of a regular and a singular soliton ($k = 2$)

Example 4.2. Let $N = 2$. There are two possibilities:

a) $V = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$ with $k_1^2, k_2^2 \neq 1$ and $k_1 k_2 \neq 1$.

Then

$$p(m,t) = 1 + \ell_1(m,t) + \ell_2(m,t) + \left(\frac{k_1 - k_2}{k_1 k_2 - 1}\right)^2 \ell_1(m,t)\ell_2(m,t), \tag{4.2}$$

where $\ell_j(m,t) = \frac{a_j c_j}{k_j^2 - 1} k_j^{2m} \exp\big((k_j - k_j^{-1})t\big) = \epsilon_j \exp\big((\log k_j^2)m + (k_j - k_j^{-1})t + \varphi_j\big)$ for $j = 1, 2$, where we used the same parametrization as in Example 4.1.

This results in the well-known collision of two solitons.

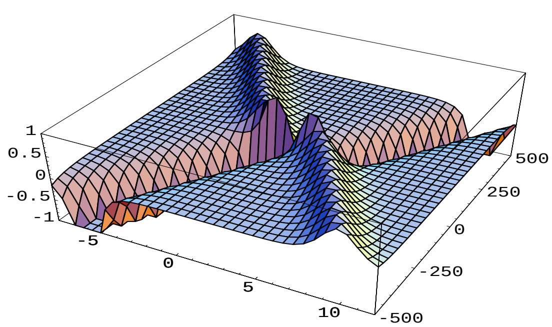
b) $V = \begin{pmatrix} k & 1 \\ 0 & k \end{pmatrix}$ with $k^2 \neq 1$.

Here the calculation of the matrices is more involved. We end up with

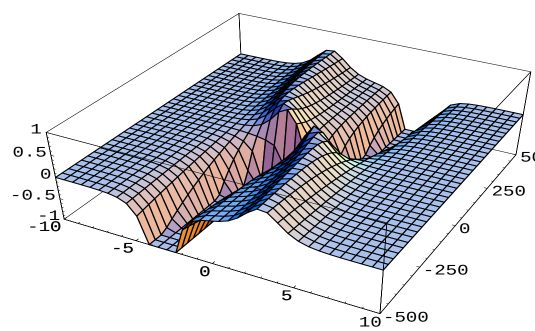
$$p(m,t) = 1 + \big(q(m,t) + A\big)\ell(m,t) - \big(\ell(m,t)\big)^2 \tag{4.3}$$

for $\ell(m,t) = \epsilon \exp\big((\log k^2)m + (k - k^{-1})t + \varphi\big)$, $q(m,t) = -2 + \big(2km + (k^2+1)t\big)/k^2$. We used the abbreviations $\epsilon \exp(\varphi) = \frac{a_1 c_2}{(k^2-1)^2}$, $A = \frac{a_1 c_1 + a_2 c_2}{k^2 - 1}\left(\frac{a_1 c_2}{(k^2-1)^2}\right)^{-1}$.

The corresponding solution is a single negaton consisting of two solitons.



Two solitons collide: one is regular ($k = 2.1$), the other one singular ($k = 1.9$).

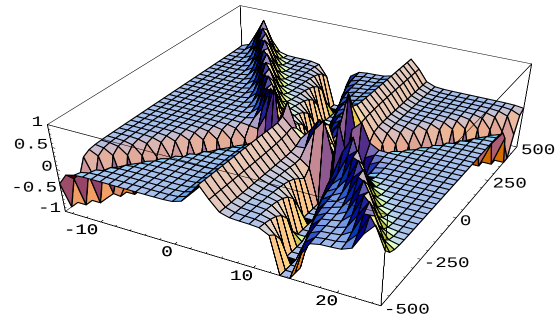


Single negaton ($k = 2$) with a regular and a singular soliton.

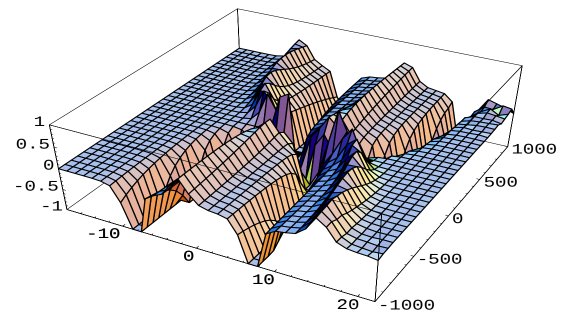
(In the pictures, the solution $f(m - \frac{2-1/2}{\log(2^2)}t, t)$ is plotted, where the variables m and t are depicted as usually. In standard coordinates all waves would drift to the right because $k > 1$).

Example 4.3. Finally we want to give an example of higher order. Of course again all matrices can be calculated explicitly, but instead of stating those (rather complicated) formulas, we provide computer graphics of some typical solutions. Again we always plot $f\big(m - \big((2 - 1/2)/\log(2^2)\big)t, t\big)$, the variables m and t as usual.

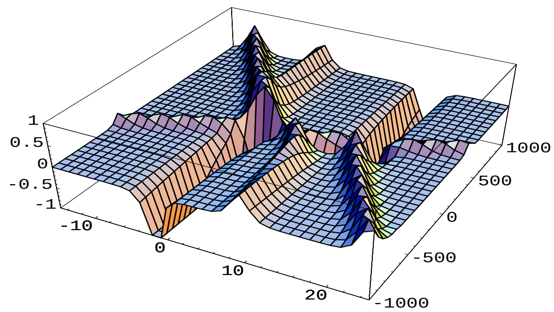
In either case $N = 4$.



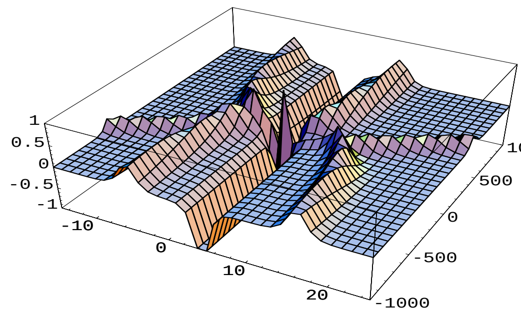
Four solitons collide: two are regular ($k = 2$, 2.2)
two are singular ($k = 1.7$, 2.1).



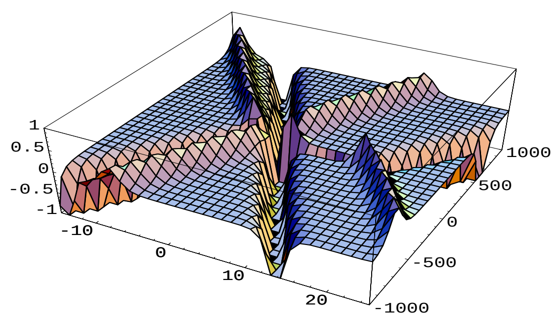
Single negaton ($k = 2$)
consisting of two regular and two singular solitons.



Negaton ($k = 2$) and two solitons ($k = 1.6$, 2.1)
collide, the former consisting of one regular
and one singular soliton.



Soliton ($k = 1.7$) and negaton ($k = 2$) collide,
the latter consisting of two regular
and one singular soliton.



Two negatons ($k = 1.9$, 2.1) collide, each
consisting of one regular and one singular soliton.

5 Asymptotics

5.1 On the solution B of the matrix equation $VBV - B = a \otimes c$

Recall that $V \in \mathcal{M}_{n,n}(\mathbb{C})$ is in Jordan form with N Jordan blocks V_j of dimension n_j respectively as assumed in Theorem 2.1. Accordingly, we have decomposed the vectors $a, c \in \mathbb{C}^n$ as

$$\begin{aligned} a &= (a_1, \ldots, a_N) \text{ with } a_j = (a_j^{(1)}, \ldots, a_j^{(n_j)}) \in \mathbb{C}^{n_j}, \\ c &= (c_1, \ldots, c_N) \text{ with } c_j = (c_j^{(1)}, \ldots, c_j^{(n_j)}) \in \mathbb{C}^{n_j}. \end{aligned} \tag{5.1}$$

Proposition 5.1. $\Phi_V^{-1}(a\otimes c) = (C_i W_{ij} A_j)_{ij=1}^N$ with

$$W_{ij} = \left(\frac{1}{(\mu-1)!}\frac{1}{(\nu-1)!}\frac{\partial^{\mu-1}}{\partial k_i^{\mu-1}}\frac{\partial^{\nu-1}}{\partial k_j^{\nu-1}}\Big(\frac{1}{k_ik_j-1}\Big)\right)_{\substack{\mu=1,\dots,n_i\\ \nu=1,\dots,n_j}} \in \mathcal{M}_{n_i,n_j}(\mathbb{C}), \tag{5.2}$$

$$A_j = \begin{pmatrix} a_j^{(1)} & & a_j^{(n_j)} \\ & \ddots & \\ 0 & & a_j^{(1)} \end{pmatrix}, C_j = \begin{pmatrix} c_j^{(1)} & & c_j^{(n_j)} \\ & \cdot^{\cdot^{\cdot}} & \\ c_j^{(n_j)} & & 0 \end{pmatrix} \in \mathcal{M}_{n_j,n_j}(\mathbb{C}). \tag{5.3}$$

(Note that the expressions for A_j and C_j differ!)

Proof a) First consider two Jordan blocks V_1, V_2 of dimensions n_1, n_2. Define an operator on $\mathcal{M}_{n_1,n_2}(\mathbb{C})$ by $\Phi_{V_1,V_2}X := V_1XV_2 - X$. Then we can check explicetly

$$V_1\big(D_1W_{12}\big)V_2 - \big(D_1W_{12}\big) = \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \text{ with } D_1 = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \in \mathcal{M}_{n_1,n_1}(\mathbb{C}), \tag{5.4}$$

which is essentially due to the chain rule. Therefore, rewriting the one-dimensional matrix in terms of two vectors, we get

$$\Phi_{V_1,V_2}^{-1}\Big(\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \otimes \begin{pmatrix} \vdots \\ 0 \\ 1 \end{pmatrix}\Big) = D_1W_{12}. \tag{5.5}$$

b) Next, from $a_2\otimes c_1 = \left(a_2^{(\nu)}c_1^{(\mu)}\right)_{\substack{\mu=1,\dots,n_1\\ \nu=1,\dots,n_2}} = C_1\begin{pmatrix} 1 & & 0 \\ & \cdot^{\cdot^{\cdot}} & \\ 0 & & 0 \end{pmatrix} A_2 = C_1D_1\begin{pmatrix} 0 & & 0 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} A_2,$ we find by a)

$$a_2\otimes c_1 = C_1D_1\Big(V_1\big(D_1W_{12}\big)V_2 - \big(D_1W_{12}\big)\Big)A_2 = V_1\big(C_1W_{12}A_2\big)V_2 - \big(C_1W_{12}A_2\big), \tag{5.6}$$

where we used $D_1^2 = 1$ and $[C_1D_1, V_1] = 0$, $[A_2, V_2] = 0$ (the matrices commute since they are Toeplitz matrices). Thus $\Phi_{V_1,V_2}^{-1}(a_2\otimes c_1) = C_1W_{12}A_2$.

c) Finally, observe $\Phi_V^{-1}(a\otimes c) = \Big(\Phi_{V_i,V_j}^{-1}(a_j\otimes c_i)\Big)_{ij=1}^N$. ■

Theorem 5.2.

$$\det\big(\Phi_V^{-1}(a\otimes c)\big) = \prod_{j=1}^N \Big(\frac{1}{k_j^2-1}\Big)^{n_j^2} \prod_{\substack{ij=1\\ i<j}}^N \Big(\frac{k_i-k_j}{k_ik_j-1}\Big)^{2n_in_j} \tag{5.7}$$

For the proof of this theorem we refer to [8] since it requires a great deal of work. In principle, it is a tricky combination of what has to be done in the two special cases that V is a diagonal matrix or a pure Jordan block. The former is quite simple and can already be found in [6].

5.2 Some technical reductions

Define $\ell_j(m,t) = k_j^{2m}\exp\big((k_j - k_j^{-1})t\big)$. For the Jordan block V_j of dimension n_j, it is straightforward to calculate

$$V_j^{2m}\exp\big((V_j - V_j^{-1})t\big) = \begin{pmatrix} \ell_j^{(0)} & & \ell_j^{(n_j-1)} \\ & \ddots & \\ 0 & & \ell_j^{(0)} \end{pmatrix} \text{ with } \ell_j^{(\nu)} = \frac{1}{\nu!}\frac{\partial^\nu}{\partial k_j^\nu}\ell_j, \tag{5.8}$$

and of course $V^{2m}\exp\big((V - V^{-1})t\big) = \mathrm{diag}\{V_j^{2m}\exp\big((V_j - V_j^{-1})t\big)|j = 1,\ldots,N\}$.

Thus

$$p(m,t) = \det\Big(I + \Big(V_i^{2m}\exp\big((V_i - V_i^{-1})t\big)\Phi^{-1}_{V_i,V_j}\big(a_j \otimes c_i\big)\Big)_{ij=1}^{N}\Big), \tag{5.9}$$

and, using Proposition 5.1, we finally arrive at

Proposition 5.3. $p(m,t) = \det\Big(I + \big(L_iW_{ij}\big)_{ij=1}^N\Big)$, where

$$L_j = \begin{pmatrix} L_j^{(n_j-1)} & & L_j^{(0)} \\ & \cdot^{\cdot^{\cdot}} & \\ L_j^{(0)} & & 0 \end{pmatrix}, \qquad \begin{aligned} L_j^{(\nu)} &= \sum_{\kappa=0}^{\nu} A_j^{(n_j-1-\nu+\kappa)}\frac{1}{\kappa!}\frac{\partial^\kappa}{\partial k_j^\kappa}\ell_j, \\ A_j^{(\nu)} &= \sum_{\kappa=1}^{n_j-\nu} a_j^{(\kappa)}c_j^{(\kappa+\nu)}. \end{aligned} \tag{5.10}$$

5.3 Asymptotic estimates

We devide our arguments in two steps. In the first step we show, that the N-negaton asymptotically is a superposition of N single negatons. Then, in the second step, we investigate how a single negaton behaves.

We only consider $t \to +\infty$, since the case $t \to -\infty$ is completely symmetric.

Step 1: To distinguish the single negatons properly, we recall the notion of the velocity of a negaton. Namely, to the j-th negaton (the negaton corresponding to the Jordan block V_j), we have associated the velocity $v_j = -(k_j - k_j^{-1})/\log(k_j^2)$.

Thus the index set $\Lambda_j^+ = \{i|v_i > v_j\}$ characterizes all negatons which move faster and will hence overtake the j-th negaton, $\Lambda_j^- = \{i|v_i < v_j\}$ the slower negatons, which will be overtaken by the j-th negaton.

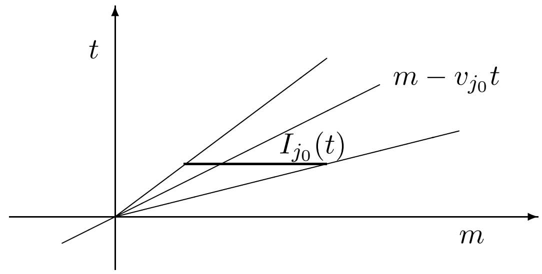
Proposition 5.4. $f(m,t) \approx \sum_{j_0=1}^{N} f_{j_0}(m,t)$ for $t \approx +\infty$,

$f_{j_0} = [Tp_{j_0}]$ with $p_{j_0} = \det\Big(Z_{ij}^{(j_0)}\Big)_{ij\in\Lambda_{j_0}^+\cup\{j_0\}}$ and $Z_{ij}^{(j_0)} = \begin{cases} \delta_{ij}I_{n_in_j} + L_iW_{ij},\ j = j_0 \\ W_{ij},\ j \in \Lambda_{j_0}^+. \end{cases}$

For the proof we need the following elementary lemma.

Lemma 5.5. Let Z, $\Delta \in \mathcal{M}_{n,n}(\mathbb{C})$. If $|Z| \le \exp(\zeta t)$, $|\Delta| \le \exp(-\delta t)$ for some positive constants ζ, δ (by definition, a matrix is bounded by a constant if the corresponding estimate holds simultaneously for all its entries), then $|\det(Z+\Delta) - \det(Z)| \le \exp\big((n\zeta - \delta)t\big)$.

Proof of Proposition 5.4 Fix j_0. Let $m \in I_{j_0}(t) = [(v_{j_0} - \delta_{j_0})t, (v_{j_0} + \delta_{j_0})t]$. This interval has the center $v_{j_0}t$, and its diameter grows linearly with t. We show that asymptotically a) the only contribution to the N-negaton in $I_{j_0}(t)$ is due to the j_0-th negaton, and b) outside $I_{j_0}(t)$ the j_0-th negaton vanishes. Of course we have to choose $\delta_{j_0} < \min_{j \neq j_0} |v_j - v_{j_0}|$ in order not to cross the path of another negaton.



From (5.10), we immediately get ($\log k_j^2 > 0$ since $|k_j| > 1$)

$$\exists \beta_{j_0} > 0 \quad \exists t_{j_0} : \forall t \geq t_{j_0} \quad \begin{cases} |L_j| < \exp(-\beta_{j_0} t), & j \in \Lambda_{j_0}^-, \\ |L_j^{-1}| < \exp(-\beta_{j_0} t), & j \in \Lambda_{j_0}^+. \end{cases} \tag{5.11}$$

This motivates to replace $p =: \det(Z_{ij})_{ij=1}^N$ by a determinant $\widehat{p} = \det(\widehat{Z}_{ij})_{ij=1}^N$ where $\widehat{Z}_{ij} = L_i^{-1} Z_{ij}$ whenever $i \in \Lambda_{j_0}^+$. Thus $\widehat{p}$ has vanishing entries, namely

$$\widehat{Z}_{ij} = \begin{cases} \delta_{ij} I_{n_j,n_j} + L_i W_{ij} & i \in \Lambda_{j_0}^- \cup \{j_0\} \\ \delta_{ij} L_i^{-1} + \quad W_{ij} & i \in \Lambda_{j_0}^+. \end{cases} \tag{5.12}$$

Note that the above manipulation of p does not alter $f = [Tp]$. Indeed, $[Tp] = [T\widehat{p}]$ since $p = \widehat{p} \prod_{j \in \Lambda_{j_0}^+} \det(L_j)$ and, by (5.10), $\det(L_j) = (-1)^{n_j(n_j+3)/2} (A_j^{(n_j-1)} \ell_j)^{n_j}$.

Now $\widehat{Z}_{ij} = Z_{ij}^{(j_0)} + \Delta_{ij}$, where we extend the definition of $Z_{ij}^{(j_0)}$ by $Z_{ij}^{(j_0)} = \delta_{ij} I_{n_j,n_j}$ for $i \in \Lambda_{j_0}^-$, and $\Delta_{ij} = L_i W_{ij}$ $(i \in \Lambda_{j_0}^-)$, 0 $(i = j_0)$, $\delta_{ij} L_i^{-1}$ $(i \in \Lambda_{j_0}^+)$. Using Lemma 5.5 we obtain $|\widehat{p} - p_{j_0}| < \exp\big((n\delta_{j_0} - \beta_{j_0})t\big)$ $\forall t \geq t_{j_0}$, and we can achieve $n\delta_{j_0} - \beta_{j_0} < 0$ for δ_{j_0} small enough. Thus $\widehat{p} - p_{j_0}$ converges uniformly to zero with respect to t.

Similar arguments show:

$$\begin{aligned} &\text{for } m < (v_{j_0} - \delta_{j_0})t : \quad p_{j_0} \longrightarrow \det(W_{ij})_{ij \in \Lambda_{j_0}^+} && \text{uniformly,} \\ &\text{for } m > (v_{j_0} + \delta_{j_0})t : \quad \widehat{p}_{j_0} \longrightarrow \det(W_{ij})_{ij \in \Lambda_{j_0}^+ \cup \{j_0\}} && \text{uniformly,} \end{aligned} \tag{5.13}$$

where in the latter case p_{j_0} is replaced by $\widehat{p}_{j_0} = \det(\widehat{Z}_{ij}^{(j_0)})_{ij=1}^N$ with $\widehat{Z}_{ij}^{(j_0)} = L_i^{-1} Z_{ij}^{(j_0)}$ only for $i = j_0$.

The result for $f_{j_0}(m,t) = [Tp_{j_0}](m,t) = p_{j_0}(m+1,t) p_{j_0}(m-1,t)/p_{j_0}(m,t)^2 - 1$ follows from the asymptotic behaviour of $p_{j_0}(m,t)$ because there are no cancellation phenomena between numerator and denominator for t large. For the formal argument we have to consider strips along the zero locus of $p_{j_0}(m,t)$ and estimate on each strip the functions $p_{j_0}(m \pm 1, t)$ from below. The zero locus of $p_{j_0}(m,t)$ will be precisely described in the proof of Proposition 5.7. The estimates themselves are straightforward.

Thus $\mathrm{d}^\infty(f, f_{j_0}) \longrightarrow 0$ uniformly on $I_{j_0}(t)$ and $\mathrm{d}^\infty(f_{j_0}, 0) \longrightarrow 0$ uniformly on $\mathbb{R} \backslash I_{j_0}(t)$ as $t \to \infty$, and we have shown a), b).

Finally it can be shown in a completely analogous manner that the N-negaton vanishes outside $\cup_{j=1}^{N} I_j(t)$, that is f converges to zero as $t \to \infty$ uniformly on $\mathbb{R} \setminus \cup_{j=1}^{N} I_j(t)$ with respect to d^∞. ■

Step 2: First we give the basic estimates for the finer analysis of the j_0-th negaton. To this end we consider what happens if we deviate from its path logartihmically on the curve $\gamma_\rho(t) = v_{j_0} t + \big(\rho \log |t|\big)/\log k_{j_0}^2$ (which just means $(\log k_{j_0}^2)(\gamma_\rho(t) - v_{j_0} t) = \rho \log |t|$).

Proposition 5.6. $p_{j_0}\big(\gamma_\rho(t), t\big) = X\Big(1 + \sum_{\kappa=1}^{n_{j_0}} X_\kappa\, |t|^{\rho\kappa} (t)^{(n_{j_0}-\kappa)\kappa} \Big[1 + \mathcal{O}\Big(\frac{\log|t|}{t}\Big)\Big]\Big)$ with $X = \det\big(W_{ij}\big)_{ij \in \Lambda_{j_0}^+}$ and

$$X_\kappa = (-1)^{\frac{\kappa(\kappa+3)}{2}} \frac{\prod_{\widehat{\kappa}=1}^{\kappa-1} \widehat{\kappa}!}{\prod_{\widehat{\kappa}=1}^{\kappa} (n_{j_0} - \widehat{\kappa})!} \left(A_{j_0}^{(n_{j_0}-1)} \frac{(d_{j_0})^{n_{j_0}-\kappa}}{(k_{j_0}^2 - 1)^\kappa} \right)^{\kappa} \prod_{j \in \Lambda_{j_0}^+} \left[\frac{k_j - k_{j_0}}{k_j k_{j_0} - 1} \right]^{2\kappa n_j} .$$

Proof If we set $\partial^\kappa \ell_{j_0}/\partial k_{j_0}^\kappa =: q_{j_0}^{(\kappa)} \ell_{j_0}$, then the $q_{j_0}^{(\kappa)}$ are polynomials satisfying the recursion $q_{j_0}^{(\kappa+1)} = q_{j_0}^{(\kappa)} q_{j_0}^{(1)} + \partial q_{j_0}^{(\kappa)}/\partial k_{j_0}$ with $q_{j_0}^{(0)} = 1$, $q_{j_0}^{(1)} = 2m/k_{j_0} + (1 + 1/k_{j_0}^2)t$. With this in mind, we get from (5.10)

$$L_{j_0}^{(\kappa)}\big(\gamma_\rho(t), t\big) = A_{j_0}^{(n_{j_0}-1)} \frac{(d_{j_0} t)^\kappa}{\kappa!} |t|^\rho \ \Big[1 + \mathcal{O}\Big(\frac{\log|t|}{t}\Big)\Big] \tag{5.14}$$

for $d_{j_0} = (1 + 1/k_{j_0}^2) - (1 - 1/k_{j_0}^2)/\log|k_{j_0}|$.

Consider $T = (T_{ij})_{ij \in \Lambda_{j_0}^+ \cup \{j_0\}}$ with $T_{ij} = (T_{ij}^{(\mu\nu)})_{\substack{\mu=1,\ldots,n_i \\ \nu=1,\ldots,n_j}}$. For $J \subseteq \{1, \ldots, n_{j_0}\}$ define $T[J]$ as the matrix with the blocks $T[J]_{ij}$, where 1) $T[J]_{ij} = T_{ij}$ if $i \neq j_0$, $j \neq j_0$, 2) $T[J]_{j_0 j}$ is obtained from $T_{j_0 j}$ by maintaining only the rows indexed by J, and 3) $T[J]_{i j_0}$ is obtained from $T_{i j_0}$ by maintaining only the columns indexed by J. Then

$$\begin{aligned} &\det\Big(\delta_{ij_0}\delta_{jj_0} I_{n_{j_0} n_{j_0}} + T_{ij}\Big)_{ij \in \Lambda_{j_0}^+ \cup \{j_0\}} = \\ &= \ \det(T_{ij})_{ij \in \Lambda_{j_0}^+} + \sum_{\kappa=1}^{n_{j_0}} \sum_{\sigma_1 < \ldots < \sigma_\kappa} \det\Big(T[\{\sigma_1, \ldots, \sigma_\kappa\}]_{ij}\Big)_{ij \in \Lambda_{j_0}^+ \cup \{j_0\}}. \end{aligned} \tag{5.15}$$

Application to the matrix $\widehat{W}$ with the blocks $\widehat{W}_{ij} = W_{ij}$ for $i \in \Lambda_{j_0}^+$ and $\widehat{W}_{j_0 j} = L_{j_0} W_{j_0 j}$ for $i = j_0$ yields an expansion for p_{j_0} (see Proposition 5.4). Observe

$$L_{j_0} W_{j_0 j} = \Big(\sum_{\kappa=1}^{n_{j_0}-(\mu-1)} W_{j_0 j}^{(\kappa\nu)} L_{j_0}^{(n_{j_0}-(\mu-1)-\kappa)} \Big)_{\substack{\mu=1,\ldots,n_{j_0} \\ \nu=1,\ldots,n_j}}. \tag{5.16}$$

Expanding once more, we get for $\det\Big(\widehat{W}[\{\sigma_1, \ldots, \sigma_\kappa\}]\Big)$

$$\sum_{\widehat{\sigma}_1=1}^{n_{j_0}-\sigma_1+1} \cdots \sum_{\widehat{\sigma}_\kappa=1}^{n_{j_0}-\sigma_\kappa+1} \det\Big(W[\{\widehat{\sigma}_1, \ldots, \widehat{\sigma}_\kappa\}]\Big) \prod_{\widehat{\kappa}=1}^{\kappa} L_{j_0}^{(n_{j_0}-(\sigma_{\widehat{\kappa}}-1)-\widehat{\sigma}_{\widehat{\kappa}})} \tag{5.17}$$

By (5.14), the power of the leading term in t is $\kappa(\rho + (n_{j_0}+1)) - \sum_{\widehat{\kappa}=1}^{\kappa}(\sigma_{\widehat{\kappa}} + \widehat{\sigma}_{\widehat{\kappa}})$, which is maximized precisely by the choice $\sigma_{\widehat{\kappa}} = \widehat{\kappa}$, $\widehat{\sigma}_{\widehat{\kappa}} = \pi(\widehat{\kappa})$ for any permutation π of $\{1, \ldots, \kappa\}$.

Hence $\det\left(\widehat{W}[\{1,\ldots,\kappa\}]\right) = H_\kappa \left((d_{j_0})^{n_{j_0}-\kappa} A_{j_0}^{(n_{j_0}-1)}\right)^\kappa |t|^{\kappa\rho}(t)^{\kappa(n_{j_0}-\kappa)} \left[1+\mathcal{O}\left(\frac{\log|t|}{t}\right)\right]$, where

$$\begin{aligned} H_\kappa &= \sum_\pi \left(\prod_{\widehat{\kappa}=1}^{\kappa} \mathrm{fac}_{\widehat{\kappa}\pi(\widehat{\kappa})}\right) \det\left(W[\{\pi(1),\ldots,\pi(\kappa)\}]\right) \\ &= \det\left(\mathrm{fac}_{\mu\nu}\right)_{\mu\nu=1}^{\kappa} \det\left(W[\{1,\ldots,\kappa\}]\right), \end{aligned} \tag{5.18}$$

for $\mathrm{fac}_{\mu\nu} = \frac{1}{(n_{j_0}+1-(\mu+\nu))!}$ if $\mu + \nu \leq n_{j_0} + 1$ and 0 else. Since $\det\left(\mathrm{fac}_{\mu\nu}\right)_{\mu\nu=1}^{\kappa} = (-1)^{\kappa(\kappa+3)/2} \prod_{\widehat{\kappa}=1}^{\kappa-1} \widehat{\kappa}! / \prod_{\widehat{\kappa}=1}^{\kappa}(n_{j_0} - \widehat{\kappa})!$ (for a proof we refer to [10]) and by Theorem 5.2, H_κ can be calculated explicitly.

Inserting the expression for $\det\left(\widehat{W}[\{1,\ldots,\kappa\}]\right)$ in the expansion (5.15) for $\widehat{W}$, the proof is completed. ■

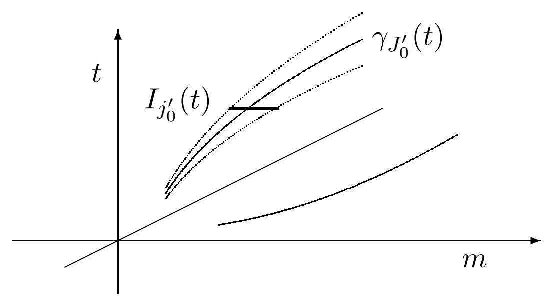
Remark 5.7. Slight modfications in the arguments above show that the same result is true for $p_{j_0}\left(\gamma_\rho(t)\pm 1, t\right)$ where the coefficients X_κ have to be replaced by $X_\kappa^\pm = (k_{j_0})^{\pm 2\kappa} X_\kappa$.

Proposition 5.8. $f_{j_0}(m,t) \approx \sum_{j_0'=0}^{n_{j_0}-1} f_{j_0 j_0'}(m,t)$ for $t \approx \infty$,

$f_{j_0 j_0'} = [T p_{j_0 j_0'}]$ with $p_{j_0 j_0'} = 1 + (-1)^{j_0'} \epsilon_{j_0} \exp(\Gamma^+_{j_0 j_0'})$, where the curve $\Gamma^+_{j_0 j_0'} = \Gamma^+_{j_0 j_0'}(m,t)$ (and all the data associated to it), the sign ϵ_{j_0} were defined in Theorem 3.1.

Proof Fix j_0'. $J_0' := -(n_{j_0} - 1) + 2j_0'$.

Let $m \in I_{j_0'}(t) = [\gamma_{J_0'-\delta_{j_0'}}(t), \gamma_{J_0'+\delta_{j_0'}}(t)]$ for $0 < \delta_{j_0'} < \frac{1}{2}$. This interval has the center $\gamma_{J_0'}(t) = v_{j_0} t + J_0' \log|t| / \log k_{j_0}^2$, and its diameter grows logarithmically with t. We show that asymptotically a) the only contribution to the j_0-th negaton in $I_{j_0'}(t)$ is due to the soliton $f_{j_0 j_0'}$ and b) outside $I_{j_0'}(t)$ the soliton $f_{j_0 j_0'}$ vanishes.



Parametrize $m = \gamma_{J_0'+\rho}(t)$ with $\rho \in (-\delta_{j_0'}, \delta_{j_0'})$. Then the summands of leading order in t in Proposition 5.6 are $\kappa = j_0', j_0' + 1$, and we get

$$p_{j_0}\left(\gamma_{J_0'+\rho}(t), t\right) = X\left(X_{j_0'} + X_{j_0'+1} t^{-J_0'} |t|^{\rho+J_0'}\right) t^{(n_{j_0}-j_0')j_0'} |t|^{(\rho+J_0')j_0'} \left[1+\mathcal{O}\left(\frac{\log|t|}{t}\right)\right]. \tag{5.19}$$

From $f_{j_0} = [T p_{j_0}]$ and Remark 5.7,

$$f_{j_0}\left(\gamma_{J_0'+\rho}(t), t\right) = (k_{j_0} - k_{j_0}^{-1})^2 \frac{(X_{j_0'+1}/X_{j_0'})|t|^\rho}{\left(1 + (X_{j_0'+1}/X_{j_0'})|t|^\rho\right)^2} + \mathcal{O}\left(\frac{\log|t|}{t}\right) \tag{5.20}$$

where $(X_{j_0'+1}/X_{j_0'}) = (-1)^{j_0'} \epsilon_{j_0} \exp\left(\varphi_{j_0} + \varphi^+_{j_0} + \varphi^+_{j_0 j_0'}\right)$. Thus the first term on the right-hand side of (5.20) equals $f_{j_0 j_0'}\left(\gamma_{J_0'+\rho}(t), t\right)$. This shows $\mathrm{d}^\infty(f_{j_0}, f_{j_0 j_0'}) \to 0$ uniformly on $I_{j_0'}(t)$.

It is easy to verify $f_{j_0 j_0'} \to 0$ uniformly outside $I_{j_0'}(t)$.

At last, we check analogously that the j_0-th negaton vanishes outside $\cup_{j_0'=0}^{n_{j_0}-1} I_{j_0'}(t)$, which completes the proof. ■

6 Appendix

Here we briefly sketch the proof that N-negatons are solutions of the Toda lattice. To see this, proceed as follows:

Step 1: By explicit calculations verify that $V_m(t) = \big(I + L_m(t)\big)^{-1}\big(VL_m(t)V - L_m(t)\big)$ for $L_m(t)$ as defined in Theorem 2.1 is a solution of the equation

$$\frac{\partial}{\partial t}\left((1+V_m)^{-1}\frac{\partial}{\partial t}V_m\right) = (1+V_m)^{-1}(1+V_{m+1}) - (1+V_{m-1})^{-1}(1+V_m), \tag{6.1}$$

which is just the matrix-valued analogue of the Toda lattice (2.1).

Step 2: Use the fact that the trace is multiplicative on one-dimensional matrices in the sense that $\mathrm{tr}(T_1T_2) = \mathrm{tr}(T_1)\mathrm{tr}(T_2)$ whenever $T_1 = a_1 \otimes c$, $T_2 = a_2 \otimes c$ to see that

$$v_m = \mathrm{tr}(V_m) \tag{6.2}$$

is a solution of the Toda lattice (2.1).

Step 3: The well-known relation $\det(I + T) = 1 + \mathrm{tr}(T)$ for matrices T which are one-dimensional then yields the reformulation

$$v_m(t) = p_{m+1}(t)/p_m(t) - 1. \tag{6.3}$$

(For details see [9]).

Remark 6.1. We want to point out that an extension of these ideas lead to quite abstract solution formulas of the Toda lattice. The main point is that in these formulas we even can plug in almost arbitrary (say bounded) operators V instead of matrices.

For a systematic explanation we refer the reader to [2], for an overview on applications (including solutions by the inverse scattering method and countable superpositions of solitons) to [3].

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