A type of the travelling wave solutions of higher-order Camassa-Holm equations

Fenggang Xue[1] Danping Ding[1]

[1]College of Sciences, Jiangsu University, Zhenjiang 212013, China.

Abstract

This paper researches the wave equation of higher-order Camassa-Holm equations, the general expression of a type of travelling wave solutions is obtained.

Keywords: higher-order Camassa-Holm equations,the wave equation,the travelling wave solutions

1. Introduction

In 2003,Adrian Constantin and Boris Kolev first got the higher-order Camassa-Holm equations [1] . Its concrete form is as follows:

$$\partial_t u = B_k(u,u), \tag{1}$$

where $k \in \{0\} \cup N$,

$$B_k(u,u) := A_k^{-1} C_k(u) - u\partial_x u ,$$

$$A_k(u) := \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u ,$$

$$C_k(u) := -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u)$$

In 2009, G.M.Coclite,H.Holden and K.H.Karlsen first researched the global Well-Posedness of the higher-order Camassa-Holm equations, and Got a weak solution [2] .

In 2010, Dingdanping, Lvpeng researched the existence of global solutions to the higher-order Camassa-Holm equations. Global solution is constructed by the small viscosity method for the frequency localized equations, especially global solution is energy conservative for given finite band initial data [3] .

The structure of the paper is organized as follows. In Section 2, the formal solutions of higher-order C-H equations are obtained, In Section 3,Energy Conservation of the formal solutions is discussed. In Section 4, a type of travelling wave solutions are given.

2. The formal solution

According to the equation (1),we have

$$\sum_{j=0}^{k} (-1)^j [\partial_x^{2j}(\partial_t u) + u\partial_x^{2j}(\partial_x u) + 2\partial_x u \partial_x^{2j} u] = 0 \tag{2}$$

Let $u = \varphi(\xi) = \varphi(x - ct)$, the travelling wave equation of (2) is

$$\sum_{j=0}^{k} (-1)^j [(-c\varphi')^{(2j)} + \varphi(\varphi')^{(2j)} + 2\varphi'\varphi^{(2j)}] = 0 \tag{3}$$

After twice-integration of equation (3),it was differentiated to yield

$$\varphi'(\varphi^{(2k)} - \varphi^{(2k-2)} + \varphi^{(2k-4)} - \cdots - (-1)^k \varphi'' + (-1)^k \varphi) = 0 ,$$

if $\varphi' = 0$,then $u = a$ (a is an arbitrary real number), otherwise,

$$\varphi^{(2k)}-\varphi^{(2k-2)}+\cdots-(-1)^k\varphi''+(-1)^k\varphi=0. \quad (4)$$

Equation (4) is ODE,when $k=0$,the solution is $u=a$ (a is an arbitrary real number).For $k\geq 1$, its characteristic equation is

$$r^{2k}-r^{2k-2}+r^{2k-4}-\cdots-(-1)^k r^2 +(-1)^k=0. \quad (5)$$

According to references [4-6], the roots of (5) have the form as follows:

$$r_i=\pm\alpha_i\pm i\beta_i,$$

where $\|r_i\|=\sqrt{\alpha_i^2+\beta_i^2}=1,a_i>0,\beta_i>0$,a ndwhen k is even, $i=1,2,\cdots\frac{k}{2}$,when k is odd, $i=1,2,\cdots\frac{k+1}{2}$.

So when k is even,the formal solution of (2) is

$$u(x,t)=\sum_{i=1}^{k/2}\Big[e^{\alpha_i(x-ct)}(c_{i1}\cos\beta_i(x-ct)+ c_{i2}\sin\beta_i(x-ct))+e^{-\alpha_i(x-ct)}(c_{i3}\cos\beta_i(x-ct) +c_{i4}\sin\beta_i(x-ct))\Big]$$

where $c_{i1},c_{i2},c_{i3},c_{i4}$ are coefficients,

when k is odd,the formal solution of (2) is

$$u(x,t)=\sum_{i=1}^{(k+1)/2}\Big[e^{\alpha_i(x-ct)}(c_{i1}\cos\beta_i(x-ct)+ c_{i2}\sin\beta_i(x-ct))+e^{-\alpha_i(x-ct)}(c_{i3}\cos\beta_i(x-ct) +c_{i4}\sin\beta_i(x-ct))\Big]$$

where $c_{i1},c_{i2},c_{i3},c_{i4}$ are coefficients.

3. Energy Conservation

According to attenuation of the travelling wave solutions, the formal solution can be reduced to

$$u(x,t)=\sum_{i=1}^{k/2}e^{-|\alpha_i(x-ct)|}\big[c_{i1}\cos\beta_i(x-ct) +c_{i2}\sin\beta_i(x-ct)\big] \ (k \text{ is even}),$$

$$u(x,t)=\sum_{i=1}^{(k+1)/2}e^{-|\alpha_i(x-ct)|}\big[c_{i1}\cos\beta_i(x-ct) +c_{i2}\sin\beta_i(x-ct)\big] \ (k \text{ is odd}).$$

if $u(x,t)$ obeys conservation of energy while k is even,then when k is odd, the same conclusion can be obtained.

Let $I(t)=\int_R u^2+(\partial_x u)^2+\cdots+(\partial_x^k u)^2dx$,

then $$\frac{dI(t)}{dt}=\sum_{l=0}^{k}\int_R\partial_x^l u\partial_x^l(\partial_t u)dx \ (\partial_t u=-c\partial_x u) =-c\cdot\sum_{l=0}^{k}\int_R\partial_x^l u\partial_x^{l+1}udx,$$

if $u(x,t)$ satisfies the condition of $\frac{dI(t)}{dt}=0$ in a certain assumptions, a type of travelling wave solutions can be obtained. So two independent assumed conditions are given.

3.1. $c_{i2}=0$

when k is even ($k\neq 0$),then

$$\int_{-\infty}^{ct}\partial_x^l u\partial_x^{l+1}udx =\sum_{1\leq i,j\leq\frac{k}{2}}c_{i1}c_{j1}\int_{-\infty}^{ct}e^{(\alpha_i+\alpha_j)(x-ct)}\cos(l\theta_{i1}+\beta_i(x-ct))\cdot \cos(\theta_{j1}+l\theta_{j1}+\beta_j(x-ct))dx,$$

where $\cos\theta_{i1} = a_i, \sin\theta_{i1} = \beta_i$, $l = 0,1,2,\cdots,k$,

$$\int_{-\infty}^{ct} e^{(\alpha_i+\alpha_j)(x-ct)} \cos(l\theta_{i1} + \beta_i(x-ct)) \cdot \cos(\theta_{j1} + l\theta_{j1} + \beta_j(x-ct))dx$$

$$= \frac{1}{2}\cos(l\theta_{i1} + \theta_{j1} + l\theta_{j1}) \frac{\alpha_i + \alpha_j}{(\alpha_i+\alpha_j)^2 + (\beta_i+\beta_j)^2}$$

$$+\frac{1}{2}\sin(l\theta_{i1} + \theta_{j1} + l\theta_{j1}) \frac{(\beta_i + \beta_j)}{(\alpha_i+\alpha_j)^2 + (\beta_i+\beta_j)^2}$$

$$+\frac{1}{2}\cos(l\theta_{i1} - \theta_{j1} - l\theta_{j1}) \frac{\alpha_i + \alpha_j}{(\alpha_i+\alpha_j)^2 + (\beta_i-\beta_j)^2}$$

$$+\frac{1}{2}\sin(l\theta_{i1} - \theta_{j1} - l\theta_{j1}) \frac{(\beta_i - \beta_j)}{(\alpha_i+\alpha_j)^2 + (\beta_i-\beta_j)^2}$$

.

$$\int_{ct}^{+\infty} \partial_x^l u \partial_x^{l+1} u dx = \sum_{1\le i,j\le \frac{k}{2}} c_{i1}c_{j1} \int_{ct}^{+\infty} e^{-(\alpha_i+\alpha_j)(x-ct)} \cos(l\theta_{i2} + \beta_i(x-ct)) \cdot$$

$$\cos(\theta_{j2} + l\theta_{j2} + \beta_j(x-ct))dx,$$

where $\cos\theta_{i2} = -a_i, \sin\theta_{i2} = \beta_i, \theta_{i2} = \pi - \theta_{i1}$.

$$\int_{ct}^{+\infty} e^{-(\alpha_i+\alpha_j)(x-ct)} \cos(l\theta_{i2} + \beta_i(x-ct)) \cdot \cos(\theta_{j2} + l\theta_{j2} + \beta_j(x-ct))dx$$

$$= -\frac{1}{2}\cos(l\theta_{i1} + \theta_{j1} + l\theta_{j1}) \frac{\alpha_i + \alpha_j}{(\alpha_i+\alpha_j)^2 + (\beta_i+\beta_j)^2}$$

$$-\frac{1}{2}\sin(l\theta_{i1} + \theta_{j1} + l\theta_{j1}) \frac{(\beta_i + \beta_j)}{(\alpha_i+\alpha_j)^2 + (\beta_i+\beta_j)^2}$$

$$-\frac{1}{2}\cos(l\theta_{i1} - \theta_{j1} - l\theta_{j1}) \frac{\alpha_i + \alpha_j}{(\alpha_i+\alpha_j)^2 + (\beta_i-\beta_j)^2}$$

$$-\frac{1}{2}\sin(l\theta_{i1} - \theta_{j1} - l\theta_{j1}) \frac{(\beta_i - \beta_j)}{(\alpha_i+\alpha_j)^2 + (\beta_i-\beta_j)^2}$$

,

then
$$\int_R \partial_x^l u \partial_x^{l+1} u dx = \int_{-\infty}^{ct} \partial_x^l u \partial_x^{l+1} u dx + \int_{ct}^{+\infty} \partial_x^l u \partial_x^{l+1} u dx = \sum_{1\le i,j\le \frac{k}{2}} c_{i1}c_{j1} \cdot 0 = 0,$$

then
$$\frac{dI(t)}{dt} = -c \cdot \sum_{l=0}^{k} \int_R \partial_x^l u \partial_x^{l+1} u dx = 0.$$

So when k is even, $u(x,t)$ obeys conservation of energy. the same conclusion is obtained while k is odd.

3.2. $c_{i1} = 0$

In this case, the same method is used to proof that the formal solution is also obeys conservation of energy.

4. Conclusions

A type of the travelling wave solutions of higher-order Camassa-Holm equations is as follows:

when k is even,

$$u(x,t) = \sum_{i=1}^{k/2} c_i e^{-|\alpha_i(x-ct)|} \cos\beta_i(x-ct),$$

or
$$u(x,t) = \sum_{i=1}^{k/2} c_i e^{-|\alpha_i(x-ct)|} \sin\beta_i(x-ct);$$

when k is odd,

$$u(x,t) = \sum_{i=1}^{(k+1)/2} c_i e^{-|\alpha_i(x-ct)|} \cos\beta_i(x-ct),$$

or

$$u(x,t) = \sum_{i=1}^{(k+1)/2} c_i e^{-|\alpha_i(x-ct)|} \sin\beta_i(x-ct),$$

where c_i is an arbitrary real number, $\alpha_i > 0$ and $\beta_i > 0$, them are determined by the following equation:

$$r^{2k} - r^{2k-2} + r^{2k-4} - \cdots - (-1)^k r^2 + (-1)^k = 0 \cdot$$

The solution of this equation is $r_i = \pm\alpha_i \pm i\beta_i$ ($\alpha_i^2 + \beta_i^2 = 1$), when k is even, the subscript $i = 1, 2, 3, \cdots, \frac{k}{2}$,when k is odd, the subscript $i = 1, 2, 3, \cdots, \frac{k+1}{2}$ and $r_{(k+1)/2} = \pm\alpha_{(k+1)/2} \pm \beta_{(k+1)/2} = \pm 1$.

Moreover, when $k = 1$,the travelling wave solution is $u(x,t) = c_1 e^{-|x-ct|}$ (c_1 is an arbitrary real number).It is the soliton solution[7] of Camassa-Holm equation which is obtained by Camassa and Holm.

5. References

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