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Discrete KP Equation and Momentum Mapping of Toda System

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Abstract

A new approach to discrete KP equation is considered, starting from the Gelfand-Zakhharevich theory for the research of Casimir function for Toda Poisson pencil. The link between the usual approach through the use of discrete Lax operators, is emphasized. We show that these two different formulations of the discrete KP equation are equivalent and they are different representations of the same equations. The relation between the two approaches to the KP equation is obtained by a change of frame in the space $\mathcal{L}_n$ of upper truncated Laurent series and translated into the space $\mathcal{D}_n$ of shift operators.

1 Introduction

In the classical picture (see [17, 18]), the discrete KP equation is described by a *discrete Lax* operator of the form

$$L = \Delta + \sum_{j\geq 1} q_j(n)\Delta^{-j}, \tag{1.1}$$

where Δ is the *shift* operator, defined by the following relation

$$\Delta^k\Big(q(n)\Big) = q(n+k) \tag{1.2}$$

for every integer k. Clearly, L is an element of the non commutative algebra $\mathcal{D}_n$ of formal series in Δ, with multiplication given by

$$\Delta^k \cdot \Delta^l = \Delta^{k+l}.$$

The operator L defines a *discrete Lax equation*, if

$$\frac{\partial}{\partial t_i} L = \left[\Big(L^i\Big)_+, L\right], \tag{1.3}$$

where $\left(L^i\right)_+$ is the positive part of the expansion of the i-th power of L respect to Δ. Equation (1.3) is the discrete KP equation in the discrete Lax formalisms.

We consider a different approach to (1.3), based on the *Gelfand-Zakhharevich* (or GZ) theory [14, 15] of bi-Hamiltonian geometry. In analogy with the theory developed in the continuous case for KdV [19], the starting point is the study of the Casimir functions of the Poisson pencil [2]

$$\begin{aligned} \dot{a}_n &= a_n\left(\frac{\partial H}{\partial b_n}(b_n+\lambda) - \frac{\partial H}{\partial b_{n+1}}(b_{n+1}+\lambda) + \frac{\partial H}{\partial a_{n-1}}a_{n-1} - \frac{\partial H}{\partial a_{n+1}}a_{n+1}\right) \\ \dot{b}_n &= \left(\frac{\partial H}{\partial a_{n-1}}a_{n-1} - \frac{\partial H}{\partial a_n}a_n\right)(b_n+\lambda) + \frac{\partial H}{\partial b_{n+1}}a_n - \frac{\partial H}{\partial b_{n-1}}a_{n-1} \end{aligned} \tag{1.4}$$

associated to the periodic Toda system [26] in $\mathbb{R}^{2N}$, in the formalism of Flaschka [11, 12]. The main result proved in [22, 23] is that the Casimir functions can be expressed as a product of N Hamiltonian densities $h(n) = h(n, a, b; \lambda)$

$$\mathcal{J} = h(1)\cdot h(2)\cdot\dots\cdot h(N), \tag{1.5}$$

where $h(n, a, b; \lambda)$ are solutions of the following Riccati system

$$h(n)\cdot h(n+1) = (b_{n+1}+\lambda)\cdot h(n) + a_n. \tag{1.6}$$

There are two fundamental solution of the Riccati system which differ by their expansion as Laurent series in λ. They are:

$$h(n,\lambda) = \lambda + \sum_{l=0}^{\infty}\frac{h_l(n)}{\lambda^l}, \qquad \text{and} \qquad k(n,\lambda) = a_n\left(\frac{1}{\lambda} + \sum_{l=2}^{\infty}\frac{k_l(n)}{\lambda^l}\right). \tag{1.7}$$

The coefficients $h_l(n)$ and $k_l(n)$ of the above series can be computed iteratively by substitution into the Riccati equation (1.6). We are interested to the monic solution $h(n,\lambda)$, because it is related to the construction of the discrete KP hierarchy; while the second density $k(n,\lambda)$ is related to the modified discrete KP hierarchy [25]. The Riccati system defines a map which at every point of the Toda-Flaschka phase space associates an n-uples of functions $h(n, a, b; \lambda)$ in the algebra of upper truncated Laurent series. This allows to compute the Casimir functions of the Poisson pencil and therefore the integrals of motion of the Toda equations, according to the GZ scheme (we refer to [22] for details and relations with the polynomial Casimir functions for Toda).

The map $\mathcal{J} : (a_n, b_n) \to h(1)\cdot\dots\cdot h(N)$ is the *momentum mapping* for Toda [2, 10, 13] and it gives the link between the Toda system and the discrete KP hierarchy. In fact, when we move in the phase space along a trajectory of the Toda system, the value of the moment mapping $\mathcal{J}$ does not change in time, while the values of the Hamiltonian density $h(n)$ change in time and verify a local conservation law of the form

$$\frac{\mathrm{d}h(n)}{\mathrm{d}t} = h(n)\big(H(n+1) - H(n)\big), \tag{1.8}$$

which is compatible with the invariance of the momentum mapping $\mathcal{J}$. Equation (1.8) suggests to interpret the difference $H(n+1) - H(n)$ as the discrete analog of the (one dimensional) divergence and permits to call the quantities $H(n, \lambda)$ the *currents* associated to the conservation law. The previously described property of the momentum mapping for Toda equation holds in general for every vector field of its hierarchy. In other words, there exists an infinite series of local currents $H^{(i)}(n)$ associated to GZ hierarchy for Toda, such that

$$\frac{\partial h(n)}{\partial t_i} = h(n)\big(H^{(i)}(n+1) - H^{(i)}(n)\big), \tag{1.9}$$

if the temporal derivative is evaluated along the i-th vector field of the Toda hierarchy.

As in the continuous case [3, 9], it is possible to introduce the *Faá di Bruno* monomials $h^{(i)}(n)$, associated to the point $h(n)$, also in the discrete case. This permits to consider (1.9) as the equations of the moving frame of the Faá di Bruno monomials in the space $\mathcal{L}_n$ of upper truncated Laurent series.

At this point, equations (1.9) change their significate. They may not any more be considered as the temporal evolutions of the moment mapping of the Toda system, but as a new system of equations defined automatically in the space of Laurent series. Interpreted from this new perspective, these equations can be called the *discrete KP equations* [20]. We show that these two different formulations of the discrete KP equation are equivalent and they are different representations of the same equations. The relation between the two approaches to the KP equation is given by a change of frame in the space $\mathcal{L}_n$ of upper truncated Laurent series and translated into the space $\mathcal{D}_n$ of difference operator through the introduction of a linear map ϕ_n defined by

$$\phi_n(h^{(i)}(n)) = \Delta^i; \tag{1.10}$$

on the elements of the Faá di Bruno basis.

Moreover, we introduce the potentials $\psi_n(\lambda)$ from the *zero curvature* [7] condition

$$\frac{\partial H^{(i)}(n)}{\partial t_j} = \frac{\partial H^{(j)}(n)}{\partial t_i}. \tag{1.11}$$

We will show that these functions $\psi_n(\lambda)$, up to an opportune normalization, are the well known Baker-Akhiezer functions for Toda system [27, 28], and the relation between the two approaches is complete.

2 The discrete KP equation

In this section we recall the main steps which lead to obtain the discrete KP equations from the temporal evolution of the moment mapping for Toda system, along a generic Hamiltonian vector field of Toda Poisson pencil (1.4). For further details we remand to [20, 22, 23, 25].

If the coefficients (a_n, b_n) in the Riccati equation (1.6) evolve in time according to (1.4), it is clear that the solution $h(n, \lambda)$ evolves in time according to the law

$$\dot{h}(n)h(n+1) + h(n)\dot{h}(n+1) - \dot{h}(n)(b_{n+1} + \lambda) = \dot{b}_{n+1}h(n) + \dot{a_n}, \tag{2.1}$$

and using the Riccati equation, we can write the previous equation in the form

$$(h(n+1)-(b_{n+1}+\lambda))\left[\dot h(n)-h(n)\left(H(n+1)-H(n)\right)\right]$$
$$+h(n)\left[\dot h(n+1)-h(n+1)\left(H(n+2)-H(n+1)\right)\right]=0$$

where

$$H(n)=[h(n)-(b_{n+1}+\lambda)]\,\frac{\partial H}{\partial b_n}-\frac{\partial H}{\partial a_{n-1}}a_{n-1}. \tag{2.2}$$

We then find that $h(n,\lambda)$ evolves according to the law (1.8). We now characterize a class of Hamiltonian functions which are compatible with the asymptotic expansion of the Hamiltonian densities $h(n;\lambda)$, in such a way that the previous equation is univocally determinate. These Hamiltonian functions are given by the GZ method when we take, as Casimir functions of the pencil, the Laurent series

$$\begin{aligned} H(\lambda) &= \log(h_1\cdot\dots\cdot h(n)) = N\log\lambda + H_1\lambda^{-1}+H_2\lambda^{-2}+\dots,\\ K(\lambda) &= \log(k_1\cdot\dots\cdot k_N) = -N\log\lambda + K_0 + K_1\lambda^{-1}+K_2\lambda^{-2}+\dots \quad ; \end{aligned}$$

with relations

$$K_0=\log(a_1\cdot\dots\cdot a_N) \qquad \textit{and} \qquad K_i=-H_i. \tag{2.3}$$

Hence, they define the same hierarchy

$$X_i=QdK_i=-PdK_{i+1}, \tag{2.4}$$

which is the *Toda hierarchy* according to the GZ scheme. It is easy to prove that it is not a really infinite hierarchy. In fact, only the first N vector fields are functionally independent, and the other are combination of them. The advantage to consider the Toda hierarchy above described is that the vector fields X_i of this hierarchy can be expressed as Hamiltonian vector fields with respect to the Poisson pencil with hamiltonian function given by

$$K^{(i)}(\lambda)=\lambda^i K_0+\lambda^{i-1}K_1+\dots+K_i. \tag{2.5}$$

At this point, we can say that the Hamiltonian density $h(n)(\lambda)$ evolves, along the orbits of the vector fields X_i of the Toda hierarchy (2.4), according to the local conservation law

$$\frac{\partial h(n)}{\partial t_i}=h(n)\Big(H^{(i)}(n+1)-H^{(i)}(n)\Big) \tag{2.6}$$

where the current $H^{(i)}(n)$ is given by

$$H^{(i)}(n)=a_{n-1}\Big(\frac{\partial K^{(i)}(\lambda)}{\partial b_n}\frac{1}{h(n-1)}-\frac{\partial K^{(i)}(\lambda)}{\partial a_{n-1}}\Big). \tag{2.7}$$

To conclude the argument, it remains to prove only that (with the chosen currents) the derivative $\dot{h(n)}(\lambda)$ has the correct asymptotic expansion in λ. This result follows from the following Lemma.

Lemma 1. *The currents $H^{(i)}(n)$, which appear in the local conservation laws for the densities $h(n)$ along the i-th flow of the Toda hierarchy, have the following asymptotic expansion:*

$$H^{(i)}(n) = \lambda^i + \mathcal{O}(\lambda^{-1}), \tag{2.8}$$

Proof. Remembering that $\Big(\lambda^i K(\lambda)\Big)_- = \mathcal{O}(\lambda^{-1})$, we can write the expression of the currents as

$$H^{(i)}(n) = \lambda^i a_{n-1}\left[\frac{\partial K(\lambda)}{\partial b_n}\frac{1}{h(n-1)} - \frac{\partial K(\lambda)}{\partial a_{n-1}}\right] + \mathcal{O}(\lambda^{-1}) \tag{2.9}$$

Clearly, the Lemma is proved from the former expression (2.3) for $K^{(i)}(\lambda)$ and the following *Pfaffian* relation

$$\frac{\partial H(\lambda)}{\partial b_n} = h(n-1)\frac{\partial H(\lambda)}{\partial a_{n-1}} \tag{2.10}$$

(see [23]) between the Hamiltonian density and the Casimir function. ■

The previous Lemma gives the expressions of the currents $H^{(i)}(n)$ in the usual basis of the space of Laurent series, i.e. as a series in the parameter λ. As we said, we want to give a closed expression of the currents in terms of the densities themselves. In order to give independence to equations (2.6) in the space $\mathcal{L}_n$ of Laurent series, we have to determinate the pencil parameter λ (and its powers) in terms of the densities $h(n)$. Once again, the main idea comes from the Riccati equation. In fact we interpret the Riccati system as a change of basis between the powers of $\lambda, \lambda^2, \ldots$ and the products of the densities $h(n)h(n+1), h(n), \frac{1}{h(n-1)}, \ldots$. This remark permits to define the basis of discrete Faá di Bruno monomials $h^{(i)}(n)$ associated to $h(n)$ as

$$\begin{aligned} h^{(0)}(n) &= 1 \\ h^{(i+1)}(n) &= h(n)h^{(i)}(n+1). \end{aligned} \tag{2.11}$$

In this way, the Riccati system expresses the link between the natural basis of the powers of λ and the basis of Faá di Bruno monomials in the affine space of Laurent series in λ. This link can be written as $\lambda^s \in \mathrm{span}\Big(\mathrm{h}^{(\mathrm{s})}(\mathrm{n}), \ldots, \mathrm{h}^{(-\mathrm{s})}(\mathrm{n})\Big)$. This means that there exists opportune functions $\alpha_{n,j}(a,b)$ not depending of λ such that

$$\lambda^s = \sum_{j=-s}^{s} \alpha_{n,j} h^{(j)}(n). \tag{2.12}$$

At this point the following Lemma is easily proved.

Lemma 2. *The currents $H^{(i)}(n)$, associated to the local conservation laws for the densities $h(n)$, along the i-th vector field of the Toda hierarchy, satisfy the following relations:*

$$H^{(i)}(n) = c_i(n)h^{(i)}(n) + \cdots + c_0(n)h^{(0)}(n), \tag{2.13}$$

where the coefficients $c_s(n)$ are independents from λ.

This means that the current $H^{(i)}(n)$ are elements of the subspace

$$\mathcal{H}_+(n) = \text{span}\Big(\text{h}^{(0)}(\text{n}), \text{h}^{(1)}(\text{n}), \text{h}^{(2)}(\text{n}), \dots\Big)$$

of the space of Laurent series $\mathcal{L}_n$. This determines a direct decomposition of the space of Laurent series

$$\mathcal{L}_n = \mathcal{H}_+(n) \oplus \mathcal{H}_-(n), \tag{2.14}$$

where clearly $\mathcal{H}_-(n)$ is the subspace generated by the negative Faá di Bruno monomials of $h(n)$. Using the trivial projection π_+ over the subspace $\mathcal{H}_+(n)$, we can express (2.13) as:

$$\pi_+\Big(\lambda^i\Big) = H^{(i)}(n), \tag{2.15}$$

where we made use of the asymptotic expansion of the currents $H^{(i)}(n) = \lambda^i + \mathcal{O}(\lambda^{-1})$ and of the linearity of the projector π_+ and property $\pi_+\Big(\mathcal{O}(\lambda^{-1})\Big) = 0$ (since every positive Faá di Bruno monomial has the asymptotic expansion $h^{(s)}(n) = \lambda^s + \mathcal{O}(\lambda^{s-1})$, and no linear combination of them can generate a series without positives powers of λ). Formula (2.15) gives a method to compute the currents in terms of the coefficients of the series $h(n)$. The passage from the local conservation laws of the Toda hierarchy to the KP hierarchy, is now easy. At this point, we consider the density

$$h(n) = \lambda + h_0(n) + h_1(n)\lambda^{-1} + \dots, \tag{2.16}$$

as an arbitrary monic Laurent series, no more forced to be a solution of the Riccati system. This permits to consider the coefficients $h_j(n)$ of the expansion of $h(n)$ as independent coordinates in the infinite dimensional manifold of the Laurent series in λ. Formula (2.15) of the currents, formulated before, is now considered as the definition of the currents and the expression (2.6) is the *discrete KP equation*. The previous definition of the currents, permits to determine $H^{(i)}(n)$ as functions of the point $h(n)$. In fact, the currents $H^{(i)}(n)$ are the unique Laurent series with the expansion in λ given by (2.8) and the expansion in the Faá di Bruno basis given by (2.13). The main difference with the picture delineated before consists in the fact that the coefficients $c_l(n)$ of the former expansion are functions of the coefficients $h_l(n)$ of the arbitrary Laurent series (2.16).

3 Lax formulation

So far we have used a *space representation*, where the Faá di Bruno basis $h^{(i)}(n)$ is moving in $\mathcal{L}_n$ and the standard basis λ^i is fixed. Now we pass to a *body representation*, where the Faá di Bruno basis is considered as fixed and the standard basis becomes moving. The KP equation is considered as the equation of moving frame of the Faá di Bruno monomials. The relation between the two approach to the KP equation is given by this change of representation in the space $\mathcal{L}_n$ and translated into the space $\mathcal{D}_n$ of difference operator through a linear map ψ_n. We introduce a linear map ϕ_n from the space of Laurent series

$\mathcal{L}_n$ in λ and the algebra $\mathcal{D}_n$ of shift operators. On the elements of the Faá di Bruno basis it is defined by

$$\phi_n(h^{(i)}(n)) = \Delta^i, \tag{3.1}$$

and it is extended by linearity over the entire space. The relation between the two approaches is given by the following Theorem.

Theorem 1.

$$\phi_n(\lambda) = L \tag{3.2}$$

Proof. The proof develops in two parts: in the first part we will show that $\phi_n(\lambda)$ has the same expansion in Δ of the operator L (and we obtain an invertible relation between the components of the expansion of the density $h(n)$ and the components of the expansion of the operator L). In the second part we will show that if the components of $h(n)$ evolve according to the KP equation:

$$\frac{\partial h(n)}{\partial t_i} = h(n)\Big(H^{(i)}(n+1) - H^{(i)}(n)\Big), \tag{3.3}$$

then the components of L evolve according to equation

$$\frac{\partial L}{\partial t_i} = \left[\left(L^i\right)_+, L\right]. \tag{3.4}$$

We denote with $q_j(n)$ the components of the expansion of λ with the Faá di Bruno monomials

$$\lambda = h^{(1)}(n) - \sum_{j\geq 0} q_j(n)h^{(-j)}(n). \tag{3.5}$$

Formula (3.5) express the change of representation before discussed. At the first orders we have explicitly:

$$\begin{aligned}
q_0(n) &= h_0(n) \\
q_1(n) &= h_1(n) \\
q_2(n) &= h_2(n) + h_0(n-1)h_1(n) \\
q_3(n) &= h_3(n) + h_0(n-1)h_2(n) + h_0(n-2)h_2(n) \\
&\quad +h_1(n-1)h_1(n) + h_0(n-2)h_0(n-1)h_1(n) \\
&\quad \vdots
\end{aligned} \tag{3.6}$$

If we apply the map ψ_n to (3.5), we have

$$\phi_n(\lambda) = \Delta - \sum_{j\geq 0} q_j(n)\Delta^{-j}. \tag{3.7}$$

In this way $\phi_n(\lambda)$ is an element L of $\mathcal{D}_n$ and the relation between the two expression (1.1) and (3.7) is given by (3.6).

At this point, we have to verify that the evolution of the coefficients $q_j(n)$ of $L = \phi_n(\lambda)$ satisfies equation (3.4) and this proves the theorem.

For this purpose, we give some important properties of the map ϕ_n. The first one concerns the values of the multiplication of an elements of the Faá di Bruno basis by a power of λ. We will show that:

$$\phi_n\Big(\lambda^k \cdot h^{(j)}(n)\Big) = \Delta^j(L^k). \tag{3.8}$$

To prove (3.8), we consider the case of $j = 1$ and $k = 1$:

$$\begin{aligned}
\phi_n\Big(\lambda \cdot h^{(1)}(n)\Big) &= \phi_n\Bigg(\Bigg[h^{(1)}(n+1) - \sum_{j\geq 0} q_j(n+1)h^{(-j)}(n+1)\Bigg] \cdot h^{(j)}(n)\Bigg) \\
&= \phi_n\Bigg(\Bigg[h^{(2)}(n) - \sum_{j\geq 0} q_j(n+1)h^{(-j+1)}(n)\Bigg] \cdot h^{(j)}(n)\Bigg) \\
&= \Delta^2 - \sum_{j\geq 0} q_j(n+1)\Delta^{-j+1} \\
&= \Delta \cdot \Big(\Delta - \sum_{j\geq 0} q_j(n)\Delta^{-j}\Big) = \Delta \cdot L
\end{aligned}$$

where we used the definition of the Faá di Bruno monomials. The general proof of (3.8) proceeds by induction with similar arguments.

The second property of ϕ_n concerns its action on the temporal derivatives of the Faá di Bruno monomials. In fact, using the evolution law (3.3) for $h(n)$, it is possible to obtain the expression of the evolution law for generic Faá di Bruno monomials. If $h(n)$ evolves according to

$$\dot{h(n)} = h(n)\Big(H(n+1) - H(n)\Big), \tag{3.9}$$

with $H(n)$ a generic currents, then

$$\dot{h^{(j)}}(n) = h^{(j)}(n)\Big(H(n+j) - H(n)\Big). \tag{3.10}$$

For example, for $j = 2$ we have:

$$\begin{aligned}
\dot{h^{(2)}}(n) &= \Big(h(n+1) \cdot h(n)\Big)^{\bullet} \\
&= \dot{h(n)} \cdot h(n+1) + h(n) \cdot \dot{h}(n+1) \\
&= h(n) \cdot h(n+1)\Big(H(n+2) - H(n+1)\Big) + h(n+1) \cdot h(n)\Big(H(n+1) - H(n)\Big) \\
&= h^{(2)}(n)\Big(H(n+2) - H(n)\Big),
\end{aligned}$$

and analogously for a generic integer j. If now apply the function ϕ_n to (3.10) and suppose that the current $H(n)$ does not depend on λ, we obtain:

$$\begin{aligned}
\phi_n\Big(\dot{h^{(j)}}(n)\Big) &= \phi_n\Big(h^{(j)}(n)\Big(H(n+j) - H(n)\Big)\Big) \\
&= H(n+j)\Delta^j - H(n)\Delta^j \\
&= \big[\Delta^j, H(n)\big].
\end{aligned}$$

Now, let us suppose that the current in (3.10) is a polynomial in λ

$$H(n) = \sum_k \lambda^k H_k(n)$$

then we have

$$\begin{aligned}\phi_n\Big(h^{\dot{(j)}}(n)\Big) &= \phi_n\Big(\sum_k \lambda^k h^{(j)}(n)\Big(H_k(n+j) - H_k(n)\Big)\Big) \\ &= \sum_k \left[\Delta^j, H(n)\right] L^k\end{aligned}$$

where we have used the previous equation and the first property of ϕ_n equation (3.8).

The third property of ϕ_n which we use is the following:

$$\phi_n \cdot \pi_+ = \Pi_+ \cdot \phi_n \tag{3.11}$$

where π_+ is the projection of the space $\mathcal{L}_n$ onto the subspace $< 1, h^{(1)}(n), h^{(2)}(n), \cdots >$ generated by the positive Faá di Bruno monomials, while Π_+ is the projection of $\mathcal{D}_n$ on the subspace $< 1, \Delta, \Delta^2, \cdots >$ generated by the positive powers of Δ.

Now, we are ready to conclude the proof of (3.4). Let us consider the relation (3.5)

$$\lambda = h^{(1)}(n) - \sum_{l\geq 0} q_l(n) h^{(-l)}(n), \tag{3.12}$$

and let us differentiate it with respect to the j-th flux of the KP equation (3.3)

$$\sum_{l\geq 0} \frac{\partial q_l(n)}{\partial t_j} h^{(-l)}(n) = \frac{\partial h^{(1)}(n)}{\partial t_j} - \sum_{l\geq 0} q_l(n) \frac{\partial h^{(-l)}(n)}{\partial t_j}. \tag{3.13}$$

Applying the map ϕ_n to both members of this equation and using the first two properties of the map ϕ_n, one gets:

$$\sum_{l\geq 0} \frac{\partial q_l(n)}{\partial t_j} \Delta^{-l} = \sum_{k\geq 1} \Big[\Delta, H_k^j(n)\Big] L^{-k} - \sum_{k\geq 1}\sum_{l\geq 0} q_l(n) \Big[\Delta^{-l}, H_k^j(n)\Big] L^{-k}$$

(where the expression of the currents $H^{(j)}(n) = \lambda^j + \sum_{k\geq 1} H_k^j(n)\lambda^{-k}$ has been used). The previous expression is equivalent to

$$\frac{\partial L}{\partial t_j} + \sum_{k\geq 1} \Big[L, H_k^j(n)\Big] L^{-k} = 0. \tag{3.14}$$

If we now consider

$$Q^{(j)}(n) = \phi_n(H^{(j)}(n)) = \phi_n(\lambda^j + \sum_{k\geq 1} H_k^j(n)\lambda^{-k}) = L^j + \sum_{k\geq 1} H_k^j(n) L^{-k}, \tag{3.15}$$

we observe that

$$Q^{(j)}(n) = \phi_n(\pi_+(\lambda^j)) = \Pi_+(\phi_n(\lambda^j)) = \Big(L^j\Big)_+ \tag{3.16}$$

using the formula of the currents $H^{(j)}(n) = \pi_+(\lambda^j)$ and the third property (3.11) of ϕ_n.

At this point the proof is complete if we substitute the previous expression (3.16) into (3.14), obtaining

$$\frac{\partial L}{\partial t_j} + \left[L, \left(L^j\right)_+\right] = 0. \tag{3.17}$$

which is the discrete KP equation in the Lax formalism. ■

4 The Baker-Akhiezer function

We want to conclude by showing some properties of the KP equation which come from the bi-Hamiltonian structure of the Toda equation. First of all, we prove the commutation of the fluxes of the KP hierarchy:

$$\frac{\partial H^{(i)}(n)}{\partial t_j} = \frac{\partial H^{(j)}(n)}{\partial t_i}. \tag{4.1}$$

This property comes from the following Theorem, where the expression of the evolution of the currents $H^{(i)}(n)$ along the vector fields of the KP hierarchy is given.

Theorem 2. *Along the trajectories of the discrete KP hierarchy, the currents $H^{(i)}(n)$ obey the equations*

$$\left(\frac{\partial}{\partial t_i} + H^{(i)}(n)\right)(\mathcal{H}_+(n)) \subset \mathcal{H}_+(n), \tag{4.2}$$

where $\mathcal{H}_+(n) = \mathrm{span}\left(\mathrm{h}^{(0)}(\mathrm{n}), \mathrm{h}^{(1)}(\mathrm{n}), \mathrm{h}^{(2)}(\mathrm{n}), \dots\right) =< \mathrm{H}^{(0)}(\mathrm{n}), \mathrm{H}^{(1)}(\mathrm{n}), \mathrm{H}^{(2)}(\mathrm{n}), \cdots >$*, or equivalently:*

$$\frac{\partial H^{(i)}(n)}{\partial t_j} = -H^{(i)}(n)H^{(j)}(n) + H^{(i+j)}(n) + \sum_{l=1}^{j} H_l^i(n)H^{(j-l)}(n) + \sum_{l=1}^{i} H_l^j(n)H^{(i-l)}(n), \tag{4.3}$$

which is the Central System [20][25] of the discrete KP theory.

Proof. We proceed by induction using the Faá di Bruno formula and the KP equation written in the following equivalent form:

$$h(n) \cdot \left(\frac{\partial}{\partial t_i} + H^{(i)}(n+1)\right) = \left(\frac{\partial}{\partial t_i} + H^{(i)}(n+1)\right) \cdot h(n). \tag{4.4}$$

We prove that

$$\left(\frac{\partial}{\partial t_i} + H^{(i)}(n)\right) \cdot h^{(j)}(n) \subset \mathcal{H}_+(n). \tag{4.5}$$

Starting with $j = 1$:

$$\begin{aligned}
\left(\frac{\partial}{\partial t_i} + H^{(i)}(n)\right) \cdot h^{(1)}(n) &= \frac{\partial h(n)}{\partial t_i} + H^{(i)}(n)h(n) = h(n)H^{(i)}(n+1) \\
&= h(n)\Big(h^{(i)}(n+1) + c_{i-1}(n+1)h^{(i-1)}(n+1) + \dots \\
&\qquad \cdots + c_0(n+1)h^{(0)}(n+1)\Big) \\
&= h^{(i+1)}(n) + c_{i-1}(n+1)h^{(i)}(n) + \cdots + c_0(n+1)h^{(1)}(n) \in \mathcal{H}_+(n).
\end{aligned}$$

Now let us suppose that (4.5) is true for j and let us prove it for $j+1$:

$$\begin{aligned}\left(\frac{\partial}{\partial t_i}+H^{(i)}(n)\right)\cdot h^{(j+1)}(n) &= \left(\frac{\partial}{\partial t_i}+H^{(i)}(n)\right)\cdot h(n)h^{(j)}(n+1)\\ &= h(n)\cdot\left(\frac{\partial)}{\partial t_i}+H^{(i)}(n+1)\right)\cdot h^{(j)}(n+1)\\ &= h(n)\cdot\left(\sum c_s h^{(s)}(n+1)\right)\\ &= \sum c_s(n+1)h^{(s+1)}(n)\in\mathcal{H}_+(n).\end{aligned}$$

Clearly (4.3) is the expression of (4.2) with respect to the basis $H^{(0)}(n), H^{(1)}(n), H^{(2)}(n), \ldots$ of the space $\mathcal{H}_+(n)$. ■

The closure condition (4.1) allows us to introduce a potential $\psi(n,\lambda)$ by

$$H^{(i)}(n)=\frac{\partial}{\partial t_i}\log\psi(n). \tag{4.6}$$

We will show that the function $\psi(n)$ is the Baker-Akhiezer function [3, 7, 27, 28] for the discrete Lax operator L. First of all we establish a relation between $\psi(n)$ and the Hamiltonian density $h(n)$:

$$h(n)=\frac{\psi(n+1)}{\psi(n)}. \tag{4.7}$$

The proof of (4.7) follows by the definition (4.6) of the potential $\psi(n)$ and the KP equation itself:

$$\begin{aligned}\frac{\partial}{\partial t_i}\log\left(h(n)\psi(n)\right) &= \frac{\partial}{\partial t_i}\log\psi(n)+\frac{\partial}{\partial t_i}\log h(n)\\ &= H^{(i)}(n)+H^{(i)}(n+1)-H^{(i)}(n)\\ &= H^{(i)}(n+1)=\frac{\partial}{\partial t_i}\log\psi(n+1).\end{aligned}$$

From (4.7) follows that

$$h^{(i)}(n)=\frac{\psi(n+i)}{\psi(n)}, \tag{4.8}$$

for every elements of the Faá di Bruno basis. Now, we are ready to prove that the function $\psi(n)$ is the Baker-Akhiezer function for L, i.e. $\psi(n)$ is an eigenvector of L with eigenvalue λ:

$$\psi(n;\lambda)\lambda=L\psi(n;\lambda), \tag{4.9}$$

and

$$\frac{\partial}{\partial t_j}\psi(n;\lambda)=\left(L^j\right)_+\psi(n;\lambda). \tag{4.10}$$

The first one is easily proved, remembering the link between the function $\psi(n)$ and the Faá di Bruno monomials:

$$\begin{aligned}
\lambda\psi(n) &= \Big(h^{(1)}(n) - \sum_l q_l(n)h^{(-l)}(n)\Big)\psi(n) \\
&= \Big(\frac{\psi(n+1)}{\psi(n)} - \sum_l q_l(n)\frac{\psi(n-l)}{\psi(n)}\Big)\psi(n) \\
&= \psi(n+1) - \sum_l q_l(n)\psi(n-l) \\
&= \Big(\Delta - \sum_l q_l(n)\Delta^{-l}\Big)\psi(n) = L\psi(n).
\end{aligned}$$

With the same arguments, we prove that

$$\begin{aligned}
\frac{\partial}{\partial t_j}\psi(n) &= H^{(j)}(n)\psi(n) = \Big(h^{(j)}(n) + \sum_{l=0}^{j-1} c_l(n)h^{(l)}(n)\Big)\psi(n) \\
&= \Big(\frac{\psi(n+j)}{\psi(n)} + \sum_{l=0}^{j-1}\frac{\psi(n+l)}{\psi(n)}\Big)\psi(n) = \psi(n+j) + \sum_{l=0}^{j-1}\psi(n+l) \\
&= \Big(\Delta^j + \sum_{l=0}^{j-1} c_l(n)\Delta^l\Big)\psi(n);
\end{aligned}$$

and remembering that

$$\begin{aligned}
\Big(L^j\Big)_+ &= \phi_n\Big(H^{(j)}(n)\Big) = \phi_n\Big(h^{(j)}(n) + \sum_{l=0}^{j-1} c_l(n)h^{(l)}(n)\Big) \\
&= \Delta^j + \sum_{l=0}^{j-1} c_l(n)\Delta^l,
\end{aligned}$$

as seen in the previous section; we get the equation (4.10), which is the evolution law of the eigenfuction of the discrete Lax operator L.

A general solution of (4.6) is given by $\widehat{\psi}(n;\lambda)e^{\sum_{i\geq 1} t_i\lambda^i}$, where $\widehat{\psi}(n;\lambda) = 1 + \mathcal{O}(\lambda^{-1})$, up to an opportune normalization. Relation (4.6), fixes this normalization with

$$\psi(n;\lambda) = \mu^n\widehat{\psi}(n;\lambda)e^{\sum_{i\geq 1} t_i\lambda^i}, \tag{4.11}$$

where $\mu = \mathcal{O}(\lambda)$ is an invariant for all the fluxes of the KP hierarchy, because $\mu^N = h(1)\cdot h(2)\cdot\dots\cdot h(N)$ is the Casimir (1.5) of Toda Poisson pencil. Formula (4.6) is the well known Baker-Akhiezer function for Toda.

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