Discretization of Toroidal Soliton Equations

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Abstract

We propose a way of discretization for the soliton equations associated with the toroidal Lie algebra based on the direct method. By the discretization, the symmetry of the system is modified so that the discrete time evolutions are no longer compatible with the original continuous ones. The solutions of the discrete soliton equations are explicitly given in terms of the Casorati determinants.

1   Introduction

As is well known, the hierarchies of soliton equations have the symmetries described by the infinite dimensional Lie algebras, affine or toroidal. All of the KP hierarchies are derived from the symmetries of affine Lie algebras[6], while the toroidal Lie algebras are associated with the 2+1 dimensional generalization of the 1+1 dimensional reduced KP hierarchies[1, 5, 4, 7]. For example, the 2+1 dimensional KdV equation[3]

$$u_t = u_{xxy} + 2uu_y + u_x \int^x u_y dx$$

and the 2+1 dimensional nonlinear Schrödinger equation[12]

$$iu_t = u_{xy} + u \int^x (|u|^2)_y dx$$

belong to the 2-toroidal hierarchy. In the above equations, the variable x is derived from the action of the usual affine part of the toroidal Lie algebra, while the evolutions in y and t are induced by the action of the genuine toroidal part. The weight of u and the relative weight of y and t are balanced with that of x, so we have two freedoms to determine the weights for all of the variables, the weight of x itself and the absolute value of weight of y (or t), which is one of the features of the 2-toroidal hierarchy. Another famous example is the self-dual Yang-Mills equation[11]

$$(J_y J^{-1})_{\bar{y}} + (J_z J^{-1})_{\bar{z}} = 0$$

which has the 3-toroidal Lie algebra symmetry, where we have three freedoms to balance all of the weights.

The integrable discretizations for the equations in KP hierarchies, i.e. affine case, have been investigated widely and deeply for many years from various standing points of view. For example, the Miwa transformation[9]

$$\partial_k = a\partial_{x_1} + \frac{a^2}{2}\partial_{x_2} + \frac{a^3}{3}\partial_{x_3} + \cdots$$

$$\partial_l = b\partial_{x_1} + \frac{b^2}{2}\partial_{x_2} + \frac{b^3}{3}\partial_{x_3} + \cdots$$

$$\partial_m = c\partial_{x_1} + \frac{c^2}{2}\partial_{x_2} + \frac{c^3}{3}\partial_{x_3} + \cdots$$

$$\vdots$$

can always produce integrable discrete analogues for a given continuous soliton equation in KP hierarchies, because the discrete evolutions in k, l, m, $\cdots$ are nothing but the chains of Bäcklund transformations.

On the other hand, nothing has been studied for the integrable discretization in the case of toroidal Lie algebra. In this paper, we investigate how to discretize the soliton equations associated with the toroidal Lie algebra keeping integrability and propose a way of discretization by using the Hirota's direct method of soliton theory. The discrete equations are bilinearized by introducing the auxiliary independent variables which come from affine symmetry and the solutions are directly constructed in the form of Casorati determinants.

2   Discretization for toroidal case

In the KP case, i.e. affine case, the solution depends on all of the continuous independent variables x_1, x_2, x_3, $\cdots$ and the discrete ones k, l, m, $\cdots$ simultaneously, and all of the flows induced by these continuous and discrete variables are compatible on the space of solution. Both the continuous and the discrete flows are described by the action of the same affine Lie algebra. Thus in which form we write down the equation is just a matter of choice of the variables. If we choose the continuous independent variables, we get the (hierarchy of) continuous soliton equations, and if we choose the discrete variables, we get the discrete analogues of soliton equations, and of course by choosing both continuous and discrete variables, we get the partial differential difference equations. Therefore in the KP case, we can easily derive integrable discrete analogues for a given continuous equation, because it is sufficient to change the choice of independent variables without changing the structure of the space of solution.

On the other hand in the case of toroidal symmetry, the solution depends on the continuous and discrete variables, x_1, x_2, x_3, $\cdots$ and k, l, m, $\cdots$ which come from the affine part and the continuous variables only, y_1, y_2, y_3, $\cdots$ which come from the genuine toroidal part. The point is that there is no discrete compatible flow associated with the genuine toroidal part because all of the Bäcklund transformations can be described by the affine part only and no discrete variable associated with y_1, y_2, y_3, $\cdots$ can be introduced. For example, there is no Miwa transformation for y_1, y_2, y_3, $\cdots$. The essential difficulty

to construct discrete anologues for the case of toroidal Lie algebra comes from the above fact.

In order to discretize y_1, y_2, y_3, $\cdots$, we give up keeping compatibility of all of the flows and break the affine symmetry. In other words, in order to introduce the discrete independent variables K, L, M, $\cdots$ associated with the toroidal part, with keeping the continuous toroidal variables y_1, y_2, y_3, $\cdots$, we modify the structure of the space of solution so that the original affine symmetry is killed and only $A_\infty^{(1)}$ symmetry survives. In the following sections, we are going to explain concretely how to modify by taking a typical example of the soliton equations with toroidal symmetry.

3   Modified Bogoyavlenskii-Schiff equation

The modified Bogoyavlenskii-Schiff equation[2, 10]

$$4u_{y_3} = u_{x_1 x_1 y_1} - 4u^2 u_{y_1} - 2u_{x_1} \int^{x_1} (u^2)_{y_1} dx$$

which has sl_2^{tor} symmetry, is bilinearized into the bilinear form

$$D_{x_1}^2 g \cdot f = 0$$
$$(D_{y_1} D_{x_1}^2 - 4D_{y_3})g \cdot f = 0$$
$$(D_{x_1}^3 - 4D_{x_3})g \cdot f = 0$$

through the dependent variable transformation

$$u = \left(\log \frac{g}{f}\right)_{x_1}$$

where x_3 is the auxiliary independent variable. The evolutions along the variables x_k are described by the affine symmetry $A_1^{(1)}$ and those for y_k are induced by the genuine toroidal part.

The solution of the bilinear equations is given in the form of the Wronski determinant

$$f = \det\left(\varphi_i^{(j-1)}\right) \qquad g = \det\left(\varphi_i^{(j)}\right)$$

where $\varphi_i^{(j)}$ are arbitrary functions satisfying the following dispersion relations associated with the affine part and genuine toroidal part respectively

$$\partial_{x_k} \varphi_i^{(j)} = \varphi_i^{(j+k)}$$
$$\partial_{y_3} \varphi_i^{(j)} = \partial_{y_1} \varphi_i^{(j+2)}$$

and also satisfying the condition of 2-reduction

$$\varphi_i^{(j)} \doteqdot \varphi_i^{(j+2)}$$

where $\doteqdot$ means 'equal up to gauge', i.e., equal up to multiplication of exponential function of linear combination of the independent variables.

4   Discrete modified Bogoyavlenskii-Schiff equation

In the above modified Bogoyavlenskii-Schiff equation, discretizing the affine variables x_k is quite simple and straightforward[8], because we can just apply the Miwa transformation on x_k and replace the continuous affine variables by the compatible discrete ones. Our purpose here is to discretize the genuine toroidal variables y_k.

In order to discretize y_k, the affine symmetry $A_1^{(1)}$ is modified to $A_\infty^{(1)}$. Under this modification, we will have all of the even and odd affine variables X_1, X_2, X_3, $\cdots$, instead of just odd ones x_1, x_3, x_5, $\cdots$ for $A_1^{(1)}$. Let us denote the discrete independent variables corresponding to y_k as n_k, then this modification is simply given by introducing some shifts into the components of determinant in the following manner.

$$f_{n_1 n_3} = \det\big(\varphi_i^{(j-1)}(n_1 - j, n_3 + j)\big) \qquad g_{n_1 n_3} = \det\big(\varphi_i^{(j)}(n_1 - j, n_3 + j)\big)$$

where $\varphi_i^{(j)}$ are arbitrary functions satisfying the dispersion relations of continuous affine and discrete toroidal types

$$\partial_{X_k}\varphi_i^{(j)}(n_1, n_3) = \varphi_i^{(j+k)}(n_1 - k, n_3 + k)$$
$$\Delta_3\varphi_i^{(j)}(n_1, n_3) = \Delta_1\varphi_i^{(j+2)}(n_1, n_3)$$

where Δ_k is the backward difference operator with respect to n_k defined as $\Delta_k F(n_k) = F(n_k) - F(n_k - 2)$. And $\varphi_i^{(j)}$ should also satisfy

$$\varphi_i^{(j)}(n_1, n_3) \eqsim \varphi_i^{(j+2)}(n_1, n_3)$$

We should note that this is not the condition of 2-reduction, because the components of determinant have shifted indices. Therefore the original $A_1^{(1)}$ flows are no longer compatible and the solution has only $A_\infty^{(1)}$ symmetry as the affine part.

The bilinear equations satisfied by the above Casorati determinants with the shifted indices of components are

$$(D_{X_1}^2 - D_{X_2})g_{n_1-1,n_3+1} \cdot f_{n_1 n_3} = 0$$
$$(D_{X_1}^2 + D_{X_2})g_{n_1+1,n_3-1} \cdot f_{n_1 n_3} = 0$$
$$g_{n_1+1,n_3+1}f_{n_1 n_3} - g_{n_1+1,n_3-1}f_{n_1,n_3+2} = \frac{1}{2}(D_{X_1}^2 - D_{X_2})g_{n_1+1,n_3+1} \cdot f_{n_1 n_3}$$
$$g_{n_1 n_3}f_{n_1+1,n_3+1} - g_{n_1,n_3+2}f_{n_1+1,n_3-1} = \frac{1}{2}(D_{X_1}^2 + D_{X_2})g_{n_1 n_3} \cdot f_{n_1+1,n_3+1}$$

Here the later two bilinear equations are new. Finally by introducing the dependent variable transformation

$$u_{n_1 n_3} = \begin{cases} \big(\log \dfrac{g_{n_1-1,n_3+1}}{f_{n_1 n_3}}\big)_{X_1} & n_1, n_3 : \text{even} \\[2ex] \big(\log \dfrac{f_{n_1-1,n_3+1}}{g_{n_1 n_3}}\big)_{X_1} & n_1, n_3 : \text{odd} \end{cases}$$

$$Q_{n_1 n_3} = \begin{cases} \big(\log \dfrac{g_{n_1+1,n_3}}{g_{n_1-1,n_3}}\big)_{X_1} & n_1 : \text{even}, \ n_3 : \text{odd} \\[2ex] \big(\log \dfrac{f_{n_1+1,n_3}}{f_{n_1-1,n_3}}\big)_{X_1} & n_1 \text{odd}, \ n_3 : \text{even} \end{cases}$$

we obtain the discrete modified Bogoyavlenskii-Schiff equation

$$\Delta e^{\int^{X_1} \Delta_3 u_{n_1 n_3} dX_1} = (Q_{n_1 n_3})_{X_1} - Q_{n_1 n_3}(u_{n_1+1,n_3} - u_{n_1,n_3-1})$$

$$\Delta Q_{n_1 n_3} = \Delta_1 u_{n_1+2,n_3}$$

where Δ is the difference operator defined as $\Delta F(n_1, n_3) = F(n_1, n_3 + 1) - F(n_1 + 1, n_3)$.

5   Concluding remarks

The way to discretize the genuine toroidal independent variables of the soliton equations associated with the toroidal Lie algebra symmetry was given by modifying the structure of the space of solution, i.e., by introducing the shifts of discrete variables into the components of the Casorati determinant solutions. Since the structure is changed, the symmetry of the system is not preserved, which is quite different from the case of usual discretization of affine flows. This method of discretization is applicable to any 2-toroidal Lie algebras but not to the 3-toroidal case, thus the discretization of the self-dual Yang-Mills equation would be a future work.

The author thanks to Dr. Kakei for valuable discussions.

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