Lattice Geometry of the Discrete Darboux, KP, BKP and CKP Equations. Menelaus' and Carnot's Theorems

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Abstract

Möbius invariant versions of the discrete Darboux, KP, BKP and CKP equations are derived by imposing elementary geometric constraints on an (irregular) lattice in a three-dimensional Euclidean space. Each case is represented by a fundamental theorem of plane geometry. In particular, classical theorems due to Menelaus and Carnot are employed. An interpretation of the discrete CKP equation as a permutability theorem is also provided.

1 Introduction

The KP, BKP and CKP hierarchies [1] and their multi-component analogues constitute the fundamental hierarchies of integrable equations in soliton theory. Proto-typical examples of integrable equations which may be located in these hierarchies are the sine-Gordon, Korteweg-de Vries, nonlinear Schrödinger, Kadomtsev-Petviashvili (KP) and Davey-Stewartson equations. Remarkably, the entire KP and BKP hierarchies may be retrieved from Hirota's [2] and Miwa's [3] integrable discrete master equations respectively by taking sophisticated continuum limits. The classical Darboux system governing conjugate coordinate systems is likewise integrable and contains as reductions a variety of nonlinear equations such as the Maxwell-Bloch and 2+1-dimensional sine-Gordon equations [4]. In fact, the Darboux system has been identified as a 'squared-eigenfunction' symmetry of the 2-component KP hierarchy [5]. The Darboux system possesses an integrable discrete counterpart which, in fact, represents an analogous geometric situation in the framework of discrete geometry [6].

In the present paper, we record a novel geometric characterization of the discrete Darboux system and show how the discrete KP, BKP and CKP equations may be obtained as specializations by imposing elementary geometric constraints. These constraints on an (irregular) lattice in a three-dimensional Euclidean space naturally lead to Möbius invariant versions of the discrete KP, BKP and CKP equations by employing fundamental theorems

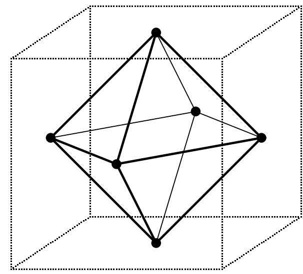

Figure 1. An octahedron embedded in an elementary cube

of plane geometry. Specifically, the Möbius invariant discrete Darboux system is encoded in a generalization of Menelaus' classical theorem [7, 8]. In a canonical degenerate geometric situation, Menelaus' theorem yields the discrete KP equation. The discrete BKP equation is encapsulated in a theorem related to Maxwell's reciprocal quadrangles [9, 10]. Carnot's classical theorem [11] is shown to enshrine the discrete CKP equation. The latter may also be interpreted as a permutability theorem associated with a binary Darboux transformation applied to the CKP hierarchy.

2 Geometric preliminaries and notation

In the present paper, we are concerned with the geometry of maps $\boldsymbol{v}$ of the type

$$\boldsymbol{v} : \mathbb{F} \to \mathbb{R}^3,$$

where $\mathbb{F}$ constitutes the set of face centres of a simple cubic lattice $\mathbb{Z}^3$. The set $\mathbb{F}$ does not represent a proper but 'irregular' lattice since it is not translationally invariant. The six face centres of any elementary cube of the cubic lattice may be regarded as the vertices of an octahedron as displayed in Figure 1. The twelve face centres which are inside a cube composed of eight adjacent elementary cubes are taken as the vertices of a cubo-octahedron. The eight octahedra inscribed in the elementary cubes are linked to the cubo-octahedron as indicated in Figure 2 so that the set $\mathbb{F}$ may be associated with the 'tiling' of Euclidean space by octahedra and cubo-octahedra. In this sense, we may refer to an edge, triangle or square of an octahedron or cubo-octahedron as an edge, triangle or square in $\mathbb{F}$.

In the context of difference equations, it proves convenient to label the three types of face centres $\boldsymbol{f}^i$, $i = 1, 2, 3$ by

$$\boldsymbol{f}^1(\boldsymbol{n}) = (1, \tfrac{1}{2}, \tfrac{1}{2}) + \boldsymbol{n}, \quad \boldsymbol{f}^2(\boldsymbol{n}) = (\tfrac{1}{2}, 1, \tfrac{1}{2}) + \boldsymbol{n}, \quad \boldsymbol{f}^3(\boldsymbol{n}) = (\tfrac{1}{2}, \tfrac{1}{2}, 1) + \boldsymbol{n},$$

where $\boldsymbol{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$, and identify a map $\boldsymbol{v}$ with three maps

$$\boldsymbol{v}^i : \mathbb{Z}^3 \to \mathbb{R}^3, \quad \boldsymbol{v}^i(\boldsymbol{n}) = \boldsymbol{v}(\boldsymbol{f}^i(\boldsymbol{n})).$$

In connection with the discrete BKP equation, it is useful to introduce auxiliary maps of the type

$$\boldsymbol{w} : \mathbb{M} \to \mathbb{R}^3$$

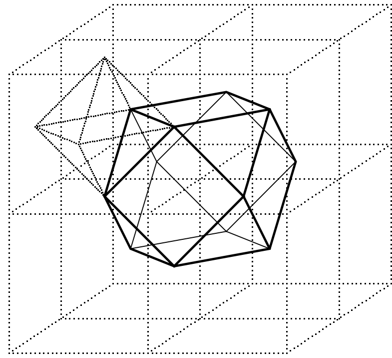

Figure 2. The 'tiling' of space by octahedra and cubo-octahedra

which are defined on the set $\mathbb{M}$ of centres of the elementary cubes. If the centres $\boldsymbol{m}$ are labelled by

$$\boldsymbol{m}(\boldsymbol{n}) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) + \boldsymbol{n}$$

then we may make the natural identification of $\boldsymbol{w}$ with

$$\boldsymbol{w} : \mathbb{Z}^3 \to \mathbb{R}^3, \quad \boldsymbol{w}(\boldsymbol{n}) = \boldsymbol{w}(\boldsymbol{m}(\boldsymbol{n})).$$

We usually suppress the argument of a function $g(\boldsymbol{n})$ but indicate increments of the independent variables n_i by, for instance,

$$g = g(n_1, n_2, n_3), \quad g_1 = g(n_1 + 1, n_2, n_3)$$
$$g_{11} = g(n_1 + 2, n_2, n_3), \quad g_{23} = g(n_1, n_2 + 1, n_3 + 1).$$

Moreover, throughout the paper, it is assumed that $i, k, l \in \{1, 2, 3\}$ are distinct.

The multi-ratio of $2n$ complex numbers is defined by

$$\mathsf{M}_{2n} = \mathsf{M}(P_1, \ldots, P_{2n}) = \frac{(P_1 - P_2)(P_3 - P_4) \cdots (P_{2n-1} - P_{2n})}{(P_2 - P_3)(P_4 - P_5) \cdots (P_{2n} - P_1)}.$$

It is invariant under the group of Möbius transformations acting on the complex plane. In the context of integrable systems, the geometric importance of multi-ratios has recently been revealed by various authors [10, 12, 13, 14, 15].

3 Darboux lattices. A generalization of Menelaus' theorem

We now select particular maps $\boldsymbol{v}$ by imposing a simple geometric condition:

Definition 1. A map $\boldsymbol{v} : \mathbb{F} \to \mathbb{R}^3$ is termed a *discrete Darboux map* if the four images of the vertices of any square in $\mathbb{F}$ are collinear.

In the case of a generic map $\boldsymbol{v}$, the squares of the cubo-octahedra are mapped to the quadrilaterals $(\boldsymbol{v}^i, \boldsymbol{v}^k, \boldsymbol{v}^i_k, \boldsymbol{v}^k_i)$. These quadrilaterals degenerate to (parts of) straight lines

$L^{ik} = L^{ki}$ if $\boldsymbol{v}$ constitutes a discrete Darboux map. The latter is therefore characterized by the six linear relations

$$\Delta_k \boldsymbol{v}^i = \rho^{ik}(\boldsymbol{v}_i^k - \boldsymbol{v}_k^i), \tag{3.1}$$

where the difference operators Δ_k are defined by $\Delta_k g = g_k - g$. However, the coefficients ρ^{ik} are constrained by the compatibility conditions $\Delta_l \Delta_k \boldsymbol{v}^i = \Delta_k \Delta_l \boldsymbol{v}^i$ which imply that there exist potentials H^i such that

$$\rho^{ik} = \frac{\Delta_k H^i}{H^i}. \tag{3.2}$$

It is then readily shown that the system (3.1) is compatible if and only if the functions H^i obey the nonlinear difference equations

$$\Delta_{ik} H^l = \frac{\Delta_k H_l^i}{H_l^i} \Delta_i H^l + \frac{\Delta_i H_l^k}{H_l^k} \Delta_k H^l \tag{3.3}$$

with the second-order difference operator $\Delta_{ik} = \Delta_i \Delta_k$. This system is integrable and has come to be known as the 'discrete Darboux system' since it governs conjugate lattices in Euclidean space [16, 17]. Nevertheless, as established in the preceding, the discrete Darboux system affords an alternative geometric interpretation. Darboux lattices $\boldsymbol{v}(\mathbb{F})$ as delineated in Definition 1 may therefore be considered integrable.

In order to proceed, we recall an important theorem [8] of planar geometry. Thus, consider a planar polygon $(P_1, \ldots, P_n)$, $n \in \mathbb{N}$ and denote by Q_m the points of intersection of a coplanar straight line L with the (extended) edges (P_m, P_{m+1}), $m = 1, \ldots, n$. Here, we make the natural identification $P_{n+1} = P_1$. Then, the ratios of the directed lengths $\overline{P_m Q_m}$ and $\overline{Q_m P_{m+1}}$ obey the algebraic relation

$$\prod_{m=1}^{n} \frac{\overline{P_m Q_m}}{\overline{Q_m P_{m+1}}} = (-1)^n.$$

In the case of a triangle corresponding to $n = 3$, Menelaus' classical theorem is retrieved [7]. The link with Darboux lattices is now established as follows:

It is evident that the vertices $\boldsymbol{v}^i, \boldsymbol{v}_k^i, \boldsymbol{v}_{kl}^i$ of a Darboux lattice corresponding to the twelve vertices of a cubo-octahedron are coplanar. These vertices lie on the six straight lines L^{ik}, L_l^{ik} which we may associate with (the squares of) that cubo-octahedron. A prototypical planar configuration of six lines and twelve points is displayed in Figure 3. Now, as indicated in Figure 3 for $(i, k, l) = (1, 2, 3)$, the four lines $L^{ik}, L_k^{il}, L_l^{ik}, L^{il}$ give rise to the quadrilateral $(P_1, P_2, P_3, P_4) = (\boldsymbol{v}^i, \boldsymbol{v}_k^i, \boldsymbol{v}_{kl}^i, \boldsymbol{v}_l^i)$. The points of intersection with the line L_i^{kl} are given by $(Q_1, Q_2, Q_3, Q_4) = (\boldsymbol{v}_i^k, \boldsymbol{v}_{ik}^l, \boldsymbol{v}_{il}^k, \boldsymbol{v}_i^l)$. Thus, if we identify the plane with the complex plane and regard the eight points P_m, Q_m as complex numbers then the above theorem for $n = 4$ implies that

$$\frac{(\boldsymbol{v}^i - \boldsymbol{v}_i^k)(\boldsymbol{v}_k^i - \boldsymbol{v}_{ik}^l)(\boldsymbol{v}_{kl}^i - \boldsymbol{v}_{il}^k)(\boldsymbol{v}_l^i - \boldsymbol{v}_i^l)}{(\boldsymbol{v}_i^k - \boldsymbol{v}_k^i)(\boldsymbol{v}_{ik}^l - \boldsymbol{v}_{kl}^i)(\boldsymbol{v}_{il}^k - \boldsymbol{v}_l^i)(\boldsymbol{v}_i^l - \boldsymbol{v}^i)} = 1$$

or

$$\mathsf{M}(\boldsymbol{v}^i, \boldsymbol{v}_i^k, \boldsymbol{v}_k^i, \boldsymbol{v}_{ik}^l, \boldsymbol{v}_{kl}^i, \boldsymbol{v}_{il}^k, \boldsymbol{v}_l^i, \boldsymbol{v}_i^l) = 1. \tag{3.4}$$

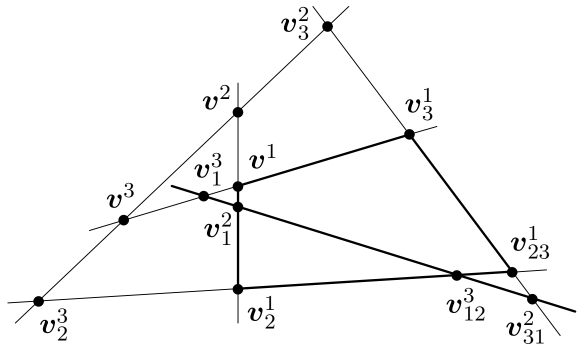

Figure 3. The image of a cubo-octahedron under a discrete Darboux map

It is evident that the above multi-ratio condition is invariant under the symmetry group of the quadrilateral formed by the lines $L^{ik}, L_k^{il}, L_l^{ik}, L^{il}$. Thus, there essentially exists one multi-ratio condition $\mathsf{M}_8 = 1$ corresponding to any five squares of a cubo-octahedron. Moreover, it may be shown [15] that only three of the six multi-ratio conditions associated with any cubo-octahedron are independent.

The above multi-ratio conditions are only valid locally since we have identified the coplanar vertices with complex numbers. In order to obtain a set of difference equations governing Darboux lattices, we now consider scalar-valued solutions v^i of the linear system (3.1), that is

$$\frac{v_i^k - v^i}{v_i^k - v_k^i} = \frac{H_k^i}{H^i}. \tag{3.5}$$

This system is compatible modulo the discrete Darboux system. Equivalently, it may be regarded as a linear system for the coefficients H^i. The compatibility conditions $H_{kl}^i = H_{lk}^i$ then deliver the three multi-ratio lattice equations

$$\mathsf{M}(v^i, v_i^k, v_k^i, v_{ik}^l, v_{kl}^i, v_{il}^k, v_l^i, v_i^l) = 1. \tag{3.6}$$

By construction, this system constitutes a Möbius invariant version of the discrete Darboux system (3.3). Its geometric origin has been demonstrated to reside in the generalization ($n = 4$) of Menelaus' theorem.

4 KP lattices. Menelaus' theorem

There exist particular Darboux lattices for which the configurations of six lines and twelve points degenerate to configurations consisting of four lines and six points. In order to define lattices of this type, it is noted that $\mathbb{F}$ contains eight families of parallel triangles of the same orientation. Each family corresponds to one of the triangles of a cubo-octahedron. This gives rise to the following definition:

Definition 2. Let $\mathbb{T}$ be one of the eight families of triangles in $\mathbb{F}$ which are parallel and of the same orientation. Then, a discrete Darboux map $\boldsymbol{v} : \mathbb{F} \to \mathbb{R}^3$ is termed a *discrete KP map* if the three vertices of any triangle in $\mathbb{T}$ are mapped to the same point.

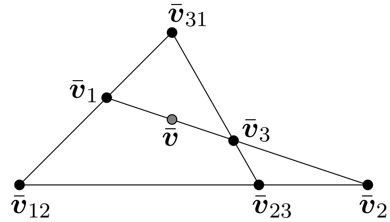

Figure 4. A Menelaus figure

Since the eight families of triangles appear on an equal footing, we may assume without loss of generality that the vertices $\boldsymbol{v}^1, \boldsymbol{v}^2$ and $\boldsymbol{v}^3$ coincide. Thus, if we set

$$\boldsymbol{v}^i = \bar{\boldsymbol{v}}$$

then the vertices $\{\bar{\boldsymbol{v}}_i, \bar{\boldsymbol{v}}_{ik}, \bar{\boldsymbol{v}}_{il}\} \subset L^i$ are collinear and there exists a line L passing through the points $\bar{\boldsymbol{v}}_1, \bar{\boldsymbol{v}}_2, \bar{\boldsymbol{v}}_3$. The vertex $\bar{\boldsymbol{v}}$ also lies on the line L but is insignificant since it does not constitute a point of intersection of lines. Accordingly, in the case of discrete KP maps, the planar configurations consist of the four lines L, L^i and the six points $\bar{\boldsymbol{v}}_i, \bar{\boldsymbol{v}}_{ik}$ (cf. Figure 4). Menelaus' theorem then implies that the points $(P_1, P_2, P_3) = (\bar{\boldsymbol{v}}_{31}, \bar{\boldsymbol{v}}_{12}, \bar{\boldsymbol{v}}_{23})$ and $(Q_1, Q_2, Q_3) = (\bar{\boldsymbol{v}}_1, \bar{\boldsymbol{v}}_2, \bar{\boldsymbol{v}}_3)$ regarded as complex numbers obey the algebraic relation

$$\frac{(\bar{\boldsymbol{v}}_{31} - \bar{\boldsymbol{v}}_1)(\bar{\boldsymbol{v}}_{12} - \bar{\boldsymbol{v}}_2)(\bar{\boldsymbol{v}}_{23} - \bar{\boldsymbol{v}}_3)}{(\bar{\boldsymbol{v}}_1 - \bar{\boldsymbol{v}}_{12})(\bar{\boldsymbol{v}}_2 - \bar{\boldsymbol{v}}_{23})(\bar{\boldsymbol{v}}_3 - \bar{\boldsymbol{v}}_{31})} = -1$$

or

$$\mathsf{M}(\bar{\boldsymbol{v}}_{31}, \bar{\boldsymbol{v}}_1, \bar{\boldsymbol{v}}_{12}, \bar{\boldsymbol{v}}_2, \bar{\boldsymbol{v}}_{23}, \bar{\boldsymbol{v}}_3) = -1. \tag{4.1}$$

The discrete symmetries of the multi-ratio condition $\mathsf{M}_6 = -1$ guarantee that this multi-ratio condition is equivalent to the multi-ratio conditions

$$\mathsf{M}(\bar{\boldsymbol{v}}_k, \bar{\boldsymbol{v}}_{ik}, \bar{\boldsymbol{v}}_{kl}, \bar{\boldsymbol{v}}_{il}, \bar{\boldsymbol{v}}_l, \bar{\boldsymbol{v}}_i) = -1,$$

which are obtained by setting $\boldsymbol{v}^i = \bar{\boldsymbol{v}}$ in (3.4). The linear system (3.1) simplifies to

$$\bar{\boldsymbol{v}}_k - \bar{\boldsymbol{v}} = \mu^{ik}(\bar{\boldsymbol{v}}_i - \bar{\boldsymbol{v}}), \quad \mu^{ik} = \frac{\rho^{ik}}{\rho^{ik} + 1} \tag{4.2}$$

which is compatible if and only if the coefficients μ^{ik} may be parametrized according to

$$\mu^{ik} = \frac{\phi_k}{\phi_i},$$

where the 'potential' ϕ obeys the difference equation

$$\frac{\phi_3 - \phi_1}{\phi_{31}} + \frac{\phi_1 - \phi_2}{\phi_{12}} + \frac{\phi_2 - \phi_3}{\phi_{23}} = 0.$$

The latter constitutes nothing but the discrete modified Kadomtsev-Petviashvili (mKP) equation which is related to the discrete KP equation by a discrete Miura-type transformation (see, e.g., [13]).

A Möbius invariant form of the discrete (m)KP equation may be obtained by considering a scalar-valued solution $\bar{v}$ of the linear system (4.2), that is

$$\frac{\bar{v}_k - \bar{v}}{\bar{v}_i - \bar{v}} = \frac{\phi_k}{\phi_i}.$$

Once again, this system is compatible modulo the discrete mKP equation but may also be regarded as a linear system for the potential ϕ. The associated compatibility conditions lead to multi-ratio conditions which are equivalent to the scalar version of (4.1), that is

$$\mathsf{M}(\bar{v}_{31}, \bar{v}_1, \bar{v}_{12}, \bar{v}_2, \bar{v}_{23}, \bar{v}_3) = -1.$$

The latter integrable lattice equation therefore governs KP lattices $\boldsymbol{v}(\mathbb{F})$ as delineated in Definition 2. It may be regarded as a 'Schwarzian' version of the discrete KP equation and has been discussed in detail in [13] in connection with Menelaus' theorem and the inversive geometry of the plane.

5 BKP lattices. Reciprocal quadrangles

Another class of particular Darboux lattices is obtained by relating a discrete Darboux map $\boldsymbol{v}$ to a map $\boldsymbol{w}$ defined on the centres $\mathbb{M}$ of the cubic lattice. We first note that any face centre $\boldsymbol{f} \in \mathbb{F}$ constitutes the centre of the line segment $(\boldsymbol{m}, \boldsymbol{m}')$, where $\boldsymbol{m}, \boldsymbol{m}' \in \mathbb{M}$ are the centres of the two adjacent elementary cubes. Thus, if we regard $\mathbb{M}$ itself as a cubic lattice then it is natural to state that the vertex $\boldsymbol{v}(\boldsymbol{f})$ and the edge $(\boldsymbol{w}(\boldsymbol{m}), \boldsymbol{w}(\boldsymbol{m}'))$ *correspond*. The following definition is based on the assumption that the relation between $\boldsymbol{f}$ and $\boldsymbol{m}, \boldsymbol{m}'$ is preserved by the maps $\boldsymbol{v}$ and $\boldsymbol{w}$:

Definition 3. A discrete Darboux map $\boldsymbol{v} : \mathbb{F} \to \mathbb{R}^3$ is termed a *discrete BKP map* if the vertices $\boldsymbol{v}$ constitute the centres of the corresponding edges associated with a map $\boldsymbol{w} : \mathbb{M} \to \mathbb{R}^3$.

If $\boldsymbol{v}$ constitutes a discrete Darboux map and $\boldsymbol{w}$ is a map defined on the centres $\mathbb{M}$ then the condition for $\boldsymbol{v}$ being a discrete BKP map is given by

$$\boldsymbol{v}^i = \frac{\boldsymbol{w} + \boldsymbol{w}_i}{2}. \tag{5.1}$$

This specialization reduces the linear system (3.1) to

$$\boldsymbol{w}_{ik} - \boldsymbol{w} = \rho^{ik}(\boldsymbol{w}_k - \boldsymbol{w}_i). \tag{5.2}$$

The associated compatibility conditions $\boldsymbol{w}_{ikl} = \boldsymbol{w}_{kli} = \boldsymbol{w}_{lik}$ imply that the coefficients $\rho^{ik} = -\rho^{ki}$ may be parametrized according to

$$\rho^{21} = \frac{\tau_1\tau_2}{\tau\tau_{12}}, \quad \rho^{32} = \frac{\tau_2\tau_3}{\tau\tau_{23}}, \quad \rho^{13} = \frac{\tau_3\tau_1}{\tau\tau_{31}},$$

where the potential τ obeys the integrable discrete BKP equation [3]

$$\tau\tau_{123} + \tau_1\tau_{23} + \tau_2\tau_{31} + \tau_3\tau_{12} = 0.$$

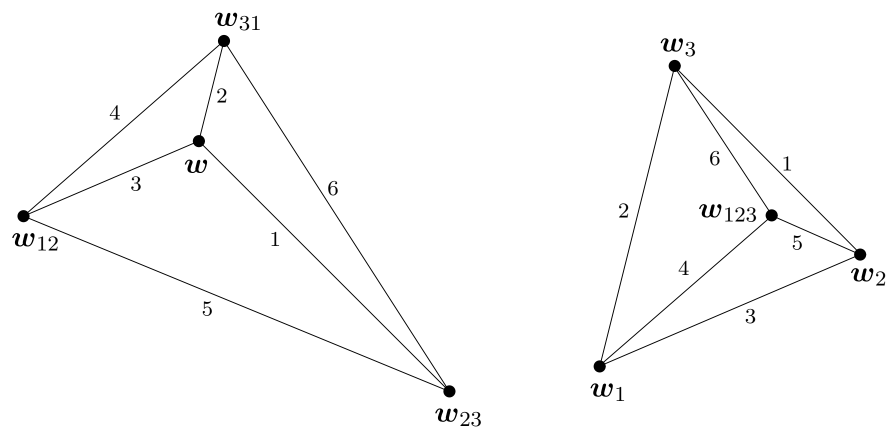

Figure 5. Reciprocal quadrangles

The latter encodes the complete hierarchy of KP equations of B type.

BKP lattices $\boldsymbol{v}(\mathbb{F})$ and their associated lattices $\boldsymbol{w}(\mathbb{M})$ possess interesting geometric properties. Thus, if we regard the points $(\boldsymbol{w}_{23}, \boldsymbol{w}_{31}, \boldsymbol{w}_{12}, \boldsymbol{w})$ and $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_{123})$ as the vertices of two (planar) quadrangles then the linear system (5.2) together with

$$\boldsymbol{w}_{ikl} - \boldsymbol{w}_l = \rho_l^{ik}(\boldsymbol{w}_{kl} - \boldsymbol{w}_{il})$$

show that any of the six edges of one quadrangle corresponds to a parallel edge of the other quadrangle as indicated in Figure 5. In fact, the six edges are parallel to the six lines L^{ik}, L_l^{ik}. In the terminology of Maxwell [9], the two quadrangles constitute 'reciprocal' quadrangles. Accordingly, under a discrete BKP map, each cubo-octahedron is mapped to a pair of reciprocal quadrangles. Reciprocal figures were investigated in detail by Maxwell in connection with graphical statics and diagrams of forces [9]. In [10], various reciprocal figures have been related to integrable lattice equations of BKP type. In particular, the inversive geometry of reciprocal quadrangles has been discussed. These considerations have led to the observation that the cross-ratios associated with any two reciprocal quadrangles coincide, that is

$$\frac{(\boldsymbol{w}_{23} - \boldsymbol{w}_{31})(\boldsymbol{w}_{12} - \boldsymbol{w})}{(\boldsymbol{w}_{31} - \boldsymbol{w}_{12})(\boldsymbol{w} - \boldsymbol{w}_{23})} = \frac{(\boldsymbol{w}_1 - \boldsymbol{w}_2)(\boldsymbol{w}_3 - \boldsymbol{w}_{123})}{(\boldsymbol{w}_2 - \boldsymbol{w}_3)(\boldsymbol{w}_{123} - \boldsymbol{w}_1)}$$

or

$$\mathsf{M}(\boldsymbol{w}_{23}, \boldsymbol{w}_{31}, \boldsymbol{w}_{12}, \boldsymbol{w}) = \mathsf{M}(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_{123}). \tag{5.3}$$

Here, the vertices $\boldsymbol{w}, \ldots, \boldsymbol{w}_{123}$ have been identified with complex numbers. This is admissible since the two planes spanned by the two quadrangles are parallel. It is evident that the ordering of the four points in one of the multi-ratios M_4 is irrelevant as long as the ordering of the arguments in the second multi-ratio corresponds. At an algebraic level, this is due to the discrete symmetries of the classical cross-ratio. For this reason, the multi-ratio conditions (3.4) are equivalent to (5.3) if the reduction (5.1) is made.

Scalar-valued solutions w of the linear system (5.2) obeying

$$
\begin{aligned}
w_{12} - w &= \frac{\tau_1\tau_2}{\tau\tau_{12}}(w_1 - w_2)\\
w_{23} - w &= \frac{\tau_2\tau_3}{\tau\tau_{23}}(w_2 - w_3)\\
w_{31} - w &= \frac{\tau_3\tau_1}{\tau\tau_{31}}(w_3 - w_1)
\end{aligned}
$$

give rise to a Möbius invariant avatar of the discrete BKP equation. Indeed, elimination of the 'τ-function' via compatibility leads to multi-ratio conditions which are equivalent to a scalar version of (5.3), that is

$$\mathsf{M}(w_{23}, w_{31}, w_{12}, w) = \mathsf{M}(w_1, w_2, w_3, w_{123}).$$

This lattice equation may be regarded as a discrete 'Schwarzian' BKP equation since it constitutes a compact form of the hierarchy of singular manifold equations [18] associated with the BKP hierarchy. Thus, the Schwarzian BKP hierarchy may, in principle, be obtained from the geometrically defined discrete BKP maps considered in this section.

6 CKP lattices. Carnot's theorem

The final section is concerned with a reduction of the discrete Darboux system associated with a theorem due to Carnot. Thus, based on the observation that there exist four planar hexagons formed by the edges of a cubo-octahedron, each of which divides the cubo-octahedron into two halves, we adopt the following definition:

Definition 4. A discrete Darboux map $\boldsymbol{v} : \mathbb{F} \to \mathbb{R}^3$ is termed a *discrete CKP map* if the six images of the vertices of any planar hexagon belonging to a cubo-octahedron lie on a conic.

In order to derive the difference equations governing CKP lattices $\boldsymbol{v}(\mathbb{F})$, we first recall Carnot's theorem [11]. Thus, consider a triangle (A, B, C) and denote by (P_m, Q_m, R_m), $m = 1, \ldots, n$ the points of intersection of a curve of degree $n \in \mathbb{N}$ with the (extended) edges $(A, B), (B, C), (C, A)$ respectively. Then, the algebraic relation

$$\prod_{m=1}^{n} \frac{\overline{AP_m}}{\overline{P_mB}}\frac{\overline{BQ_m}}{\overline{Q_mC}}\frac{\overline{CR_m}}{\overline{R_mA}} = (-1)^n$$

obtains. If the curve is a straight line ($n = 1$) then Menelaus' theorem is retrieved. Moreover, since a curve of second degree (conic) is determined by five points, the converse of the above theorem is valid in the case $n = 2$. Accordingly, if $\boldsymbol{v}$ constitutes a discrete Darboux map and the choice $(A, B, C) = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ is made then the six points $(P_1, P_2) = (\boldsymbol{v}_2^1, \boldsymbol{v}_1^2)$, $(Q_1, Q_2) = (\boldsymbol{v}_3^2, \boldsymbol{v}_2^3)$, $(R_1, R_2) = (\boldsymbol{v}_1^3, \boldsymbol{v}_3^1)$ lie on a conic (cf. Figure 6) if and only if

$$\frac{(\boldsymbol{v}^1 - \boldsymbol{v}_2^1)(\boldsymbol{v}^2 - \boldsymbol{v}_3^2)(\boldsymbol{v}^3 - \boldsymbol{v}_1^3)}{(\boldsymbol{v}_2^1 - \boldsymbol{v}^2)(\boldsymbol{v}_3^2 - \boldsymbol{v}^3)(\boldsymbol{v}_1^3 - \boldsymbol{v}^1)} = \frac{(\boldsymbol{v}_1^2 - \boldsymbol{v}^2)(\boldsymbol{v}_2^3 - \boldsymbol{v}^3)(\boldsymbol{v}_3^1 - \boldsymbol{v}^1)}{(\boldsymbol{v}^2 - \boldsymbol{v}_2^3)(\boldsymbol{v}^3 - \boldsymbol{v}_3^1)(\boldsymbol{v}^1 - \boldsymbol{v}_1^2)}$$

or

$$\mathsf{M}(\boldsymbol{v}^1, \boldsymbol{v}_2^1, \boldsymbol{v}^2, \boldsymbol{v}_3^2, \boldsymbol{v}^3, \boldsymbol{v}_1^3) = \mathsf{M}(\boldsymbol{v}_1^2, \boldsymbol{v}^2, \boldsymbol{v}_2^3, \boldsymbol{v}^3, \boldsymbol{v}_3^1, \boldsymbol{v}^1). \tag{6.1}$$

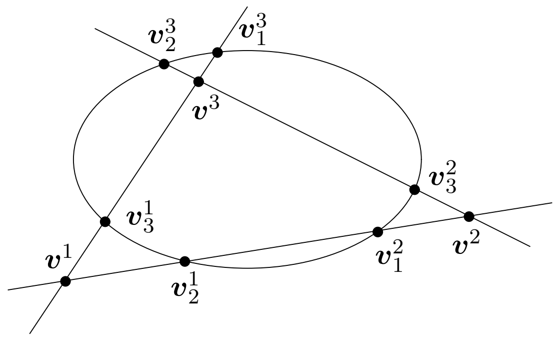

Figure 6. A Carnot figure ($n = 2$)

Here, the usual identification of the vertices with complex numbers has been made. It may be shown that the multi-ratio conditions $\mathsf{M}_6 = \mathsf{M}_6$ associated with the remaining three families of hexagons are equivalent to the multi-ratio condition (6.1). The specialization of discrete Darboux maps to discrete CKP maps is therefore encoded in the multi-ratio condition (6.1).

In terms of the coefficients ρ^{ik}, the above multi-ratio condition may be expressed as

$$\frac{\rho^{12}(\rho^{12}+1)}{\rho^{21}(\rho^{21}+1)}\frac{\rho^{23}(\rho^{23}+1)}{\rho^{32}(\rho^{32}+1)}\frac{\rho^{31}(\rho^{31}+1)}{\rho^{13}(\rho^{13}+1)} = 1 \tag{6.2}$$

by virtue of the linear system (3.1). The scalar version (3.5) of the latter system then implies that CKP lattices are governed by the three lattice equations (3.6), that is

$$\mathsf{M}(v^i, v^k_i, v^i_k, v^l_{ik}, v^i_{kl}, v^k_{il}, v^i_l, v^l_i) = 1, \tag{6.3}$$

subject to the constraint

$$\mathsf{M}(v^1, v^1_2, v^2, v^2_3, v^3, v^3_1) = \mathsf{M}(v^2_1, v^2, v^3_2, v^3, v^1_3, v^1). \tag{6.4}$$

In the remainder of this paper, it is shown that the overdetermined system (6.3), (6.4) is compatible and constitutes a Möbius invariant version of the 'symmetric' discrete Darboux system which, in turn, may be regarded as a discrete CKP equation.

The discrete Darboux system may be cast into the form of an equivalent first-order system, namely

$$\beta^{ik}_l = \frac{\beta^{ik}+\beta^{il}\beta^{lk}}{(\Gamma^{kl})^2}, \quad (\Gamma^{kl})^2 = 1 - \beta^{kl}\beta^{lk}. \tag{6.5}$$

The functions β^{ik} are related to the coefficients H^i by

$$H^i_k = \frac{H^i + \beta^{ki}H^k}{(\Gamma^{ik})^2}. \tag{6.6}$$

If a solution β^{ik} of the nonlinear system (6.5) is prescribed then (6.6) may be regarded as a compatible linear system for the coefficients H^i which, in turn, obey the discrete Darboux system (3.3). Conversely, for any given solution H^i of the discrete Darboux system, the system (6.6) may be solved for the functions β^{ik} which then satisfy the system (6.5). The latter admits a set of potentials a^i defined by

$$a^i_k = \Gamma^{ik} a^i$$

since the associated compatibility conditions

$$\frac{\Gamma^{12}_3}{\Gamma^{12}} = \frac{\Gamma^{23}_1}{\Gamma^{23}} = \frac{\Gamma^{31}_2}{\Gamma^{31}} \tag{6.7}$$

are indeed satisfied. These potentials may be used to formulate the discrete Darboux system in a different 'gauge'. Thus, if we make the definitions

$$\bar{H}^i = a^i H^i, \quad \bar{\beta}^{ik} = \frac{a^k}{a^i}\beta^{ik}$$

then the systems (6.5), (6.6) become

$$\bar{\beta}^{ik}_l = \frac{\bar{\beta}^{ik} + \bar{\beta}^{il}\bar{\beta}^{lk}}{\Gamma^{il}\Gamma^{kl}}, \quad \bar{H}^i_k = \frac{\bar{H}^i + \bar{\beta}^{ki}\bar{H}^k}{\Gamma^{ik}}. \tag{6.8}$$

It is noted that the quantities Γ^{ik} are symmetric in their indices, that is $\Gamma^{ik} = \Gamma^{ki}$, and that $\bar{\Gamma}^{ik} = \Gamma^{ik}$.

In view of the parametrization (3.2) of the coefficients ρ^{ik}, it is now seen that

$$\frac{\rho^{ik}(\rho^{ik}+1)}{\rho^{ki}(\rho^{ki}+1)} = \frac{\beta^{ki}}{\beta^{ik}}\left(\frac{H^k}{H^i}\right)^2$$

so that, on introduction of the quantities Q^{ik} according to

$$Q^{ik} = \frac{\bar{\beta}^{ki}}{\bar{\beta}^{ik}},$$

the CKP constraint (6.2) assumes the form

$$Q^{12}Q^{23}Q^{31} = \frac{\bar{\beta}^{21}\bar{\beta}^{32}\bar{\beta}^{13}}{\bar{\beta}^{12}\bar{\beta}^{23}\bar{\beta}^{31}} = 1. \tag{6.9}$$

In this case, the coefficients Q^{ik} depend only on two variables n_i and n_k since

$$\Delta_l Q^{ik} = 0$$

modulo (6.9). The condition (6.9) then gives rise to the factorization $Q^{ik} = f^{ik}(n_i)g^{ik}(n_k)$. However, on use of the invariance $\bar{H}^i \to f^i(n_i)\bar{H}^i$, $\bar{\beta}^{ik} \to f^k(n_k)\bar{\beta}^{ik}/f^i(n_i)$ of the system (6.8), it may be achieved that

$$Q^{ik} = 1 \quad \Leftrightarrow \quad \bar{\beta}^{ik} = \bar{\beta}^{ki}.$$

This evidently constitutes an admissible reduction of the discrete Darboux system $(6.8)_1$. Thus, it has been established that the geometric system (6.3), (6.4) constitutes a Möbius invariant version of the 'symmetric' reduction [19, 20, 21] of the discrete Darboux system.

The symmetric discrete Darboux system may be written as a single equation if one exploits the relations (6.7). Indeed, the latter give rise to the parametrization

$$(\Gamma^{ik})^2 = \frac{\tau\tau_{ik}}{\tau_i\tau_k}, \quad (\bar{\beta}^{ik})^2 = 1 - \frac{\tau\tau_{ik}}{\tau_i\tau_k} \tag{6.10}$$

so that, in terms of the potential τ, the system $(6.8)_1$ becomes

$$\begin{aligned}(\tau\tau_{123} - \tau_1\tau_{23} - \tau_2\tau_{31} - \tau_3\tau_{12})^2 \\ = 4(\tau_1\tau_2\tau_{23}\tau_{31} + \tau_2\tau_3\tau_{31}\tau_{12} + \tau_3\tau_1\tau_{12}\tau_{23} - \tau_1\tau_2\tau_3\tau_{123} - \tau\tau_{12}\tau_{23}\tau_{31}).\end{aligned} \tag{6.11}$$

This quartic integrable lattice equation may also be obtained in a different manner. Thus, it is observed that the discrete KP equation may be identified with the superposition principle of solutions of the KP hierarchy generated by the classical Darboux transformation (see, e.g., [22]). In a similar manner, the generic binary Darboux transformation [23] associated with the KP hierarchy gives rise to the discrete Darboux system. The binary Darboux transformation may be specialized in such a way that it induces a Bäcklund transformation for the BKP hierarchy. The corresponding permutability theorem is precisely of the form of the discrete BKP equation [24]. The KP hierarchy of C type is generated via the compatibility of an infinite hierarchy of linear differential equations of the type

$$\psi_{t_n} = L^n(\partial_x)\psi = \sum_{m=0}^{2n+1} L^n_m \partial^m_x \psi, \tag{6.12}$$

where the differential operators L^n are skew-symmetric, that is

$$L^n + L^{n\dagger} = 0,$$

and normalized by $L^n_{2n+1} = 1$. The first non-trivial member of this linear hierarchy is given by

$$\psi_t = \psi_{xxx} + 2q\psi_x + q_x\psi \tag{6.13}$$

with $t = t_1$. If ψ^α, $\alpha \in \mathbb{N}$ constitute particular solutions of (6.13) then ‘squared eigenfunctions’ $M^{\alpha\beta}$ may be introduced via

$$M^{\alpha\beta}_x = \psi^\alpha\psi^\beta, \quad M^{\alpha\beta}_t = \psi^\alpha\psi^\beta_{xx} + \psi^\alpha_{xx}\psi^\beta - \psi^\alpha_x\psi^\beta_x + 2q\psi^\alpha\psi^\beta.$$

The latter are compatible modulo the evolution (6.13). It is now readily verified that (6.13) is invariant under the binary Darboux transformations $\mathcal{D}_\alpha : (\psi, q) \to (\psi_\alpha, q_\alpha)$, where

$$\psi_\alpha = \psi - \psi^\alpha \frac{M^{\alpha 0}}{M^{\alpha\alpha}}, \quad q_\alpha = q + \frac{3}{2}(\ln M^{\alpha\alpha})_{xx}$$

and $\psi^0 = \psi$. Here, it is required to impose the admissible constraint $M^{\alpha\beta} = M^{\beta\alpha}$. The binary Darboux transformation $\mathcal{D}$ may be iterated in a purely algebraic manner. In fact, if we define a τ-function according to

$$q = \frac{3}{2}(\ln\tau)_{xx}$$

then its iterated analogues are readily shown to be [25]

$$\tau_{\nu_1\cdots\nu_\mu} = \tau \det(M^{\alpha\beta}), \quad \alpha, \beta \in \{\nu_1, \ldots, \nu_\mu\}, \quad \mu \in \mathbb{N}.$$

In particular, elimination of the bilinear potentials $M^1, M^2, M^3, M^{12}, M^{23}, M^{31}$ between the expressions for $\tau_1, \tau_2, \tau_3, \tau_{12}, \tau_{23}, \tau_{31}, \tau_{123}$ leads to the quartic equation (6.11) which is now interpreted as a superposition principle.

The above superposition principle is readily shown to hold for the entire CKP hierarchy if the definition of the bilinear potentials $M^{\alpha\beta}$ is suitably extended to include all 'times' t_n. This also applies to the 'negative' members of the CKP hierarchy which are obtained by adding compatible hyperbolic linear equations to the hierarchy (6.12). The first two members which may be generated in this manner are the asymmetric modified Nizhnik-Veselov-Novikov (mNVN) equation [23] and

$$\Xi_{xyz}^2 = 4\Xi_{xy}\Xi_{yz}\Xi_{xz}. \tag{6.14}$$

The latter equation has been recorded by Darboux [4] and is equivalent to the natural continuum limit of the symmetric discrete Darboux system. Indeed, in the limit

$$\bar{\beta}^{ik} \to \epsilon\bar{\beta}^{ik}, \quad x_i = \epsilon n_i, \quad \epsilon \to 0,$$

the symmetric discrete Darboux system $(6.8)_1$ reduces to the classical symmetric Darboux system

$$\bar{\beta}^{ik}_{,l} = \bar{\beta}^{il}\bar{\beta}^{lk},$$

where $_{,l}$ indicates the partial derivative with respect to the continuous variable x_l. The symmetric Darboux system admits the relations $\bar{\beta}^{12}\bar{\beta}^{12}_{,3} = \bar{\beta}^{23}\bar{\beta}^{23}_{,1} = \bar{\beta}^{31}\bar{\beta}^{31}_{,2}$ which may be used to parametrize the coefficients $\bar{\beta}^{ik}$ in terms of a potential. In fact, the continuum limit of $(6.10)_2$ reads

$$(\bar{\beta}^{ik})^2 = -(\ln\tau)_{,i,k} = \Xi_{,i,k}$$

so that the symmetric Darboux system becomes (6.14) with $(x, y, z) = (x_1, x_2, x_3)$. In analogy with the KP and BKP hierarchies, one may therefore refer to (6.11) as a discrete CKP equation. By construction, the discrete CKP equation constitutes a discretization of Darboux's equation (6.14) expressed in terms of τ.

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