Moyal Deformation of 2D Euler Equation and Discretization

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Abstract

In this paper we discuss the Moyal deformed 2D Euler flows and its Lax pairs. This in turn yields the semi-discrete version of 2D Euler equation.

1 Introduction

The geometry approach to fluid mechanics has a century old long history. In post war period, this activity started again from the work of Arnold [1], who showed in 1966 that if $u(x,t)$ is a time dependent divergence free vector field on a compact Riemannian n-manifold M and if $\eta(x,t)$ is the volume preserving flow, then u satisfies the Euler equation

$$\frac{\partial u}{\partial t} + \nabla_u u = -grad\ p$$

if and only if the curve $t \longmapsto \eta(\cdot,t)$ is an L^2 geodesic in $\mathcal{D}_\mu(M)$, the group of C^∞ volume preserving diffeomorphism on M.

In a celebrated paper, Ebin and Marsden [6] developed the analytic geometrical side of Arnold's paper. They showed that the spray of the Euler equation is smooth. They also proved that on manifold without boundary, the solutions of the Navier-Stokes equation converge to those of the Euler equations when viscosity tends to zero.

The integrable discretization of an integrable mechanical system is always a challenging problem. Recently, in a series of papers, Bobenko and Suris [2,3] discussed integrable discretization based on the discretization of variational principles. They are motivated by the work of Moser and Veselov [16] in which an integrable discretization of the N-dimensional rigid body was investigated. In particular, they studied [17] an integrable discrete-time Lagrangian system on the group of area preserving plane diffeomorphisms $SDiff(\mathbf{R}^2)$.

A further development of the general theory was undertaken in [4], where general discrete-time Euler-Poisson equations on semidirect product Lie algebras were obtained as a result of a reduction procedure , applied to discrete-time Lagrangian systems on Lie groups.

Variational principles for mechanical systems with symmetry and their applications to integration algorithms were studied by Marsden and Wendlandt [13]. The general idea for these variational integrators is to directly discretize Hamilton's principle rather than the equations of motion in a way that preserves the original system's invariants, notably the symplectic form and, via a discrete version of Noether's theorem, the momentum map. The resulting mechanical integrators are second-order accurate, implicit, symplectic-momentum algorithms. In other papers, Marsden et. al. [14,15] showed geometric-variational approach to continuous and discrete mechanics and field theories. Using multisymplectic geometry, they showed that the existence of the fundamental geometric structures as well as their preservation along solutions can be obtained directly from the variational principle. In the case of mechanics, they recover the variational symplectic integrators of Veselov type [25], while for PDEs we obtain covariant spacetime integrators which conserve the corresponding discrete multisymplectic form as well as the discrete momentum mappings corresponding to symmetries.

Following the method of Strachan's [21,22] deformation of Plebanski's first heavenly equation [3], $\{\Omega_{,p}, \Omega_{,q}\}_{\rm MB} = 1$, where Ω is an unknown Kähler potential on the space-time with coordinates $p, q, \widehat{p}, \widehat{q}$, we propose the deformation of our equations. Later Takasaki [21,22] proved the integrability of Strachan's equation using the dressing operator method.

Using the close similarities between integrable systems and their q-deformations with Moyal brackets and associative $\star$-products, a-version of the discretization procedure of R. Kemmoku and S. Saito is given in [11]. This leads to a new q-Moyal type bracket [5]. Using this scheme we propose a discrete version of 2D Euler equation of incompressible fluid, this actually yields a sort of q-version of the deformation quantization programme. We have encountered such systems in string theory also [10]. This method can be applied to any geodesic equation on area preserving diffeomorphism group [9].

2 Preliminaries

The three-dimensional incompressible Euler equation can be expressed in vorticity form,

$$\partial_t \Omega + (u \cdot \nabla)\Omega - (\Omega \cdot \nabla)u = 0, \tag{2.1}$$

where $u = (u_1, u_2, u_3)$ is the velocity, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the vorticity, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\Omega = \nabla \times u$, and $\nabla \cdot u = 0$.

Dropping the third subscript of Ω, the 2D Euler equation can be written as

$$\partial_t \Omega + \{\Omega, \Psi\} = 0, \tag{2.2}$$

where the bracket is given by $\{f, g\} = \partial_x f \partial_y g - \partial_y f \partial_x g$, and Ψ is the stream function given by

$$\Omega = \Delta \Psi.$$

The Lax pair of the 2D Euler equation [12] is given by

$$L\phi = \lambda\phi, \quad \partial_t \phi + A\phi = 0, \tag{2.3}$$

where

$$L\phi = \{\Omega, \phi\}, \quad A\phi + \{\Psi, \phi\} = 0.$$

Compatibility of equation (3) yields:

Proposition 1.

$$\partial_t L + [A, L] = \partial_t \Omega + \{\Omega, \Psi\} = 0.$$

Outline of Proof:

$$(\partial_t L)\phi = \{(\partial_t \Omega), \phi\},$$
$$[A, L]\phi = -\{\Psi, \{\Omega, \phi\}\} + \{\Omega, \{\Psi, \phi\}\}$$
$$= \text{ Jacobi identity } + \{\phi, \{\Psi, \Omega\}\}.$$

Thus the proof solely depends on the Jacobi identity. □

REMARK: The barotropic quasigeotropic flow equation on the β-plane is expressible in the form

$$\frac{\partial \Omega}{\partial t} + \{\Omega, \Psi\} - \beta \frac{\partial \Psi}{\partial x} = 0, \tag{2.4}$$

where $\Psi = \Psi(x, y)$ is the stream function which is is proportional to the pressure fluctuation and $\Omega = \Delta\Psi + k_0^2$ is the vortex density. Here Δ is the two dimensional Laplace operator, k_0^{-1} is the Rossby deformation radius and β is the lattitude derivative of the Coriolis parameter which is supposed to be constant.

The Rossby equation is equivalent to 2D Euler equation under transformations

$$\Omega \longmapsto \Omega + \beta y.$$

We can apply this scheme to Rossby equation. The Lax pair of the Rossby equation is given by

$$L\phi = \{\tilde{\Omega}, \phi\}, \quad A\phi + \{\Psi, \phi\} = 0 \quad \text{with} \quad \tilde{\Omega} = \Omega + \beta y,$$

or by change of variables

$$L\phi = \{\Omega, \phi\} - \beta \frac{\partial}{\partial \theta}, \quad A\phi + \{\Psi, \phi\} = 0.$$

EXAMPLE 2: Let us give you another example of 2D Euler type equation. It is known that geodesics on $SO(N)$ is equipped with the Manakov metric. This generalizes the "classical" integrable system of Euler's equations for a top with one fixed point, and this corresponds to $N = 3$. It was shown by Ward [26] that the limit $N \longrightarrow \infty$ corresponds to replacing $SU(N)$ by a certain Lie algebra of vector fields of the area preserving diffeomorphic group, hence the geodesic flow on $SO(\infty)$ group is given by

$$\Omega_t = \{\Omega, \Psi\}, \qquad \text{where } \Psi = \xi(x)\Omega,$$

and this leads to

$$\Omega_t = -\xi' \Omega \Omega_y.$$

The Lax pair of this equation must be same as 2D Euler equation.

3 Moyal Deformation of 2D-Euler system and Lax Pair

We will study the Moyal algebraic deformation of the geodesic flows on the area preserving diffeomorphic group. Recently Strachan introduced a (formal) Moyal algebraic algebraic deformation of self dual gravity, replacing a Poisson bracket of the Plebanski equation by a Moyal bracket [18]. Later Moyal deformation of KP and Toda hierarchies have been studied by Takasaki [23, 24].

The Poisson bivector P on $\mathbf{R}^{2n}$ determines a linear operator on the tensor square of the function space

$$\mathcal{P} \ : \ F \otimes G \longmapsto P(F \otimes G) = \sum_{i=1}^{n} (F_{x_i} \otimes G_{y_i} - F_{y_i} \otimes G_{x_i}).$$

It is known that the operator tr is the restriction to the diagonal $\mathbb{R} \hookrightarrow \mathbb{R} \times \mathbb{R}$, and hence $tr(F \otimes G) = FG$.

The Moyal product is defined by

$$F \star_\hbar G = F \, exp \left[\frac{i\hbar}{2} (\overleftarrow{\partial}_x \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \overrightarrow{\partial}_x \right] G$$

$$= e^{[\frac{i\hbar}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})]} F(x_1, y_1) G(x_2, y_2)|_{(x,y)},$$

$$= \sum_0^\infty \frac{(i\hbar)^n}{n!} \sum_0^\infty (-1)^r (\partial_x^r \partial_y^{n-r} F)(\partial_x^{n-r} \partial_y^r G),$$

where $\hbar$ is a formal parameter.

Replacing $i\hbar/2$ by κ, one obtains

$$F \star G = \sum_0^\infty \frac{\kappa^m}{m!} \sum_{i=0}^{m} (-1)^i \frac{m!}{(m-i)!i!} \frac{\partial^m F}{\partial y^{m-i} \partial x^i} \frac{\partial^m G}{\partial x^{m-i} \partial y^i}.$$

Historically this $\star$ product appeared in a seminal but somewhat unappreciated paper of H. Groenewold [8]. This paper fully understands the Weyl correspondence and produces Wigner function as the classical kernel of the density matrix.

The Moyal bracket is defined by

$$\{F, G\}_{Moyal} := \frac{F \star G - G \star F}{2\kappa}. \tag{3.1}$$

Lemma 1. The $\star_\hbar$ satisfies following properties for $F, G \in C^\infty(\mathbf{R}^{2n})$ and constant c,

1. $lim_{\hbar \to 0} F \star_\hbar G = FG$,
2. $c \star_\hbar F = cF$,
3. $\star_\hbar$ is associative,
4. $lim_{\hbar \to 0} \{F, G\}_{Moyal} = \{F, G\}_{Poisson}$,
5. $\{F, G\}_{Moyal}$ is bilinear, skew-symmetric and satisfies Jacobi identity.

3.1 Deformation equation and its Lax Pair

Following Strachan and Takasaki we deform the 2D Euler flow as

$$\frac{\partial \Omega}{\partial t} + \{\Omega, \Psi\}_{Moyal} = 0. \tag{3.2}$$

This equation is the quantum Liouville equation, for which Moyal originally introduced his bracket [18]. There are three equivalent methods of quantization: Hilbert space method, path integral approach and quantization via Wigner's phase space distribution function. We are focussing on the last method.

Lax Pair. The Lax pair of the Moyal deformed 2D Euler equation is closely related to the Lax pair of the original equation.

$$L_\hbar \phi = \lambda \phi, \quad \partial_t \phi + A_\hbar \phi = 0, \tag{3.3}$$

where

$$L_\hbar \phi = \{\Omega, \phi\}_{Moyal}, \quad A_\hbar \phi + \{\Psi, \phi\}_{Moyal} = 0.$$

Compatibility yields

Proposition 2.

$$\partial_t L_\hbar + [A, L]_\hbar = \partial_t \Omega + \{\Omega, \Psi\}_{Moyal} = 0.$$

Sketch of Proof: The Moyal deformation preserves the skew symmetric condition. □

4 Discretization of 2D Euler Flows

The operator e^∂ acts as a shift operator

$$e^{n\partial} f(x) = f(x+n). \tag{4.1}$$

We demonstrate some applications [22]. Let L be an operator

$$L = e^\lambda + u + v e^{-\lambda},$$

where λ is a parameter. The evolution equations for the fields u and v are given by

$$\frac{\partial L}{\partial t} = \{B, L\}_{Moyal},$$

where $B = e^\lambda + u$. This gives equations

$$\begin{aligned}
u_t &= \frac{2}{\kappa}[\sum_{s=0}^{\infty} \frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!}\partial_x^{2s+1}]v(x) \\
&= \frac{1}{\kappa}[v(x+\kappa/2) - v(x-\kappa/2)], \\
v_t &= \frac{2v(x)}{\kappa}[\sum_{s=0}^{\infty} \frac{\kappa^{2s+1}}{2^{2s+1}(2s+1)!}\partial_x^{2s+1}]u(x) \\
&= \frac{v(x)}{\kappa}[u(x+\kappa/2) - u(x-\kappa/2)]
\end{aligned}$$

REMARK: There exists a natural isomorphism between the space of tensor densities of degree λ on S^1, $\mathcal{F}_\lambda$, and the space of functions on $\mathbf{R}^2/\{0\}$ homogeneous of degree -2λ. For the affine parameter t, this isomorphism corresponds to

$$\phi(t)(dt)^\lambda \longmapsto b^{-2\lambda}\phi(\frac{a}{b}).$$

This isomorphism lifts the Moyal-Weyl star-product to the space of tensor densities, and this yields the Gordan's transvectant [19,20].

We wish to extend the above scheme to two variables, i.e. to the original Moyal deformation. In the case of Moyal bracket for two variables one has

$$F \star G = F(x + \kappa\partial_y, y - \kappa\partial_x)G = G(x - \kappa\partial_y, y + \kappa\partial_x)F. \tag{4.2}$$

This powerful functional $\star$-eigenvalue equation was developed by Fairlie [7]. It is known that the systematic solution of time-dependent equation quantum Liouville equation is usually predicted on the spectrum of stationary ones. But time-independent pure-state Wigner functions $\star$- commute with the Hamiltonian.

Lemma 2. The semi-discrete Moyal bracket is given by

$$\{F, G\}_{semi-discrete} = \frac{2}{\kappa}F(x + \kappa\partial_p, p - \kappa\partial_x)G - G(x + \kappa\partial_p, p - \kappa\partial_x)F$$

$$= \frac{2}{\kappa}[F(x + \kappa\partial_p, p - \kappa\partial_x) - F(x - \kappa\partial_p, p + \kappa\partial_x)]G$$

Proof. It follows from the definition (5). □

Proposition 3. The semi-discrete 2D Euler equation is given by

$$\frac{\partial\Omega}{\partial t} + \frac{2}{\kappa}[\Omega(x + \kappa\partial_p, p - \kappa\partial_x) - \Omega(x - \kappa\partial_p, p + \kappa\partial_x)]\Psi = 0 \tag{4.3}$$

5 Conclusion and outlook

In this paper we have studied the semi-discrete version of 2D Euler equation of incompressible fluid using the techniques of Moyal deformation. This method can be applied to dispersionless KP and Toda hierarchies, since both hierarchies are shown to possess a quasi-classical limit with Moyal algebra structures replaced by Poisson algebra structures.

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