

# Burchnall-Chaundy Theory for $q$ -Difference Operators and $q$ -Deformed Heisenberg Algebras

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## Abstract

This paper is devoted to an extension of Burchnall-Chaundy theory on the interplay between algebraic geometry and commuting differential operators to the case of  $q$ -difference operators.

## 1 Introduction

One of the major achievements in the theory of non-linear differential equations is the algebraic-geometric method, relating integrable non-linear differential equations and their solutions to properties of algebraic curves and algebraic manifolds. It was originally developed in 1970's in connection to the inverse scattering problem [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31], but since then it has become an area of research on its own, greatly influencing developments in algebraic geometry, non-linear equations and algebra, as well as playing an increasingly important role in many applications. This interplay between algebraic geometry and integrable non-linear equations is based on the observation that many of these equations can be formulated as conditions on the coefficients of some differential operators equivalent to the property that these operators commute. Thus the main problem becomes to describe, as detailed as possible, commuting differential operators. The solution of this problem is where algebraic geometry enters the scene. The main result responsible for this connection was obtained by Burchnall and Chaundy in the beginning of the 1920's and further explored by them in a series of papers over the following decade [2, 3, 4]. This key result states that commuting differential operators satisfy an equation for a certain algebraic curve, which can be explicitly calculated for each pair of commuting operators. This correspondence has also been discretized to classical difference operators [22, 25, 26]. However not so much has been done in this direction for  $q$ -difference operators, in spite of their widespread applications and long and colorful history. Only recently have some results appeared in the direction of integrable non-linear  $q$ -difference equations [1, 6, 7, 8, 9, 10, 17]. In [11], the key Burchnall-Chaundy type theorem for  $q$ -difference equations was obtained, where it was stated as a corollary to a more

general theorem of this type for  $q$ -deformed Heisenberg algebras. The proof in [11] is an existence argument, which can be used successfully for an algorithmic implementation for computing the corresponding algebraic curves. However, since it does not give any specific information on the structure or properties of such algebraic curves, it is desirable to have a way of describing such algebraic curves by some explicit formulae. In this article we make a step in that direction by offering a number of interesting examples, indicating the method of proof in the general case [24].

### Jackson $q$ -derivative and $q$ -difference operators

This section is devoted to ordinary  $q$ -difference operators and  $q$ -difference equations, that is to  $q$ -difference operators and  $q$ -difference equations in spaces of functions of a single variable.

In 1908 F. H. Jackson [12, 13, 14, 15, 16] reintroduced and started a systematic study of the  $q$ -difference operator

$$(D_q\varphi)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}, \quad q \neq 1, \quad (1.1)$$

which is now sometimes referred to as Euler-Jackson or Jackson  $q$ -difference operator or simply the  $q$ -derivative. This operator may be applied without any problems to any function not containing  $x = 0$  in the domain of definition. By definition, the limit as  $q$  approaches 1 is the ordinary derivative, that is

$$\lim_{q \rightarrow 1} (D_q\varphi)(x) = \frac{d\varphi}{dx}(x), \quad (1.2)$$

if  $\varphi$  is differentiable at  $x$ . The  $D_q$ -constants or multiplicatively  $q$ -periodic functions are solutions of the functional equation

$$k(qx) = k(x) \text{ or } D_qk(x) = 0. \quad (1.3)$$

These functions play in the theory of  $q$ -difference equations the role of the arbitrary constants of the differential equations.

The formulas for the  $q$ -difference of a sum of functions and of a product by a constant are [5]:

$$D_q(u(x) + v(x)) = D_qu(x) + D_qv(x), \quad (1.4)$$

$$D_q(cu(x)) = cD_qu(x). \quad (1.5)$$

So the operator  $D_q$  is linear when it acts on a linear space of functions, and the general theory of linear operators developed within linear algebra, functional analysis, operator theory and operator algebras can be applied.

The formulas for the  $q$ -difference of a product and a quotient of functions are [5]:

$$D_q(f(x)g(x)) = f(qx)D_qg(x) + D_qf(x)g(x), \quad (1.6)$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_qf(x) - f(x)D_qg(x)}{g(qx)g(x)}. \quad (1.7)$$

The usual Leibniz rule for  $q$ -derivative is recovered from (1.6) when  $q$  tends to 1.

The  $q$ -analogue of the chain rule is more complicated since it involves  $q$ -derivatives for different values of  $q$  depending on the composed functions. For example if  $g(x)$  is the function  $g(x) : x \mapsto cx^k$  and  $q^k \neq 1$ , then

$$D_q(f \circ g)(x) = (D_{q^k}(f))(g(x))D_q(g)(x). \quad (1.8)$$

The chain rule for general  $g(x)$  and  $f(x)$  is

$$D_q(f \circ g)(x) = \left( D_{\frac{g(qx)}{g(x)}}(f) \right) (g(x))D_q(g)(x) \quad (1.9)$$

If  $g(x)$  and  $f(x)$  are interpreted not as formal expressions but as functions, then this formula is true for all  $x \neq 0$  such that  $g(x) \neq 0$  and  $g(qx) \neq g(x)$ , with other points  $x$  requiring separate consideration. The general chain rule (1.9) is easily proved as follows:

$$\begin{aligned} D_q(f \circ g)(x) &= \frac{f(g(x)) - f(g(qx))}{(1-q)x} = \\ &= \frac{f(g(x)) - f\left(\frac{g(qx)}{g(x)}g(x)\right) \left(1 - \frac{g(qx)}{g(x)}\right)g(x)}{\left(1 - \frac{g(qx)}{g(x)}\right)g(x)} \frac{1}{(1-q)x} = \left( D_{\frac{g(qx)}{g(x)}}(f) \right) (g(x))D_q(g)(x). \end{aligned}$$

Strangely enough, we have not been able to find this formula and the above easy proof explicitly anywhere in the literature on  $q$ -analysis.

The general Leibniz rule for action of powers of the  $q$ -derivative operator on a product of functions is

$$D_q^n(fg)(x) = \sum_{k=0}^n \binom{n}{k}_q D_q^k(f)(xq^{n-k})D_q^{n-k}(g)(x) \quad (1.10)$$

Using the multiplicative  $q$ -shift operator  $T_q : f(x) \mapsto f(qx)$  the Leibniz rules (1.6) and (1.10) can be written as follows:

$$D_q(f(x)g(x)) = T_q f(x)D_q g(x) + D_q f(x)g(x), \quad (1.11)$$

$$D_q^n(fg)(x) = \sum_{k=0}^n \binom{n}{k}_q T_q^{n-k} D_q^k(f)(x)D_q^{n-k}(g)(x). \quad (1.12)$$

Here we have used the  $q$ -binomial coefficients defined by

$$\binom{n}{k}_q = \frac{\{n\}_q!}{\{k\}_q! \{n-k\}_q!} \quad (1.13)$$

for  $k = 0, 1, \dots, n$ , where

$$\{n\}_q = \sum_{k=1}^n q^{k-1}, \quad \{0\}_q = 0, \quad (1.14)$$

$$\{n\}_q! = \prod_{k=1}^n \{k\}_q, \quad \{0\}_q! = 1. \quad (1.15)$$

are the  $q$ -analogues of the natural numbers and the factorial function. The  $q$ -binomial coefficients  $\binom{n}{k}_q$  are polynomials in  $q$  with integer coefficients. If  $q = 1$ , then  $\{n\}_q = n$ . If  $q \neq 1$ , then  $\{n\}_q = \frac{q^n - 1}{q - 1}$ .

It can be easily checked from the definition of the  $q$ -derivative that the action of  $D_q$  on the functions  $x^s$  is given by the  $q$ -analogue of the usual rule

$$D_q(x^s) = \{s\}_q x^{s-1}.$$

The linear  $q$ -difference operators, which use Jackson  $q$ -derivative operator  $D_q$  as the generator, are sums of the form

$$P = \sum_{j=0}^n p_j D_q^j,$$

where the coefficients  $p_i$  are some functions, which we assume in this article for simplicity of exposition to be polynomials in  $x$ .

## 2 Burchnall-Chaundy type theorem for $q$ -difference operators and $q$ -deformed Heisenberg algebras

The  $q$ -deformed Heisenberg algebra for  $q \in \mathbb{C} \setminus \{0\}$  is a  $\mathbb{C}$ -algebra  $H_q$  with unit element  $I$  and generators  $A$  and  $B$  satisfying defining  $q$ -deformed Heisenberg canonical commutation relation

$$AB - qBA = I. \tag{2.1}$$

The algebra can be constructed as the quotient  $H_q = \mathbb{C}\langle A, B \rangle / (AB - qBA - I)$  of the free algebra  $\mathbb{C}\langle A, B \rangle$  by the two-sided ideal generated by  $AB - qBA - I$ .

The  $q$ -deformed Heisenberg algebra is fundamental for  $q$ -difference equations, due to the fact that the Jackson  $q$ -difference operator  $D_q$  and the operator of multiplication  $M_x : f(x) \mapsto xf(x)$  satisfy the  $q$ -deformed Heisenberg canonical commutation relation

$$D_q M_x - q M_x D_q = I. \tag{2.2}$$

Indeed,

$$\begin{aligned} (D_q M_x - q M_x D_q)(f)(x) &= \frac{xf(x) - qxf(qx)}{x(1-q)} - \frac{qxf(x) - qxf(qx)}{x(1-q)} = \\ &= \frac{xf(x) - qxf(x)}{x(1-q)} = f(x) = (If)(x) \end{aligned}$$

In the terminology of representation theory this means that the operators  $D_q$  and  $M_x$  are representatives of generators in the representation of the  $q$ -deformed Heisenberg algebra  $H_q$ . Any pair, or more generally, a set of linear operators satisfying some commutation relations is also called a representation of these commutation relations. So, the pair  $(D_q, M_x)$  is a representation of the  $q$ -deformed Heisenberg commutation relation (2.1). Any algebraic identity which holds in  $H_q$  results in the corresponding identity for the operators  $D_q$  and  $M_x$ , thus having an impact on the related  $q$ -difference equations.

Using the defining commutation relations (2.1) it can be checked that

$$\begin{aligned} B^2 A^2 &= q^{-1} BA(BA - I) \\ B^3 A^3 &= q^{-3} BA(BA - I)(BA - (q + 1)I). \end{aligned}$$

An inductive argument gives

$$B^n A^n = q^{-\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} (BA - (\sum_{k=0}^{j-1} q^k)I) = q^{-\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} (BA - \{j\}_q I), \quad (2.3)$$

Using this we see, for example, that

$$\begin{aligned} B^4 A^4 &= q^{-6} BA(BA - I)(BA - (q + 1)I)(BA - (q^2 + q + 1)I) = \\ &= q^{-6} BA(BA - \{1\}_q I)(BA - \{2\}_q I)(BA - \{3\}_q I). \end{aligned}$$

Using the equality (2.3) we get the following very useful fact.

**Lemma 1.** *In the  $q$ -deformed Heisenberg algebra  $H_q$ , all monomials and linear combinations of monomials of the form  $B^n A^n$  commute with each other.*

The following theorem is a generalization of the Burchnall-Chaundy theorem to  $q$ -deformed Heisenberg algebras. The general algebraic way this result is stated is important because then the property becomes universal in its nature, being consequently applicable not only to  $q$ -difference operators arising from the specific representation  $(D_q, M_x)$ , but also to any other class of operators associated to any other representation of the  $q$ -deformed Heisenberg canonical commutation relation.

**Theorem 1.** (*L. Hellström, S. D. Silvestrov [11]*) *If  $P, Q \in H_q$  commute, that is satisfy  $PQ = QP$ , then there exists a nonzero polynomial  $F$  in two commutative variables with coefficients from the center of  $H_q$  such that  $F(P, Q) = 0$  in  $H_q$ .*

The center of  $H_q$  is the set of elements in  $H_q$  commuting with any element in  $H_q$ . If  $q$  is not a root of unity or if  $q = 1$ , then the center of  $H_q$  is trivial, that is consisting only from the elements of the form  $\lambda I$ ,  $\lambda \in \mathbb{C}$ . So in this case, to any pair of commuting elements in  $H_q$ , one can associate an algebraic curve in  $\mathbb{C}^2$  given by the corresponding polynomial with complex coefficients, the existence of which is stated in Theorem 1. In the case when  $q$  is a root of unity but not 1, the center of  $H_q$  is the subalgebra generated by  $A^p$  and  $B^p$  where  $p$  is the smallest positive integer such that  $q^p = 1$ . The coefficients in the polynomial from the theorem are some elements of this commutative subalgebra. They might be the elements of the form  $\lambda I$ , and in this case we again would get an ordinary algebraic curve. But if they are not, then we get an algebraic curve with coefficients, which are not scalars but polynomials in  $A^p$  and  $B^p$ . However, in a particular representation we might still get ordinary algebraic curves as some or all central elements might become, for example, scalar multiples of the identity operator.

The proof of Theorem 1 given in [11] is an existence proof based on dimension growth arguments. Though it can be used for construction of an algorithm for computing the annihilating polynomials for the given pair of commuting elements in  $H_q$ , it does not give

any specific formula for such polynomials that could lead to a better understanding of their structure and of the interplay with algebraic geometry.

In the classical case of differential operators, that is in the case of  $H_1$ , there is a way to construct the annihilating algebraic curves via determinants, going back to Burchall and Chaundy [2, 3, 4]. Extension of this construction to  $q$ -difference operators, or more generally to  $q$ -deformed Heisenberg algebras can be done, but due to restrictions on the size of this article, it will be presented in its full extent in [24]. Here we would like instead to give some new interesting examples strongly indicating that the determinants work well also in the  $q$ -deformed case. These examples also provide a good illustration for the general method.

Before we turn to the examples let us first review the classical Burchall-Chaundy construction for differential operators. Let  $P = \sum_{i=0}^n p_i(x)D^i$  and  $Q = \sum_{i=0}^m q_i(x)D^i$  be two differential operators of degree  $n$  and  $m$  respectively, where functions  $p_i(x)$  and  $q_i(x)$  are analytic in their common domain of definition, or just formal power series, or as in all examples in this article, polynomials in  $x$  with coefficients in  $\mathbb{C}$ . The original Burchall-Chaundy theorem states, informally, that two commuting differential operators  $P$  and  $Q$  lie on an algebraic curve, in the sense that they are annihilated by a polynomial in two variables after being substituted for the variables. One of the first consequences of this is that the eigenvalues, corresponding to a joint eigenfunction of the two operators, are coordinates of a point on that curve. There are also other deeper connections of the properties of the algebraic curve to properties of the solutions of the equations associated to these operators, for example the non-linear differential equations in the coefficient functions, obtained from commutativity of those operators [18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 29, 30, 31].

The original proof of Burchall-Chaundy theorem depends heavily on the existence of solutions of boundary value problems for ordinary differential equations, making a simple adaption of it to  $q$ -difference operators problematic. A nice feature of the proof in the differential operator case, however, is that it is constructive in the sense that it actually tells us how to compute such an annihilating curve, given the commuting operators. This is done by constructing the **resultant** (or **eliminant**) of operators  $P$  and  $Q$ . We sketch this construction, as it is important to have in mind for this article. The following row-scheme is a first stepping-stone:

$$D^k(P - sI) = \sum_{i=0}^{n+k} \theta_{i,k} D^i - sD^k, \quad k = 0, 1, \dots, m-1 \quad (2.4)$$

$$D^k(Q - tI) = \sum_{i=0}^{m+k} \omega_{i,k} D^i - tD^k, \quad k = 0, 1, \dots, n-1 \quad (2.5)$$

where  $\theta_{i,k}$  and  $\omega_{i,k}$  are certain functions built from the coefficients of  $P$  and  $Q$  respectively, whose exact form is calculated by moving  $D^k$  through to the right of the coefficients, using Leibniz rule. The coefficients of the powers of  $D$  on the right hand side in (2.4) and (2.5) build up the rows of a matrix exactly as written. That is, as the first row we take the coefficients in  $\sum_{i=0}^n \theta_{i,0} D^i - sD^0$ , and as the second row – the coefficients in  $\sum_{i=0}^{n+1} \theta_{i,1} D^i - sD$ , continuing this until  $k = m-1$ . As the  $m$ th row we take the coefficients in  $\sum_{i=0}^m \omega_{i,0} D^i - tD^0$ , and as the  $(m+1)$ th row we take the coefficients in  $\sum_{i=0}^{m+1} \omega_{i,1} D^i - tD$  and so on. In this manner we get a  $(m+n) \times (m+n)$ -matrix using

(2.4) and (2.5). The determinant of this matrix yields a bivariate polynomial  $F(s, t)$  in  $s$  and  $t$  over  $\mathbb{C}$  (sometimes called the **Burchnall-Chaundy polynomial**), defining an algebraic curve  $F(s, t) = 0$ , and annihilating  $P$  and  $Q$  when putting  $s = P$  and  $t = Q$ .

Now, generalizing this idea to the case of  $q$ -difference operators, we indicate with a number of examples, that it is in fact a sound construction even in the  $q$ -deformed case. Respecting the limitations of this article, a rigorous general proof of this will, as we mentioned above, have to stay aside until [24] for the benefit of some interesting examples to which we now turn.

**Example 2.1.** We take  $P = M_x^3 D_q^3$  and  $Q = M_x^2 D_q^2$ . Then the following formulae hold:

$$\begin{aligned} D_q^0(P - sI) &= -sI + M_x^3 D_q^3, \\ D_q(P - sI) &= -sD_q + \{3\}_q M_x^2 D_q^3 + q^3 M_x^3 D_q^4, \\ D_q^0(Q - tI) &= -tI + M_x^2 D_q^2, \\ D_q(Q - tI) &= -tD_q + \{2\}_q M_x D_q^2 + q^2 M_x^2 D_q^3, \\ D_q^2(Q - tI) &= -tD_q^2 + \{2\}_q D_q^2 + (q\{2\}_q + q^2\{2\}_q) M_x D_q^3 + q^4 M_x^2 D_q^4 = \\ &= (\{2\}_q - t) D_q^2 + q\{2\}_q^2 M_x D_q^3 + q^4 M_x^2 D_q^4. \end{aligned}$$

The coefficients in front of the powers of  $D_q$  in these equalities can be placed in an operator matrix with the determinant

$$\begin{vmatrix} -s & 0 & 0 & M_x^3 & 0 \\ 0 & -s & 0 & \{3\}_q M_x^2 & q^3 M_x^3 \\ -t & 0 & M_x^2 & 0 & 0 \\ 0 & -t & \{2\}_q M_x & q^2 M_x^2 & 0 \\ 0 & 0 & \{2\}_q - t & q\{2\}_q^2 M_x & q^4 M_x^2 \end{vmatrix}.$$

Expanding this we get

$$q^3(q^3 s^2 + q(2q + 1)st + \{2\}_q t^2 - t^3) M_x^6,$$

which gives us the curve

$$F(s, t) = q^3 s^2 + q(2q + 1)st + \{2\}_q t^2 - t^3 = 0. \quad (2.6)$$

We now show that  $P$  and  $Q$  satisfy  $F(P, Q) = 0$ . To this end we use (2.3). So we return momentarily to the notation  $M_x = B$  and  $D_q = A$ . This by the way shows that the fact remains true even more generally in  $H_q$ , not just for  $q$ -difference operators. So taking  $s = P$  and  $t = Q$  we have

$$\begin{aligned} s^2 &= (B^3 A^3)^2 = [q^{-3} B A (B A - I) (B A - (q + 1) I)]^2 = \\ &= q^{-6} (B A)^2 (B A - I)^2 (B A - (q + 1) I)^2 = \\ &= q^{-6} ((B A)^6 - 2(q + 2)(B A)^5 + (q^2 + 6q + 6)(B A)^4 - \\ &\quad - 2(q^2 + 3q + 2)(B A)^3 + (q + 1)^2 (B A)^2) \end{aligned}$$

In a similar fashion we get

$$\begin{aligned} st &= q^{-3}BA(BA - I)(BA - (q + 1)I) \cdot q^{-1}BA(BA - I) = \\ &= q^{-4}((BA)^5 - (q + 3)(BA)^4 + (2q + 3)(BA)^3 - (q + 1)(BA)^2), \\ t^2 &= q^{-2}(BA)^2(BA - I)^2 = q^{-2}((BA)^4 - 2(BA)^3 + (BA)^2) \end{aligned}$$

and, finally,

$$t^3 = q^{-3}(BA)^3(BA - I)^3 = q^{-3}((BA)^6 - 3(BA)^5 + 3(BA)^4 - (BA)^3).$$

Insertion of the above relations into (2.6) gives

$$\begin{aligned} F(P, Q) &= q^3 \left( q^{-6}((BA)^6 - 2(q + 2)(BA)^5 + (q^2 + 6q + 6)(BA)^4 + \right. \\ &\quad \left. - 2(q^2 + 3q + 2)(BA)^3 + (q + 1)^2(BA)^2) \right) + q(2q + 1) \left( q^{-4}((BA)^5 - \right. \\ &\quad \left. - (q + 3)(BA)^4 + (2q + 3)(BA)^3 - (q + 1)(BA)^2) \right) + \\ &\quad + \{2\}_q \left( q^{-2}((BA)^4 - 2(BA)^3 + (BA)^2) \right) - \\ &\quad - \left( q^{-3}((BA)^6 - 3(BA)^5 + 3(BA)^4 - (BA)^3) \right) = [\text{collecting terms}] = \\ &= (q^{-3} - q^{-3})(BA)^6 + q^{-3}(-2(q + 2) + 2q + 1 + 3)(BA)^5 + \\ &\quad + q^{-3}((q^2 + 6q + 6) - (2q + 1)(q + 3) + q(q + 1) - 3)(BA)^4 + \\ &\quad + q^{-3}(-2(q^2 + 3q + 2) + (2q + 1)(2q + 3) - 2q(q + 1) + 1)(BA)^3 + \\ &\quad + q^3((q + 1)^2 - (2q + 1)(q + 1) + q(q + 1))(BA)^2 = \\ &= 0 \end{aligned}$$

as the coefficients in front of all powers of  $BA$  vanish.

In the special classical case of differential operators when  $q = 1$ , for  $P = M_x^3 D^3$ ,  $Q = M_x^2 D^2$  we get the following equalities:

$$\begin{aligned} D^0(P - sI) &= -sI + M_x^3 D^3, \\ D(P - sI) &= -sD + 3M_x^2 D^3 + M_x^3 D^4, \\ D^0(Q - tI) &= -tI + M_x^2 D^2, \\ D(Q - tI) &= -tD + 2M_x D^2 + M_x^2 D^3, \\ D^2(Q - tI) &= (2 - t)D^2 + 4M_x D^3 + M_x^2 D^4, \end{aligned}$$

which yield the following determinant

$$\begin{vmatrix} -s & 0 & 0 & M_x^3 & 0 \\ 0 & -s & 0 & 3M_x^2 & M_x^3 \\ -t & 0 & M_x^2 & 0 & 0 \\ 0 & -t & 2M_x & M_x^2 & 0 \\ 0 & 0 & 2 - t & 4M_x & M_x^2 \end{vmatrix}.$$

Expanding this determinant results in  $(s^2 + 3st + 2t^2 - t^3)M_x^6$ , and equating it to zero we get the classical Burchnell-Chaundy curve

$$s^2 + 3st + 2t^2 - t^3 = 0.$$



Note that this curve can be obtained also within the family of algebraic curves (2.6) by taking the parameter value  $q = 1$ . ■

**Example 2.2.** Suppose now we have  $P = M_x^4 D_q^4$  and  $Q = M_x^3 D_q^3$ . Similarly to the previous example we get the following determinant to compute:

$$\begin{vmatrix} -s & 0 & 0 & 0 & M_x^4 & 0 & 0 \\ 0 & -s & 0 & 0 & \{4\}_q M_x^3 & q^4 M_x^4 & 0 \\ 0 & 0 & -s & 0 & \{4\}_q \{3\}_q M_x^2 & q^3 \{2\}_q \{4\}_q M_x^3 & q^8 M_x^4 \\ -t & 0 & 0 & M_x^3 & 0 & 0 & 0 \\ 0 & -t & 0 & \{3\}_q M_x^2 & q^3 M_x^3 & 0 & 0 \\ 0 & 0 & -t & \{3\}_q \{2\}_q M_x & q^2 \{3\}_q M_x^2 & q^6 M_x^3 & 0 \\ 0 & 0 & 0 & \{3\}_q! - t & q \{3\}_q^2 \{2\}_q M_x & q^4 \{3\}_q^2 M_x^2 & q^9 M_x^3 \end{vmatrix}.$$

Doing the determinant computation gives us, after suitable simplifications,

$$\begin{aligned} F(s, t) = & t^4 - (q^3 + 2q^2 + 2q + 1)t^3 - q^6 s^3 - q(3q^3 + 4q^2 + 3q + 1)st^2 - \\ & - q^3(3q^2 + 2q + 1)s^2 t = 0. \end{aligned} \quad (2.7)$$

Let us now look at the corresponding classical case of differential operators. We take  $P = M_x^4 D^4$  and  $Q = M_x^3 D^3$ . We get the following determinant:

$$\begin{vmatrix} -s & 0 & 0 & 0 & M_x^4 & 0 & 0 \\ 0 & -s & 0 & 0 & 4M_x^3 & M_x^4 & 0 \\ 0 & 0 & -s & 0 & 12M_x^2 & 8M_x^3 & M_x^4 \\ -t & 0 & 0 & M_x^3 & 0 & 0 & 0 \\ 0 & -t & 0 & 3M_x^2 & M_x^3 & 0 & 0 \\ 0 & 0 & -t & 6M_x & 6M_x^2 & M_x^3 & 0 \\ 0 & 0 & 0 & 6 - t & 18M_x & 9M_x^2 & M_x^3 \end{vmatrix}.$$

When this is expanded and equated to zero we get the curve

$$t^4 - 6t^3 - s^3 - 11st^2 - 6s^2 t = 0. \quad (2.8)$$

To calm our fears, note that again when  $q = 1$  we get (2.8) from the equation (2.7). Furthermore, we can see, although with considerably more effort than in the last case, that  $F(P, Q)$  vanishes identically. ■

**Example 2.3.** Let us consider now a more complicated situation of non-monomial operators  $P = M_x^2 D_q^2 + M_x^3 D_q^3$  and  $Q = M_x D_q + M_x^2 D_q^2$ . The corresponding determinant becomes

$$\begin{vmatrix} -s & 0 & M_x^2 & M_x^3 & 0 \\ 0 & -s & \{2\}_q M_x & (\{3\}_q + q^2) M_x^2 & q^3 M_x^3 \\ -t & M_x & M_x^2 & 0 & 0 \\ 0 & 1 - t & (\{2\}_q + q) M_x & q^2 M_x^2 & 0 \\ 0 & 0 & 2\{2\}_q - t & q(\{2\}_q^2 + q) M_x & q^4 M_x^2 \end{vmatrix},$$

which gives an algebraic curve when the determinant is expanded

$$\begin{aligned} F(s, t) = & -t^3 + (q^3 - 3q^2 + q + 3)t^2 + q(5q - 2q^2 + 1)st - \\ & - (q - 2)(q^2 - q - 1)t + q^3 s^2 - q^2(q - 1)(q - 2)s = 0. \end{aligned} \quad (2.9)$$

Now, let us take the classical case of differential operators  $P = M_x^2 D^2 + M_x^3 D^3$  and  $Q = M_x D + M_x^2 D^2$ . The corresponding determinant now becomes

$$\begin{vmatrix} -s & 0 & M_x^2 & M_x^3 & 0 \\ 0 & -s & 2M_x & 4M_x^2 & M_x^3 \\ -t & M_x & M_x^2 & 0 & 0 \\ 0 & 1-t & 3M_x & M_x^2 & 0 \\ 0 & 0 & 4-t & 5M_x & M_x^2 \end{vmatrix},$$

which gives after expanding the determinant the following algebraic curve:

$$s^2 + 4st - t + 2t^2 - t^3 = 0.$$

We get this curve also by letting  $q \rightarrow 1$  in (2.9). So we have consistency. However, notice that there appears a new term in the defining equation for the  $q$ -deformed curve that does not manifest itself in the ordinary differential case. The disappearance of terms happens here also for other values of  $q$ , such as  $\frac{1+\sqrt{5}}{2}$  and  $\frac{1-\sqrt{5}}{2}$ .

Let us check that  $F(P, Q) = 0$  even in this case. Using (2.3) we see that insertion of

$$\begin{aligned} P &= B^3 A^3 + B^2 A^2 = q^{-3} BA(BA - I)(BA - \{2\}I) + q^{-1} BA(BA - I) = \\ &= q^{-3} BA(BA - I)(BA - (q + 1 - q^2)I), \\ Q &= B^2 A^2 + BA = q^{-1} BA(BA - I) + BA = q^{-1} BA(BA - (1 - q)I) \end{aligned}$$

into (2.9) gives

$$\begin{aligned} F(P, Q) &= q^{-3} (BA)^2 (BA - I)^2 (BA - (q + 1 - q^2)I)^2 - \\ &\quad - q^{-3} (-1 - 5q + 2q^2) (BA)^2 (BA - (1 - q)I) (BA - 1) (BA - (q + 1 - q^2)I) - \\ &\quad - q^{-1} (q - 1)(q - 2) BA (BA - I) (BA - (q + 1 - q^2)I) - \\ &\quad - q^{-3} (BA)^3 (BA - (1 - q)I)^3 + \\ &\quad + q^{-2} (-3q^2 + q + 3 + q^3) (BA)^2 (BA - (1 - q)I)^2 - \\ &\quad - q^{-1} (q - 2)(q^2 - q - 1) BA (BA - (1 - q)I). \end{aligned}$$

Expanding this and collecting exponents of  $BA$  shows that  $F(P, Q)$  does indeed vanish identically. ■

We would be very interested to get an answer to the following question. Is it true that the genus of the Burchall-Chaundy curves resulting from operators  $M_x^n D_q^n$  and  $M_x^m D_q^m$  (and linear combination of these) is always zero irrespective of the value of  $q \in \mathbb{C} \setminus \{0\}$ ?

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