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An Integrable Coupled Toda Equation And Its Related Equation Via Hirota's Bilinear Approach

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Abstract

A coupled Toda equation and its related equation are derived from 3-coupled bilinear equations. The corresponding Bäcklund transformation and nonlinear superposition formula are presented for the 3-coupled bilinear equations. As an application of the results, solition solutions are derived. Besides, starting from the bilinear BT, Lax pairs for these two differential-difference systems are obtained.

1 Introduction

The search for new integrable equations is a difficult and challenging problem in soliton theory. Recently, several new integrable lattices have been found (see, e.g. [1, 2, 3]) by considering the following generalized bilinear equation

$$F(D_x, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n) \cdot f(n) = 0$$

or coupled generalized bilinear equations

$$F_1(D_x, D_z, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n) \cdot f(n) = 0,$$
$$F_2(D_x, D_z, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n) \cdot f(n) = 0,$$

where F and $F_i (i = 1, 2)$ are even order polynomials in $D_x, D_z, \sinh(\alpha_1 D_n), \cdots$ and $\sinh(\alpha_l D_n)$, and l is a given positive integer; the $\alpha_i, i = 1, 2, \cdots, l$, are l different constants, and

$$F_i(0, 0, \cdots, 0) = 0.$$

Here the Hirota's bilinear differential operator $D_y^m D_t^k$ and the bilinear difference operator $\exp(\delta D_n)$ are defined by [4, 5, 6]

$$D_y^m D_t^k \ a \cdot b \equiv \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k a(y, t)b(y', t')|_{y'=y, t'=t},$$

$$\exp(\delta D_n)a(n)\cdot b(n) \equiv \exp\left[\delta\left(\frac{\partial}{\partial n}-\frac{\partial}{\partial n'}\right)\right] a(n)b(n')|_{n'=n} = a(n+\delta)b(n-\delta).$$

Now it is quite natural for us to search for new integrable systems by further considering 3-coupled generalized bilinear equations

$$F_1(D_x, D_y, D_z, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n)\cdot f(n) = 0, \tag{1.1}$$

$$F_2(D_x, D_y, D_z, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n)\cdot f(n) = 0, \tag{1.2}$$

$$F_3(D_x, D_y, D_z, \sinh(\alpha_1 D_n), \cdots, \sinh(\alpha_l D_n))f(n)\cdot f(n) = 0. \tag{1.3}$$

We could search for new integrable systems of the type (1.1)-(1.3) via the following steps.

Firstly, following [1], we seek bilinear forms $F_1 f\cdot f = 0, F_2 f\cdot f = 0$ and $F_3 f\cdot f = 0$ individually by testing Bäcklund transformations. If a Bäcklund transformation for $F_1 f\cdot f = 0$ is compatible with Bäcklund transformations for $F_2 f\cdot f = 0, F_3 f\cdot f = 0$ respectively, then this coupled system is also integrable.

It is remarked that, generally speaking, the first step in this procedure is highly technical and involves a lot of guess work although description of the procedure looks simple. Therefore the procedure is not algorithmic. However, for some simple cases, we do try this procedure to seek new integrable candidates of the type (1.1)-(1.3). In practice, we usually try to search for suitable bilinear equations $F_i f\cdot f = 0, (i = 1, 2, 3)$ firstly by testing 3-soliton or 4-soliton solutions, which is comparatively easier to do than by testing Bäcklund transformations. If successful, then we further to derive as many bilinear operator identities as we need. Finally we try to find out their possible Bäcklund transformations.

The purpose of this paper is to report the following two new integrable differential-difference systems found in this way:

$$\begin{aligned} &u_{yyy}(n+1) + u_{yyy}(n) + 2(u_y(n+1) - u_y(n))(u_{yy}(n+1) - u_{yy}(n)) \\ &\quad = e^{u(n+2)+u(n)-2u(n+1)} \int^y (e^{u(n+3)+u(n+1)-2u(n+2)} - e^{u(n+1)+u(n-1)-2u(n)})\, dy' \\ &\quad\quad - e^{u(n+1)+u(n-1)-2u(n)} \int^y (e^{u(n+2)+u(n)-2u(n+1)} - e^{u(n)+u(n-2)-2u(n-1)})\, dy', \end{aligned} \tag{1.4}$$

$$\begin{aligned} U_{xx}(n) = {} & V(n+1)W(n+1)e^{U(n+1)} + V(n-1)W(n-1)e^{U(n-1)} \\ & -2V(n)W(n)e^{U(n)} + e^{U(n+2)+U(n+1)} - e^{U(n+1)+U(n)} \\ & -e^{U(n)+U(n-1)} + e^{U(n-1)+U(n-2)}, \end{aligned} \tag{1.5}$$

$$V_x(n) = W(n+1)e^{U(n+1)} - W(n-1)e^{U(n-1)}, \tag{1.6}$$

$$W_x(n) = V(n+1)e^{U(n+1)} - V(n-1)e^{U(n-1)}. \tag{1.7}$$

In fact, these two new systems are obtained from the following 3-coupled bilinear system

$$(D_x D_y - 2D_z e^{D_n})f(n)\cdot f(n) = 0, \tag{1.8}$$

$$D_y D_z f(n)\cdot f(n) = (2e^{D_n} - 2)f(n)\cdot f(n), \tag{1.9}$$

$$(D_x e^{\frac{1}{2}D_n} - D_y^2 e^{\frac{1}{2}D_n})f(n)\cdot f(n) = 0. \tag{1.10}$$

The y-flow (1.4) can be derived from the system (1.8)-(1.10) by the dependent variable transformation

$$u(n) = \ln f(n), \tag{1.11}$$

where x and z appearing in (1.8)-(1.10) are viewed as two auxiliary variables. On the other hand, the system (1.5)-(1.7) can be deduced from the system (1.8)-(1.10) by the dependent variable transformation

$$\begin{aligned} U(n) &= \ln \frac{f(n+1)f(n-1)}{f^2(n)}, \ V(n) = \frac{D_y f(n+1) \cdot f(n-1)}{f(n+1)f(n-1)} \\ W(n) &= \frac{D_z f(n+1) \cdot f(n-1)}{f(n+1)f(n-1)} \end{aligned} \tag{1.12}$$

where y and z appearing in (1.8)-(1.10) are viewed as two auxiliary variables. It is noted that if $V(n) = W(n) = 0$, then the system (1.5)-(1.7) is reduced to the Toda lattice [7]

$$Q_{xx}(n) = e^{Q(n+2)-Q(n)} - e^{Q(n)-Q(n-2)}, \tag{1.13}$$

where $U(n) = Q(n) - Q(n-1)$. Therefore we may call the system (1.5)-(1.7) the coupled Toda equation. Besides, by the dependent variable transformation

$$u(n) = \ln f(n), \ W(n) = \frac{D_y f(n+1) \cdot f(n-1)}{f(n+1)f(n-1)}, \tag{1.14}$$

we can produce from the 3-coupled system (1.8)-(1.10) the z-flow:

$$\begin{aligned} & u_{zzz}(n+1) - u_{zzz}(n-1) + \{[u_z(n+1) \\ & \qquad -u_z(n-1)][u_z(n+1) + u_z(n-1) - 2u_z(n)]\}_z \\ & \quad = W(n+1)e^{u(n+2)+u(n)-2u(n+1)} + W(n-1)e^{u(n)+u(n-2)-2u(n-1)} \\ & \qquad -2W(n)e^{u(n+1)+u(n-1)-2u(n)}, \end{aligned} \tag{1.15}$$

$$W_z(n) = e^{u(n+2)+u(n)-2u(n+1)} - e^{u(n)+u(n-2)-2u(n-1)} \tag{1.16}$$

which is equivalent to the y-flow (1.4) under the transformation $z \to y$.

The paper is organized as follows. In the next section, we will present a bilinear Bäcklund transformation for equations (1.8)-(1.10). Then in section 3, we give a nonlinear superposition formula. Soliton solutions of equations (1.8)-(1.10) are then found through this formula. Section 4 is devoted to deriving Lax pairs for equation (1.4) and system (1.5)-(1.7) respectively. The conclusion and discussion are given in section 5.

2 A Bäcklund transformation for the system (1.8)-(1.10)

In this section, we shall derive a Bäcklund transformation for the system (1.8)-(1.10). The results obtained are:

Proposition 1. *The bilinear system (1.8)-(1.10) has the following Bäcklund transformation:*

$$(D_z + \lambda^{-1}e^{-D_n} + \mu)f(n)\cdot g(n) = 0, \tag{2.1}$$

$$(D_y e^{-\frac{1}{2}D_n} - \lambda e^{\frac{1}{2}D_n} + \gamma e^{-\frac{1}{2}D_n})f(n)\cdot g(n) = 0, \tag{2.2}$$

$$\left(\frac{1}{\lambda}D_z e^{-D_n} + \frac{\mu}{\lambda}e^{-D_n} - D_x + k\right)f(n)\cdot g(n) = 0, \tag{2.3}$$

$$(D_x - D_y^2 - 2\gamma D_y + \theta)f(n)\cdot g(n) = 0, \tag{2.4}$$

where λ, μ, γ, k and θ are arbitrary constants.

Proof. Let $f(n)$ be a solution of the system (1.8)-(1.10). What we need to prove is that the function $g(n)$ satisfying (2.1)-(2.4) is another solution of the system (1.8)-(1.10), i.e.,

$$P_1 \equiv (D_xD_y - 2D_ze^{D_n})g(n)\cdot g(n) = 0,$$
$$P_2 \equiv (D_yD_z - 2e^{D_n} + 2)g(n)\cdot g(n) = 0,$$
$$P_3 \equiv (D_xe^{\frac{1}{2}D_n} - D_y^2e^{\frac{1}{2}D_n})g(n)\cdot g(n) = 0.$$

In fact, in analogy with the proof already given in [8, 9], we know that $P_i = 0 (i = 1, 2)$ can be proved by using equations (2.1)-(2.3). Thus it suffices to show that $P_3 = 0$. In this regard, by using (A1)-(A5), we have

$$\begin{aligned}
&-P_3[e^{\frac{1}{2}D_n}f(n)\cdot f(n)] = [(D_xe^{\frac{1}{2}D_n} - D_y^2e^{\frac{1}{2}D_n})f(n)\cdot f(n)][e^{\frac{1}{2}D_n}g(n)\cdot g(n)]\\
&\quad -[(D_xe^{\frac{1}{2}D_n} - D_y^2e^{\frac{1}{2}D_n})g(n)\cdot g(n)][e^{\frac{1}{2}D_n}f(n)\cdot f(n)]\\
&= 2\sinh(\frac{1}{2}D_n)(D_xf(n)\cdot g(n))\cdot f(n)g(n)\\
&\quad -D_y[(D_ye^{\frac{1}{2}D_n}f(n)\cdot g(n))\cdot(e^{-\frac{1}{2}D_n}f(n)\cdot g(n))\\
&\quad -(e^{\frac{1}{2}D_n}f(n)\cdot g(n))\cdot(D_ye^{-\frac{1}{2}D_n}f(n)\cdot g(n))]\\
&= 2\sinh(\frac{1}{2}D_n)(D_xf(n)\cdot g(n))\cdot f(n)g(n) - 2\sinh(\frac{1}{2}D_n)(D_y^2f(n)\cdot g(n))\cdot f(n)g(n)\\
&\quad +2D_y(e^{\frac{1}{2}D_n}f(n)\cdot g(n))\cdot(D_ye^{-\frac{1}{2}D_n}f(n)\cdot g(n))\\
&= 2\sinh(\frac{1}{2}D_n)[(D_x - D_y^2)f(n)\cdot g(n)]\cdot f(n)g(n)\\
&\quad -2D_y(e^{\frac{1}{2}D_n}f(n)\cdot g(n))\cdot(\gamma e^{-\frac{1}{2}D_n}f(n)\cdot g(n))\\
&= 2\sinh(\frac{1}{2}D_n)[(D_x - D_y^2 - 2\gamma D_y)f(n)\cdot g(n)]\cdot f(n)g(n)\\
&= 0.
\end{aligned}$$

Thus we have completed the proof of Proposition 1

Using (2.1)-(2.4), we can easily obtain the following solution from the trivial solution $f(n) = 1$:

$$g(n) = 1 + \exp(\eta),$$

where $\eta = pn + \lambda^2(1 - e^{-2p})x + \lambda(1 - e^{-p})y + \lambda^{-1}(e^p - 1)z + \eta^0$ and $\mu = -\lambda^{-1}, \gamma = \lambda, k = \lambda^{-2}, \theta = 0$ and $\lambda = \pm e^{\frac{1}{2}p}$.

3 A nonlinear superposition formula

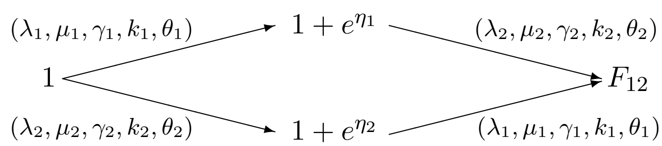
In the following, we shall simply denote, without confusion, $f(n,x,y,z) = f(n)$ or f. The results reached are given by

Proposition 2. *Let f_0 be a solution of eqns.(1.8)-(1.10). Suppose that $f_i (i = 1,2)$ are solutions of eqns.(1.8)-(1.10), which are related to f_0 under the BT eqns.(2.1)-(2.4) with parameters $(\lambda_i, \mu_i, \gamma_i, k_i, \theta_i)$, i.e., $f_0 \overset{(\lambda_i,\mu_i,\gamma_i,k_i,\theta_i)}{\longrightarrow} f_i$ $(i = 1,2)$, $\lambda_1\lambda_2 \neq 0, f_j \neq 0$ $(j = 0,1,2)$. Then f_{12} defined by*

$$\exp\left(-\frac{1}{2}D_n\right) f_0 \cdot f_{12} = c\left[\lambda_1 \exp\left(-\frac{1}{2}D_n\right) - \lambda_2 \exp\left(\frac{1}{2}D_n\right)\right] f_1 \cdot f_2 \tag{3.1}$$

is a new solution related to f_1 and f_2 under the BT (2.1)-(2.4) with parameters $(\lambda_2, \mu_2, \gamma_2, k_2, \theta_2)$, $(\lambda_1, \mu_1, \gamma_1, k_1, \theta_1)$ respectively. Here c is a nonzero constant.

This result can be proved by using Hirota's bilinear operator identities. We omit the details of the proof. Instead we are going to construct soliton solutions of the system (1.8)-(1.10). Choose, for example, $f_0 = 1, c = 1/(\lambda_1 - \lambda_2)$. It can be easily verified that



where

$$F_{12} = 1 + \frac{\lambda_1 e^{-p_1} - \lambda_2}{\lambda_1 - \lambda_2} e^{\eta_1} + \frac{\lambda_1 - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_2} + \frac{\lambda_1 e^{-p_1} - \lambda_2 e^{-p_2}}{\lambda_1 - \lambda_2} e^{\eta_1 + \eta_2}, \tag{3.2}$$

with

$$\eta_i = p_i n + \lambda_i^2 (1 - e^{-2p_i}) x + \lambda_i (1 - e^{-p_i}) y + \lambda_i^{-1} (e^{p_i} - 1) z + \eta_i^0,$$
$$\mu_i = -\lambda_i^{-1}, \gamma_i = \lambda_i, k_i = \lambda_i^{-2}, \theta_i = 0, \lambda_i = \pm e^{\frac{1}{2}p_i}.$$

In general, along this line, we can obtain multisoliton solutions for the system (1.8)-(1.10) step by step. In fact, by using BT (2.1)-(2.4) and nonlinear superposition formula (3.1), we can derive a determinantal representation of N-soliton solution given by

$$\begin{vmatrix} 1+e^{\eta_1} & (-\partial_y + \lambda_1)(1+e^{\eta_1}) & \cdots & (-\partial_y + \lambda_1)^{N-1}(1+e^{\eta_1}) \\ 1+e^{\eta_2} & (-\partial_y + \lambda_2)(1+e^{\eta_2}) & \cdots & (-\partial_y + \lambda_2)^{N-1}(1+e^{\eta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ 1+e^{\eta_N} & (-\partial_y + \lambda_N)(1+e^{\eta_N}) & \cdots & (-\partial_y + \lambda_N)^{N-1}(1+e^{\eta_N}) \end{vmatrix}.$$

4 Lax pairs for systems (1.4) and (1.5)-(1.7)

In this section, we shall derive Lax pairs for (1.4) and (1.5)-(1.7) respectively. Firstly, set

$$\psi_n = f(n)/g(n), u(n) = \ln g(n).$$

Then, from the bilinear BT (2.1)-(2.4) and after some calculations, we can obtain the following Lax pair for (1.4):

$$\psi_{n,y} + (u_y(n) - u_y(n+1))\psi_n - \lambda\psi_{n+1} + \gamma\psi_n = 0, \tag{4.1}$$

$$\begin{aligned}
&\lambda^2\psi_{n+2} + \lambda(u_y(n+2) - u_y(n))\psi_{n+1} + [u_{yy}(n+1) + u_{yy}(n) \\
&\quad +(u_y(n) - u_y(n+1))^2 - \theta - k - \gamma^2]\psi_n - \lambda^{-1}\psi_{n-1}e^{u(n+1)+u(n-1)-2u(n)} \\
&\quad \times \int^y (e^{u(n)+u(n-2)-2u(n-1)} - e^{u(n+2)+u(n)-2u(n+1)})\, dy' \\
&\quad +\lambda^{-2}\psi_{n-2}e^{u(n+1)-u(n)-u(n-1)+u(n-2)} = 0.
\end{aligned} \tag{4.2}$$

By some calculations, we can derive equation (1.4) from the compatibility condition of (4.1) and (4.2).

Next, let

$$\psi_n = f(n)/g(n),\ U(n) = \ln \frac{g(n+1)g(n-1)}{g^2(n)},$$

$$V(n) = \frac{D_y g(n+1)\cdot g(n-1)}{g(n+1)g(n-1)},\ W(n) = \frac{D_z g(n+1)\cdot g(n-1)}{g(n+1)g(n-1)}.$$

Then, from the bilinear BT (2.1)-(2.4) and after some calculations, we can obtain the following Lax pair for (1.5)-(1.7):

$$\begin{aligned}
\psi_{n,x} &= \lambda^2\psi_{n+2} + \lambda V(n+1)\psi_{n+1} \\
&\quad +\psi_n \left\{ \int^x \Big[V(n+1)W(n+1)e^{U(n+1)} - V(n)W(n)e^{U(n)} \right. \\
&\quad \left. +e^{U(n+2)+U(n+1)} - e^{U(n)+U(n-1)} \Big]\, dx' - \gamma^2 - \theta \right\},
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
&\lambda^2\psi_{n+2} + \lambda V(n+1)\psi_{n+1} + \psi_n \left\{ \int^x \Big[V(n+1)W(n+1)e^{U(n+1)} - V(n)W(n)e^{U(n)} \right. \\
&\quad \left. +e^{U(n+2)+U(n+1)} - e^{U(n)+U(n-1)} \Big]\, dx' - \gamma^2 - \theta - k \right\} \\
&\quad +\lambda^{-1}W(n)e^{U(n)}\psi_{n-1} + \lambda^{-2}e^{U(n)+U(n-1)}\psi_{n-2} = 0.
\end{aligned} \tag{4.4}$$

5 Conclusion and discussions

Starting from 3-coupled bilinear equations, two new integrable differential-difference systems have been found. One of them is a coupled Toda equation. Based on Hirota's bilinear operator identities, we have established the corresponding Bäcklund transformation and nonlinear superposition formula of the 3-coupled bilinear equations, consequently allowing one to produce solition solutions of the systems under consideration. Furthermore, Lax pairs for these two differential-difference systems are derived from the bilinear BT. Besides, set

$$\psi_n = f(n)/g(n),\ u(n) = \ln g(n),\ W(n) = \frac{D_y g(n+1)\cdot g(n-1)}{g(n+1)g(n-1)}.$$

Then, from the bilinear BT (2.1)-(2.4) and after some calculations, we can also obtain the following Lax pair for (1.15) and (1.16):

$$\psi_{n,z} + \lambda^{-1}\psi_{n-1}e^{u(n+1)+u(n-1)-2u(n)} + \mu\psi_n = 0, \tag{5.1}$$

$$\begin{aligned} &-\lambda^2\psi_{n+2} - \lambda W(n+1)\psi_{n+1} + [\theta + k + \gamma^2 \\ &\quad + \int^z (W(n)e^{u(n+1)+u(n-1)-2u(n)} - W(n+1)e^{u(n+2)+u(n)-2u(n+1)})\, dz']\psi_n \\ &\quad + \lambda^{-1}(u_z(n-1) - u_z(n+1))e^{u(n+1)+u(n-1)-2u(n)}\psi_{n-1} \\ &\quad - \lambda^{-2}\psi_{n-2}e^{u(n+1)-u(n)-u(n-1)+u(n-2)} = 0. \end{aligned} \tag{5.2}$$

(5.1) and (5.2) under the transformation $z \to y$ serves as another Lax pair for the y-flow (1.4).

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Appendix A. Hirota bilinear operator identities.

The following bilinear operator identities hold for arbitrary functions a, b, c, and d.

$$(D_x e^{\frac{1}{2}D_n} a \cdot a)(e^{\frac{1}{2}D_n} b \cdot b) - (e^{\frac{1}{2}D_n} a \cdot a)(D_x e^{\frac{1}{2}D_n} b \cdot b) = 2\sinh(\frac{1}{2}D_n)(D_x a \cdot b) \cdot ab, \tag{A1}$$

$$\begin{aligned} &(D_y^2 e^{\frac{1}{2}D_n} a \cdot a)(e^{\frac{1}{2}D_n} b \cdot b) - (e^{\frac{1}{2}D_n} a \cdot a)(D_y^2 e^{\frac{1}{2}D_n} b \cdot b) \\ &= D_y[(D_y e^{\frac{1}{2}D_n} a \cdot b) \cdot (e^{-\frac{1}{2}D_n} a \cdot b) - (e^{\frac{1}{2}D_n} a \cdot b) \cdot (D_y e^{-\frac{1}{2}D_n} a \cdot b)], \end{aligned} \tag{A2}$$

$$\begin{aligned} &D_y[(D_y e^{\frac{1}{2}D_n} a \cdot b) \cdot (e^{-\frac{1}{2}D_n} a \cdot b) + (e^{\frac{1}{2}D_n} a \cdot b) \cdot (D_y e^{-\frac{1}{2}D_n} a \cdot b)] \\ &= 2\sinh(\frac{1}{2}D_n)(D_y^2 a \cdot b) \cdot ab, \end{aligned} \tag{A3}$$

$$D_y(e^{\frac{1}{2}D_n} a \cdot b) \cdot (e^{-\frac{1}{2}D_n} a \cdot b) = 2\sinh(\frac{1}{2}D_n)(D_y a \cdot b) \cdot ab, \tag{A4}$$

$$\sinh(\frac{1}{2}D_n) a \cdot a = 0. \tag{A5}$$

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