

Multiple Hamiltonian Structures and Lax Pairs for Bogoyavlensky–Volterra Systems

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Abstract

Results on the Volterra model which is associated to the simple Lie algebra of type A_n are extended to the Bogoyavlensky–Volterra systems of type B_n , C_n and D_n . In particular we find Lax pairs, Hamiltonian and Casimir functions and multi-Hamiltonian structures. Moreover, we investigate recursion operators, higher Poisson brackets and master symmetries.

1 Introduction

The purpose of this paper is to investigate the integrable systems constructed by Bogoyavlensky in 1988 [2, 3]. These systems are connected with simple Lie algebras and are generalizations of the well known Volterra system. In particular, the Volterra system (also known as the KM system) is related to the root system of a simple Lie algebra of type A_n .

This system and its Poisson structure is treated in detail in [10]. The equations of motion are

$$\frac{dv_i}{dt} = v_i(v_{i+1} - v_{i-1}), \quad i = 1, 2, \dots, n, \quad (1.1)$$

where $v_0 = v_{n+1} = 0$. These equations were studied originally by Volterra in [29] to describe population evolution in a hierarchical system of competing individuals. The importance of this system derives from the fact that it can be considered as a discrete analogue of the Korteweg-de Vries equation. It is also associated with a lattice deformation of the Virasoro algebra [11]. This system was solved by Kac and Van Moerbeke [18] using a discrete version of inverse scattering. There is also an explicit solution by Moser in [21]. The integrability of the periodic KM system is considered in [13]. Finally, Damianou [5] constructed Multi Hamiltonian structures and master symmetries for the system.

Equations (1.1) can be written as a Lax pair $\dot{L} = [B, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} 0 & \sqrt{v_1} & 0 & \cdots & 0 \\ \sqrt{v_1} & 0 & \sqrt{v_2} & \ddots & \vdots \\ 0 & \sqrt{v_2} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & \sqrt{v_n} \\ 0 & \cdots & 0 & \sqrt{v_n} & 0 \end{pmatrix}, \quad (1.2)$$

and

$$B = \begin{pmatrix} 0 & 0 & \frac{1}{2}\sqrt{v_1 v_2} & & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sqrt{v_2 v_3} & \\ -\frac{1}{2}\sqrt{v_1 v_2} & 0 & \ddots & \ddots & \ddots \\ & -\frac{1}{2}\sqrt{v_2 v_3} & \ddots & \ddots & 0 & \frac{1}{2}\sqrt{v_{n-1} v_n} \\ 0 & & \ddots & 0 & 0 & 0 \\ & & & -\frac{1}{2}\sqrt{v_{n-1} v_n} & 0 & 0 \end{pmatrix}.$$

It follows that the functions $H_k = \frac{1}{k} \text{Tr } L^{2k}$ are constants of motion.

We present the Poisson structure of the Volterra system following [5]. We denote by π_j the bracket of degree j . The bracket π_2 is defined by

$$\{v_i, v_{i+1}\} = -\{v_{i+1}, v_i\} = v_i v_{i+1}, \quad (1.3)$$

and all other brackets are zero. In this bracket the Hamiltonian is $H_1 = \text{Tr } L^2$ and $\det L$ is the Casimir.

The Poisson bracket π_3 is defined by

$$\{v_i, v_{i+1}\} = v_i v_{i+1} (v_i + v_{i+1}), \quad \{v_i, v_{i+2}\} = v_i v_{i+1} v_{i+2}, \quad (1.4)$$

and all other brackets are zero.

We assume that n is odd. In order to define the bracket π_1 we define the vector field

$$Y_{-1} = \sum_{i=1}^n f_i \frac{\partial}{\partial v_i}, \quad (1.5)$$

where f_i is determined by

$$f_1 = -1, \quad f_{2i} = -\frac{v_{2i}}{v_{2i-1}} f_{2i-1}, \quad f_{2i-1} = -f_{2i-2} - 1, \quad (1.6)$$

and we define $\pi_1 = L_{Y_{-1}} (\pi_2)$. In this bracket the Hamiltonian is H_2 and the Casimir is H_1 .

In [5] there is a construction of an infinite sequence of vector fields Y_n , for $n \geq -1$, and an infinite sequence of Poisson brackets π_n , $n \geq 1$, satisfying:

- (i) π_j are all Poisson.
- (ii) π_i, π_j are compatible for all i, j .
- (iii) The functions H_j are in involution with respect to all of the π_i .
- (iv) $Y_i (H_j) = (i + j) H_{i+j}$.
- (v) $L_{Y_i} (\pi_j) = (i - j + 2) \pi_{i+j}$.
- (vi) $\pi_i \nabla H_j = \pi_{i-1} \nabla H_{j+1}$.
- (vii) $[Y_i, X_j] = (j - 1) X_{i+j}$, where X_j is the Hamiltonian vector field generated by H_j with respect to π_1 .

In this paper we obtain similar results for the generalized Volterra systems of Bogoyavlensky.

We would like to comment on the relation between the Volterra systems in this paper and the well known Toda systems generalized on simple Lie groups also by Bogoyavlensky [1].

There is a transformation due to Hénon which maps the Bogoyavlensky–Volterra system of type A_{2n} to the usual A_n Toda lattice. The mapping is given by

$$a_i = -\frac{1}{2}\sqrt{v_{2i}v_{2i-1}}, \quad 1 \leq i \leq n-1, \quad b_i = \frac{1}{2}(v_{2i-1} + v_{2i-2}), \quad 1 \leq j \leq n. \quad (1.7)$$

The equations satisfied by the new variables a_i, b_j are given by:

$$\dot{a}_i = a_i(b_{i+1} - b_i), \quad \dot{b}_i = 2(a_i^2 - a_{i-1}^2). \quad (1.8)$$

These are precisely the equations for the finite nonperiodic Toda lattice. Note that the number of variables for the Toda lattice is odd and this justifies our choice to consider the KM system with an odd number of variables.

In order to generalize the Hénon correspondence from generalized Volterra to generalized Toda it is necessary to work not with the original variables b_j of Bogoyavlensky but rather with some new variables v_j which also appear in [2]. However, Bogoyavlensky did not give a Lax pair for these systems in the variables v_j and this is the main construction of this paper.

The relation between the Volterra system of type B_{2n+1} (or C_{2n+1}) and Toda B_n (C_n) is due to Damianou and Fernandes [7], in 2002. The results of the present paper are essential for the calculations in [7]. The connection between Volterra D_{2n+1} and Toda D_n is still an open problem. In any case, the multi-Hamiltonian structure for the Toda D_n system is a recent development and can be found in [8].

We have to point out that since the B_n Toda lattice involves only an even number of variables it is natural to consider only Volterra B_{2n+1} systems with an even number of variables. It is not a coincidence that we have obtained results only in this particular case.

In Section 2 we present the necessary background on Poisson manifolds, bi-Hamiltonian systems and master symmetries.

In Section 3 we describe the construction of the systems. We obtain the Bogoyavlensky–Volterra (BV) system for each simple Lie algebra \mathcal{G} .

In Section 4 we investigate the BV system of type B_{n+1} . We find a Lax-pair (L, B) for every $n \geq 2$. When n is even, we define two compatible brackets π_1, π_3 which define a recursion operator $\mathcal{R} = \pi_3\pi_1^{-1}$. This recursion operator produces compatible Poisson brackets $\pi_{2j+1} = \mathcal{R}^j\pi_1$ and the constants of motion are in involution for every $j = 1, 2, 3, \dots$

In Section 5 we find master symmetries of the BV B_{n+1} system as well as the relations which they satisfy. We do not present the analogous results for the C_{n+1} system since it is equivalent to the B_{n+1} system.

In Section 6 we investigate the BV system of type D_{n+1} . We find again a Lax pair (L, B) for every $n \geq 4$ and, when n is odd, we define two compatible Poisson brackets π_1, π_3 . We also describe the Hamiltonian formulation and compute the Casimirs.

2 Background

2.1 Poisson manifolds

We begin with a brief review of Poisson Manifolds. See for example [19, 30, 28].

Let M be a C^∞ manifold and $C^\infty(M)$ the space of C^∞ real valued functions on M . A contravariant, antisymmetric tensor of order p will be called a p -tensor for short.

A manifold M with a bilinear map $\{ , \} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ which satisfies the following conditions:

$$\begin{aligned} \{f, g\} &= -\{g, f\}, & (\text{skew-symmetry}) \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0, & (\text{Jacobi identity}) \\ \{f, g \cdot h\} &= \{f, g\} \cdot h + \{f, h\} \cdot g & (\text{Leibniz rule}) \end{aligned}$$

is called a Poisson manifold. The bilinear form $\{ , \}$ is called Poisson bracket or Poisson structure on M .

The Poisson bracket gives rise to a 2-tensor π such that $\{f, g\} = \langle \pi, df \wedge dg \rangle$ where \langle , \rangle is the pairing between the 2-tensors and the differential 2-forms. In local coordinates (x_1, x_2, \dots, x_n) , π is given by

$$\pi = \sum_{1 \leq i, j \leq n} \pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \quad (2.1)$$

and

$$\{f, g\} = \langle \pi, df \wedge dg \rangle = \sum_{1 \leq i, j \leq n} \pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad (2.2)$$

where $\pi_{ij} = \{x_i, x_j\}$. Hence if we know the Poisson matrix π_{ij} we know the bracket $\{f, g\}$ of two arbitrary functions as well.

The Poisson bracket allows one to associate a vector field to each element $f \in C^\infty(M)$. Leibniz's rule implies that $\{f, \cdot\}$ is a derivation of $C^\infty(M)$. Therefore, for each $f \in C^\infty(M)$ there exists a well defined vector field X_f defined by the formula

$$X_f(g) = \{f, g\}. \quad (2.3)$$

It is called Hamiltonian vector field generated by f . A nonconstant function f such that $X_f = 0$ is called a Casimir.

A symplectic manifold is a pair (M, ω) , where ω is closed, nondegenerate 2-form. In local coordinates (x_1, \dots, x_n) we have

$$\omega = \sum_{i, j} \omega_{ij}(x) dx_i \wedge dx_j. \quad (2.4)$$

The nondegeneracy condition means that there exists an inverse (skew-symmetric) matrix ω_{ij}^{-1} , so the dimension of M is even. The condition for ω to be closed (i.e. $d\omega = 0$) is equivalent to the Jacobi identity for the tensor ω^{-1} (see [25]). Therefore we can define a Poisson bracket by

$$\{f, g\} = \omega(X_f, X_g), \quad (2.5)$$

which is called the symplectic bracket.

Let π_1, π_2 two Poisson brackets on manifold M . The two brackets are called compatible if $\pi_1 + \pi_2$ is also Poisson. For example if π_1 is Poisson and $\pi_2 = L_X \pi_1$ for some vector field X then it is easy to prove that π_1, π_2 are compatible; see [6]. If π_1 is symplectic then we can define a Recursion operator: It is the $(1, 1)$ -tensor \mathcal{R} defined by

$$\mathcal{R} = \pi_2 \pi_1^{-1}. \quad (2.6)$$

Recursion operators were introduced by Olver in [24].

A bi-Hamiltonian system is defined by specifying two Hamiltonian functions H_1, H_2 satisfying:

$$\pi_1 \nabla H_2 = \pi_2 \nabla H_1. \quad (2.7)$$

The notion of bi-Hamiltonian system is due to F Magri [20]. We have the following result, see [9, 15]:

Suppose we have a bi-Hamiltonian system on a manifold M , whose first cohomology group is trivial. Then there exists a hierarchy of mutually commuting functions H_1, H_2, \dots all in involution with respect to both brackets. They generate mutually commuting bi-Hamiltonian flows $X_i, i = 1, 2, \dots$ satisfying the Lenard recursion relations

$$X_{i+j} = \pi_i \nabla H_j, \quad (2.8)$$

where $\pi_{i+1} = \mathcal{R}^i \pi_1$ are the higher order Poisson tensors.

2.2 Master symmetries

Master symmetries were introduced by Fokas and Fuchssteiner in [14]. For further details on bi-Hamiltonian systems relevant to this paper see [12, 26, 27, 22]. The technique of generating master symmetries for bi-Hamiltonian systems can be found in [16].

We recall the definition and basic properties of master symmetries following Fuchssteiner [17].

Consider a differential equation on a manifold M

$$\dot{x} = X(x). \quad (2.9)$$

A vector field $Y = Y(x)$ is a symmetry of (2.9) if

$$[Y, X] = 0. \quad (2.10)$$

The condition for Z to be a master symmetry is:

$$[[Z, X], X] = 0, \quad \text{and} \quad [Z, X] \neq 0. \quad (2.11)$$

We consider a bi-Hamiltonian system defined by the compatible Poisson tensors J_0, J_1 and the Hamiltonians h_0, h_1 . Assume that J_0 is symplectic. We define the recursion operator $\mathcal{R} = J_1 J_0^{-1}$, the higher flows

$$X_{i+1} = \mathcal{R}^i X_1, \quad \text{where} \quad X_1 = J_1 dh_0 = J_0 dh_1, \quad (2.12)$$

and the higher order Poisson tensors

$$J_i = \mathcal{R}^i J_0, \quad i = 1, 2, \dots \quad (2.13)$$

Master symmetries preserve constants of motion, Hamiltonian vector fields and generate hierarchies of Poisson structures. For a nondegenerate bi-Hamiltonian system, master symmetries can be generated using a method due to W Oevel [23].

Theorem 1. *Suppose that Z_0 is a conformal symmetry for both J_0 , J_1 and h_0 , i.e. for some scalars α , β , and γ we have*

$$L_{Z_0} J_0 = \alpha J_0, \quad L_{Z_0} J_1 = \beta J_1, L_{Z_0} h_0 = \gamma h_0. \quad (2.14)$$

Then the vector fields

$$Z_i = \mathcal{R}^i Z_0, \quad i = 1, 2, \dots \quad (2.15)$$

are master symmetries, the J_i are Poisson and they satisfy

- (i) $[Z_i, X_j] = (\beta + \gamma + (j - 1)(\beta - \alpha)) X_{i+j},$
- (ii) $[Z_i, Z_j] = (\beta - \alpha)(j - i) Z_{i+j},$
- (iii) $L_{Z_i} J_j = (\beta + (j - i - 1)(\beta - \alpha)) J_{i+j}.$

3 Definition of the systems

We consider the system

$$\frac{du_i}{dt} = u_i(u_{i+1} - u_{i-1}), \quad i = 1, \dots, n, \quad (3.1)$$

where $u_0 = u_{n+1} = 0$. This is the Volterra system, also known as the KM system and is related to the root system of a simple Lie algebra of type A_{n+1} . Bogoyavlensky constructed integrable dynamical systems connected with simple Lie algebras that generalize the Volterra system. For more details see [2, 3].

We outline the construction of the systems:

Let \mathcal{G} be a simple Lie algebra ($\text{rank } \mathcal{G} = n$) and $\Pi = \{\omega_1, \omega_2, \dots, \omega_n\}$ the Cartan–Weyl basis of simple roots in \mathcal{G} ([4]). There are unique positive integers k_i such that

$$k_0 \omega_0 + k_1 \omega_1 + \dots + k_n \omega_n = 0, \quad (3.2)$$

where $k_0 = 1$ and ω_0 is the minimal negative root.

We consider the following Lax pairs:

$$\begin{aligned} \dot{L} &= [B, L], \\ L(t) &= \sum_{i=1}^n b_i(t) e_{\omega_i} + e_{\omega_0} + \sum_{1 \leq i < j \leq n} [e_{\omega_i}, e_{\omega_j}], \\ B(t) &= \sum_{i=1}^n \frac{k_i}{b_i(t)} e_{-\omega_i} + e_{-\omega_0}. \end{aligned} \quad (3.3)$$

Let \mathcal{H} be a Cartan subalgebra of \mathcal{G} . For every root $\omega_a \in \mathcal{H}^*$ there is a unique $H_{\omega_a} \in \mathcal{H}$ such that $\omega(h) = k(H_{\omega_a}, h) \forall h \in \mathcal{H}$, where k is the Killing form and \mathcal{H}^* is the dual space of \mathcal{H} . We also have an inner product on \mathcal{H}^* such that $\langle \omega_a, \omega_b \rangle = k(H_{\omega_a}, H_{\omega_b})$. We set

$$c_{ij} = \begin{cases} 1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i < j, \\ 0 & \text{if } \langle \omega_i, \omega_j \rangle = 0 \text{ or } i = j, \\ -1 & \text{if } \langle \omega_i, \omega_j \rangle \neq 0 \text{ and } i > j. \end{cases} \quad (3.4)$$

The vector equation (3.3) is equivalent to the dynamical system

$$\dot{b}_i = - \sum_{j=1}^n \frac{k_j c_{ij}}{b_j}. \quad (3.5)$$

We determine the skew-symmetric variables

$$x_{ij} = c_{ij} b_i^{-1} b_j^{-1}, \quad x_{ji} = -x_{ij}, \quad x_{jj} = 0, \quad (3.6)$$

which correspond to the edges of the Dynkin diagram for the Lie algebra \mathcal{G} , connecting the vertices ω_i and ω_j .

The dynamical system (3.5) in the variables x_{ij} takes the form

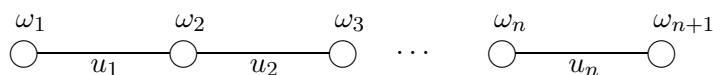
$$\dot{x}_{ij} = x_{ij} \sum_{s=1}^n k_s (x_{is} + x_{js}). \quad (3.7)$$

We recall that the vertices ω_i, ω_j of the Dynkin diagram are joined by edges only if $\langle \omega_i, \omega_j \rangle \neq 0$. Hence $x_{ij} = 0$ if there are no edges connecting the vertices ω_i and ω_j of the diagram. We call the equations (3.7) Bogoyavlensky–Volterra system associated with \mathcal{G} (BV system for short).

We shall now describe the BV system for each simple Lie algebra \mathcal{G} . The number of independent variables $x_{ij}(t)$ is equal to $n - 1$ and is one less than the number of variables $b_j(t)$. We use the standard numeration of vertices of the Dynkin diagram and define the variables $u_k(t) = x_{ij}(t)$ corresponding to the edges of the Dynkin diagram with increasing order of the vertices ($i < j$).

The phase space consists of variables u_i , $1 \leq i \leq n - 1$, with $u_i > 0$.

A_{n+1}



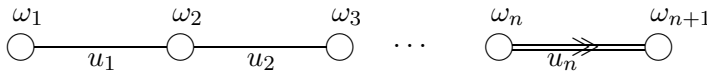
$$\begin{aligned} \omega_0 &= -(\omega_1 + \omega_2 + \cdots + \omega_{n+1}) \\ k_i &= 1, \quad i = 1, \dots, n+1 \end{aligned}$$

$(\text{BV } A_{n+1})$

$$c_{ij} = \begin{cases} 0, & |i - j| \neq 1, \\ 1, & j = i + 1, \\ -1, & j = i - 1 \end{cases}$$

$$\begin{aligned} \dot{u}_1 &= u_1 u_2 \\ \dot{u}_n &= -u_{n-1} u_n \\ \dot{u}_i &= u_i (u_{i+1} - u_{i-1}) \\ 2 \leq i &\leq n - 1 \end{aligned}$$

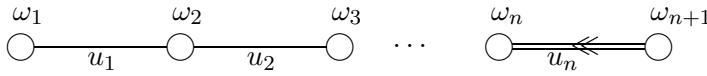
$$u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}}, \quad i = 1, \dots, n$$

B_{n+1} 

$$\begin{aligned} \omega_0 &= -(\omega_1 + 2\omega_2 + \dots + 2\omega_{n+1}) \\ k_1 &= 1, k_i = 2, \quad i = 2, \dots, n+1 \end{aligned} \quad (\underline{\text{BV } B_{n+1}})$$

$$c_{ij} = \begin{cases} 0, & |i-j| \neq 1, \\ 1, & j = i+1, \\ -1, & j = i-1 \end{cases} \quad \begin{aligned} \dot{u}_1 &= u_1(u_1 + 2u_2) \\ \dot{u}_2 &= u_2(2u_3 - u_1) \\ \dot{u}_n &= -2u_{n-1}u_n \\ \dot{u}_i &= 2u_i(u_{i+1} - u_{i-1}) \\ 3 \leq i &\leq n-1 \end{aligned}$$

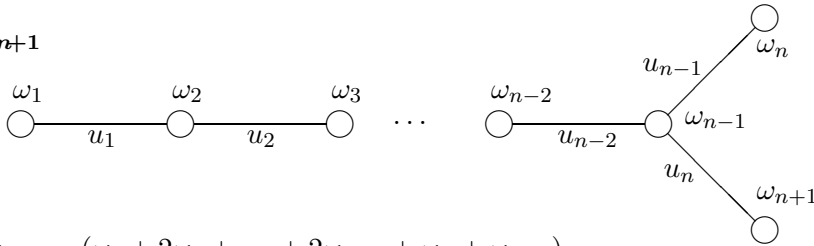
$$u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}}, \quad i = 1, \dots, n$$

 C_{n+1} 

$$\begin{aligned} \omega_0 &= -(2\omega_1 + \dots + 2\omega_n + \omega_{n+1}) \\ k_i &= 2, \quad i = 1, \dots, n, \quad k_{n+1} = 1 \end{aligned} \quad (\underline{\text{BV } C_{n+1}})$$

$$c_{ij} = \begin{cases} 0, & |i-j| \neq 1, \\ 1, & j = i+1, \\ -1, & j = i-1 \end{cases} \quad \begin{aligned} \dot{u}_1 &= 2u_1u_2 \\ \dot{u}_{n-1} &= u_{n-1}(u_n - 2u_{n-2}) \\ \dot{u}_n &= -u_n(u_n + 2u_{n-1}) \\ \dot{u}_i &= 2u_i(u_{i+1} - u_{i-1}) \\ 2 \leq i &\leq n-2 \end{aligned}$$

$$u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}}, \quad i = 1, \dots, n$$

 D_{n+1} 

$$\begin{aligned} \omega_0 &= -(\omega_1 + 2\omega_2 + \dots + 2\omega_{n-1} + \omega_n + \omega_{n+1}) \\ k_1 &= 1, k_n = 1, k_{n+1} = 1, k_i = 2, \quad 2 \leq i \leq n-1 \end{aligned}$$

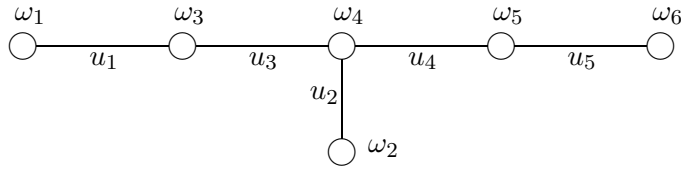
$$c_{ij} = -c_{ji} = \begin{cases} 1, & 2 \leq j = i+1 \leq n, \\ 0, & (i, j) = (n, n+1), \\ 0, & 3 \leq i+2 \leq j \leq n, \\ 1, & (i, j) = (n-1, n+1) \end{cases}$$

$$u_i = x_{i,i+1} = \frac{1}{b_i b_{i+1}}, \quad i = 1, \dots, n-1, \quad u_n = x_{n-1, n+1} = \frac{1}{b_{n-1} b_{n+1}}$$

$$\dot{u}_1 = u_1(2u_2 + u_1), \quad \dot{u}_2 = u_2(2u_3 - u_1), \quad (\underline{\text{BV } D_{n+1}})$$

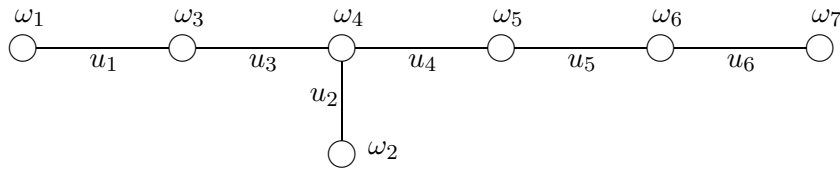
$$\begin{aligned}\dot{u}_i &= u_i(u_{i+1} - u_{i-1}), \quad 3 \leq i \leq n-3, \quad \dot{u}_{n-2} = u_{n-2}(u_n + u_{n-1} - 2u_{n-3}), \\ \dot{u}_{n-1} &= u_{n-1}(u_n - u_{n-1} - 2u_{n-2}), \quad \dot{u}_n = -u_n(u_n - u_{n-1} + 2u_{n-2}).\end{aligned}$$

$$\mathbf{E_6} \quad \omega_0 = -(\omega_1 + 2\omega_2 + 2\omega_3 + 3\omega_4 + 2\omega_5 + \omega_6)$$



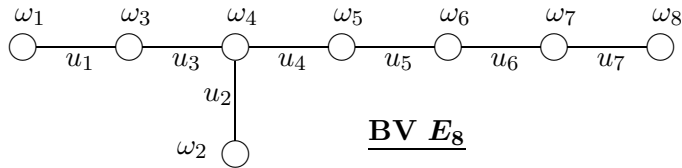
$$\begin{aligned}\dot{u}_1 &= u_1(3u_3 + u_1), \quad \dot{u}_2 = u_2(2u_4 - 2u_3 + u_2), \\ \dot{u}_3 &= u_3(2u_4 + u_3 - 2u_2 - u_1), \quad \dot{u}_4 = u_4(u_5 - u_4 - 2u_3 - 2u_2), \\ \dot{u}_5 &= -u_5(u_5 + 3u_4).\end{aligned} \quad (\mathbf{BV \ E_6})$$

$$\mathbf{E_7} \quad \omega_0 = -(2\omega_1 + 2\omega_2 + 3\omega_3 + 4\omega_4 + 3\omega_5 + 2\omega_6 + \omega_7)$$



$$\begin{aligned}\dot{u}_1 &= u_1(4u_3 + u_1), \quad \dot{u}_2 = u_2(3u_4 - 3u_3 + 2u_2), \\ \dot{u}_3 &= u_3(3u_4 + u_3 - 2u_2 - 2u_1), \quad \dot{u}_4 = u_4(2u_5 - u_4 - 3u_3 - 2u_2), \\ \dot{u}_5 &= u_5(u_6 - u_5 - 4u_4), \quad \dot{u}_6 = -u_6(3u_5 + u_6).\end{aligned} \quad (\mathbf{BV \ E_7})$$

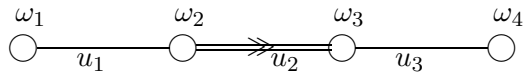
$$\mathbf{E_8} \quad \omega_0 = -(2\omega_1 + 3\omega_2 + 4\omega_3 + 6\omega_4 + 5\omega_5 + 4\omega_6 + 3\omega_7 + 2\omega_8)$$



BV E₈

$$\begin{aligned}\dot{u}_1 &= 2u_1(3u_3 + u_1), \quad \dot{u}_2 = u_2(5u_4 - 4u_3 + 3u_2), \\ \dot{u}_3 &= u_3(5u_4 + 2u_3 - 3u_2 - 2u_1), \quad \dot{u}_4 = u_4(4u_5 - u_4 - 4u_3 - 3u_2), \\ \dot{u}_5 &= u_5(3u_6 - u_5 - 6u_4), \quad \dot{u}_6 = u_6(2u_7 - u_6 - 5u_5), \quad \dot{u}_7 = -u_7(4u_6 + u_7).\end{aligned} \quad (\mathbf{BV \ E_8})$$

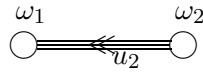
$$\mathbf{F_4} \quad \omega_0 = -(2\omega_1 + 3\omega_2 + 4\omega_3 + 2\omega_4)$$



$$\dot{u}_1 = u_1(4u_2 + u_1), \quad \dot{u}_2 = u_2(2u_3 + u_2 - 2u_1), \quad \dot{u}_3 = -u_3(2u_3 + 3u_2). \quad (\mathbf{BV \ F_4})$$

G_2

$$\omega_0 = -(3\omega_1 + 2\omega_2)$$



$$\dot{u}_1 = -u_1^2$$

(BV G_2)

Note that the systems (3.7) and the BV system for every simple Lie algebra \mathcal{G} are special cases of the corresponding periodic systems which were constructed by Bogoyavlensky ([2], Section 7), when

$$\begin{aligned} \mu_{ij} &= c_{ij}, & \mu_{k,0} &= \mu_{0,k} = 0, & 1 \leq i, j \leq n, & 0 \leq k \leq n, \\ b_0 &= 1, & u_0 &= 0. \end{aligned} \quad (3.8)$$

In this paper we restrict our attention to the Bogoyavlensky–Volterra systems associated with the classical Lie algebras.

4 The Bogoyavlensky–Volterra B_n system and its Poisson bracket

In this section we investigate the BV B_{n+1} system. We find a Lax-pair (L, B) for every $n \geq 2$. When n is even, we define two brackets π_1, π_3 which define a recursion operator $R = \pi_3 \pi_1^{-1}$ so that the Poisson brackets $\pi_{2j+1} = R^j \pi_1$ are compatible and the constants of motion are in involution in each bracket π_{2j+1} .

Recall the BV B_{n+1} system ($u_i > 0$)

$$\begin{aligned} \dot{u}_1 &= u_1 (u_1 + 2u_2), \\ \dot{u}_2 &= u_2 (2u_3 - u_1), \\ \dot{u}_i &= 2u_i (u_{i+1} - u_{i-1}), & i = 3, \dots, n-1, \\ \dot{u}_n &= -2u_{n-1}u_n. \end{aligned} \quad (4.1)$$

We rescale the coordinates

$$v_1 = u_1, \quad v_i = 2u_i, \quad i = 2, \dots, n, \quad (4.2)$$

to obtain the equivalent system

$$\begin{aligned} \dot{v}_1 &= v_1 (v_1 + v_2), \\ \dot{v}_i &= v_i (v_{i+1} - v_{i-1}), & i = 2, \dots, n-1, \\ \dot{v}_n &= -v_{n-1}v_n. \end{aligned} \quad (4.3)$$

Before giving the Lax pair for the system (4.3) we introduce some matrix notations:

$$\begin{aligned} X_i &= \begin{pmatrix} \sqrt{v_i} & 0 \\ 0 & i\sqrt{v_i} \end{pmatrix}, & O &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ Y_i &= \frac{1}{2} \begin{pmatrix} \sqrt{v_i v_{i+1}} & 0 \\ 0 & \sqrt{v_i v_{i+1}} \end{pmatrix}, & Y_0 &= \frac{i}{2} \begin{pmatrix} 0 & v_1 \\ -v_1 & 0 \end{pmatrix}. \end{aligned} \quad (4.4)$$

It turns out the equations (4.3) are equivalent to the Lax pair $\dot{L} = [L, B]$, where L, B are $(n+1) \times (n+1)$ matrices

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & \sqrt{v_1} & i\sqrt{v_1} \\ 0 & O & X_n & O & \cdots & O \\ \vdots & X_n & O & \ddots & \ddots & \vdots \\ 0 & O & \ddots & \ddots & X_3 & O \\ \sqrt{v_1} & \vdots & \ddots & X_3 & O & X_2 \\ i\sqrt{v_1} & O & \cdots & O & X_2 & O \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -\frac{1}{2}\sqrt{v_1 v_2} & -\frac{1}{2}\sqrt{v_1 v_2} & 0 & 0 \\ \vdots & O & O & Y_{n-1} & O & \cdots & \cdots & O \\ \vdots & O & O & O & \ddots & \ddots & & \vdots \\ 0 & -Y_{n-1} & O & O & \ddots & Y_4 & O & \vdots \\ \frac{1}{2}\sqrt{v_1 v_2} & O & \ddots & \ddots & \ddots & O & Y_3 & O \\ \frac{1}{2}\sqrt{v_1 v_2} & \vdots & \ddots & -Y_4 & O & O & O & Y_2 \\ 0 & \vdots & & O & -Y_3 & O & O & O \\ 0 & O & \cdots & \cdots & O & -Y_2 & O & Y_0 \end{bmatrix}. \quad (4.5)$$

We note that the elements of the matrices L, B are the 2×2 matrices X_i, Y_j and O except for the elements of the first row and the first column which are scalars.

The functions $H_{2k} = \frac{1}{2k} \text{Tr}(L^{4k})$, $k = 1, 2, \dots$ are constants of motion for the system. We use the old variables b_j appearing in the equations (3.5) in order to find a cubic bracket π_3 for the system. The equations (3.5) in the case of the Lie algebra B_{n+1} become

$$\begin{aligned} \dot{b}_1 &= -2b_2^{-1}, & \dot{b}_2 &= -2b_3^{-1} + b_1^{-1}, \\ \dot{b}_j &= -2(b_{j+1}^{-1} - b_{j-1}^{-1}), & j &= 3, \dots, n, & \dot{b}_{n+1} &= 2b_n^{-1}. \end{aligned} \quad (4.6)$$

The dynamical system (4.6) can be written in Hamiltonian form $\dot{b}_j = \{b_j, H\}$, with Hamiltonian $H = \log b_1 + 2 \sum_{j=2}^{n+1} \log b_j$ and a constant Poisson bracket

$$\{b_j, b_{j+1}\} = -\{b_{j+1}, b_j\} = 1, \quad \text{for } j = 1, 2, \dots, n. \quad (4.7)$$

All other brackets are zero. In terms of the variables v_j ($v_1 = b_1^{-1}b_2^{-1}$, $v_k = 2b_k^{-1}b_{k+1}^{-1}$, $k = 2, \dots, n$) the above skew-symmetry bracket, which we denote by π_3 , is given by

$$\begin{aligned} \{v_1, v_2\} &= v_1 v_2 (2v_1 + v_2), \\ \{v_i, v_{i+1}\} &= v_i v_{i+1} (v_i + v_{i+1}), \quad i = 2, \dots, n-1, \\ \{v_i, v_{i+2}\} &= v_i v_{i+1} v_{i+2}, \quad i = 1, \dots, n-2, \end{aligned} \quad (4.8)$$

and all other brackets are zero.

Suppose that n is even ($n = 2l$) and we look for a bracket π_1 which satisfies

$$\pi_3 \nabla H_2 = \pi_1 \nabla H_4.$$

We define the skew-symmetric matrix

$$\omega = \begin{pmatrix} 0 & -\frac{1}{v_1} & \cdots & -\frac{1}{v_1} & -\frac{1}{v_1} & -\frac{1}{v_1} \\ \frac{1}{v_1} & 0 & -\frac{1}{v_2} & \cdots & -\frac{1}{v_2} & -\frac{1}{v_2} \\ \vdots & \frac{1}{v_2} & \ddots & \ddots & \vdots & \vdots \\ \frac{1}{v_1} & \vdots & \ddots & 0 & -\frac{1}{v_{n-2}} & -\frac{1}{v_{n-2}} \\ \frac{1}{v_1} & \frac{1}{v_2} & \cdots & \frac{1}{v_{n-2}} & 0 & -\frac{1}{v_{n-1}} \\ \frac{1}{v_1} & \frac{1}{v_2} & \cdots & \frac{1}{v_{n-2}} & \frac{1}{v_{n-1}} & 0 \end{pmatrix}, \quad (4.9)$$

and we define $\pi_1 = \omega^{-1}$ (i.e. $\{v_i, v_j\}_{\pi_1} = (\omega^{-1})_{ij}$).

Theorem 2. *The brackets π_1, π_3 satisfy:*

- (i) π_1, π_3 are Poisson.
- (ii) The function $\frac{1}{4}H_2 = \frac{1}{8} \text{Tr}(L^4) = \sum_{i=2}^n \left(\frac{1}{2}v_i^2 + v_{i-1}v_i\right)$ is the Hamiltonian of the BV B_{n+1} system with respect to the bracket π_1 .
- (iii) π_1, π_3 are compatible.

Proof. (i) Changing variables in the Poisson tensor (4.7) preserves the Jacobi identity and therefore π_3 is a Poisson bracket.

In order to prove that π_1 is a Poisson bracket we consider the 2-form

$$\omega = \frac{1}{2} \sum_{i,j=1}^n \omega_{ij} dv_i \wedge dv_j = \sum_{1 \leq i < j \leq n} -\frac{1}{v_i} dv_i \wedge dv_j. \quad (4.10)$$

Since the 2-form ω is closed, (i.e. $d\omega = 0$), $\pi_1 = \omega^{-1}$ satisfies the Jacobi identity (see [25], page 11) and therefore π_1 is Poisson.

(ii) follows from simple calculations.

(iii) It is well-known, see [6], that if a Poisson tensor is a Lie derivative of another, then the two tensors are compatible. We will see later, in the next section, that π_3 is the Lie derivative of π_1 in the direction of a master symmetry and this fact makes π_1, π_3 compatible. ■

Finally, we define a sequence of Poisson brackets π_{2j-1} , $j = 1, 2, \dots$ which are compatible and the constants of motion are in involution with respect to each π_{2j-1} . Since the 2-tensor π_1 is invertible we can define the recursion operator $R = \pi_3 \pi_1^{-1}$. We define the higher order Poisson tensors

$$\pi_{2j+1} = R^j \pi_1, \quad j = 1, 2, \dots \quad (4.11)$$

Using standard theory of recursion operators [6, 20, 23] we obtain the following theorem.

Theorem 3. *The sequence of higher Poisson tensors and invariants satisfy:*

- (i) $\pi_{2j+1} \nabla H_{2i} = \pi_{2j-1} \nabla H_{2i+2}, \forall i, j.$
- (ii) H_{2i} are in involution with respect to all Poisson brackets.
- (iii) π_{2j+1} are all compatible Poisson brackets.

Remark. Since the functions H_2, H_4, \dots, H_{2l} are independent and in involution the BV B_{2l+1} system is integrable.

5 Master symmetries of the Bogoyavlensky–Volterra B_n system

In this section we find master symmetries for the system (4.3) and derive the relations which they satisfy.

We consider

$$\begin{aligned} \pi_3 &= v_1 v_2 (2v_1 + v_2) \frac{\partial}{\partial v_1} \wedge \frac{\partial}{\partial v_2} + \sum_{i=2}^{n-1} v_i v_{i+1} (v_i + v_{i+1}) \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_{i+1}} \\ &\quad + \sum_{i=2}^{n-2} v_i v_{i+1} v_{i+2} \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_{i+2}}, \\ \pi_1^{-1} &= \sum_{1 \leq i < j \leq n} -\frac{1}{v_i} dv_i \wedge dv_j, \quad H_2 = \sum_{i=2}^n (2v_i^2 + 4v_{i-1} v_i). \end{aligned}$$

The recursion operator is then

$$R = \pi_3 \pi_1^{-1} = \sum_{i,j=1}^n \alpha_{ij} dv_j \otimes \frac{\partial}{\partial v_i}, \quad (5.1)$$

where

$$\begin{aligned} \alpha_{11} &= v_2 (2v_1 + v_2 + v_3), & \alpha_{12} &= v_1 v_3, \\ \alpha_{13} &= -v_1 (2v_1 + v_2), & \alpha_{21} &= \frac{v_2 v_3}{v_1} (v_2 + v_3 + v_4), \\ \alpha_{ii} &= v_i^2 + v_{i+1}^2 + 2v_{i-1} v_i + v_i v_{i+1} + v_{i+1} v_{i+2}, & i &= 2, 3, \dots, n, \\ \alpha_{i,i+1} &= v_i (v_{i+2} + v_i + 2v_{i-1}), & i &= 2, 3, \dots, n-1, \\ \alpha_{i,i+2} &= -v_i (v_{i+1} - 2v_{i-1}), & i &= 2, 3, \dots, n-2, \\ \alpha_{i,i-1} &= \frac{v_i}{v_{i-1}} (v_{i+1}^2 + v_{i-1}^2 + v_i v_{i+1} + v_{i+1} v_{i+2}), & i &= 3, 4, \dots, n, \\ \alpha_{i,i-2} &= \frac{v_i}{v_{i-2}} (v_{i+1}^2 - v_{i-1}^2 - v_{i-1} v_i + v_i v_{i+1} + v_{i+1} v_{i+2}), & i &= 3, 4, \dots, n, \\ \alpha_{ij} &= \frac{v_i}{v_j} (v_{i+1}^2 - v_{i-1}^2 - v_{i-2} v_{i-1} - v_{i-1} v_i + v_i v_{i+1} + v_{i+1} v_{i+2}), & i-j &> 2, \\ \alpha_{ij} &= -2v_i, & j-i &> 2. \end{aligned}$$

(We assume $v_{n+1} = v_{n+2} = v_{n+3} = \dots = 0$.)

We now prove that π_1 and π_3 are compatible. It is enough to show that $\pi_3 = L_{Z_1}\pi_1$ for some vector field Z_1 . We define

$$\begin{aligned} Z_1 &= R(Z_0) = \left(\sum_{i,j}^n \alpha_{ij} dv_j \otimes \frac{\partial}{\partial v_i} \right) (Z_0) = \sum_{i,j}^n \alpha_{ij} dv_j (Z_0) \otimes \frac{\partial}{\partial v_i} \\ &= \sum_{i,j}^n \alpha_{ij} v_j \frac{\partial}{\partial v_i} = \sum_{i=1}^n \left(\sum_{j=1}^n v_j a_{ij} \right) \frac{\partial}{\partial v_i}, \end{aligned} \quad (5.2)$$

where Z_0 is the Euler vector field

$$Z_0 = \sum_{i=1}^n v_i \frac{\partial}{\partial v_i}. \quad (5.3)$$

Using the formula

$$\{f, g\}_{L_X \pi} = X \{f, g\}_\pi - \{f, X(g)\}_\pi - \{X(f), g\}_\pi \quad (5.4)$$

it is easy to check that

$$L_{Z_1}(\pi_1) = -3\pi_3, \quad (5.5)$$

and therefore π_3 is the Lie-derivative of π_1 in the direction of the vector field Z_1 . This makes π_1 compatible with π_3 and completes the proof of Theorem 2.

Using the recursion operator we generate the master symmetries

$$Z_i = R^i Z_0. \quad (5.6)$$

One calculates that

$$L_{Z_0}(\pi_1) = -\pi_1, \quad L_{Z_0}(\pi_3) = \pi_3, \quad L_{Z_0}(H_2) = 2H_2. \quad (5.7)$$

Therefore Z_0 is a conformal symmetry for π_1 , π_3 , and H_2 . The constants appearing in Oevel's Theorem are

$$\alpha = -1, \quad \beta = 1, \quad \gamma = 2. \quad (5.8)$$

We use the notation $h_0 = H_2$, $J_0 = \pi_1$, $X_1 = J_1 dh_0 = J_0 dh_1$ and in general $h_i = H_{2i+2}$, $J_j = \pi_{2j+1}$, $X_i = R^{i-1} X_1$. It follows from Theorem 1 that the higher order Poisson tensors satisfy the following deformation relations:

$$\begin{aligned} (i) \quad & [Z_i, X_j] = (1 + 2j) X_{i+j}, \\ (ii) \quad & [Z_i, Z_j] = 2(j - i) Z_{i+j}, \\ (iii) \quad & L_{Z_i} J_j = (2j - 2i - 1) J_{i+j} \iff L_{Z_i}(\pi_{2j+1}) = (2j - 2i - 1) \pi_{2(i+j)+1}. \end{aligned} \quad (5.9)$$

We also have

$$Z_i(H_{2j}) = 2(i + j) H_{2(i+j)}. \quad (5.10)$$

We will not present the results for the BV C_{n+1} system. In fact the BV C_{n+1} system is equivalent to the BV B_{n+1} system through the transformation

$$u_1 \longmapsto u_n, \quad u_2 \longmapsto u_{n-1}, \quad \dots, \quad u_{n-1} \longmapsto u_2, \quad u_n \longmapsto u_1. \quad (5.11)$$

6 The Bogoyavlensky–Volterra D_n system and its Poisson bracket

We recall the BV D_{n+1} system ($u_i > 0$)

$$\begin{aligned}
 \dot{u}_1 &= u_1(u_1 + 2u_2), \\
 \dot{u}_2 &= u_2(2u_3 - u_1), \\
 \dot{u}_i &= 2u_i(u_{i+1} - u_{i-1}), \quad i = 3, \dots, n-3, \\
 \dot{u}_{n-2} &= u_{n-2}(u_n + u_{n-1} - 2u_{n-3}), \\
 \dot{u}_{n-1} &= u_{n-1}(u_n - u_{n-1} - 2u_{n-2}), \\
 \dot{u}_n &= -u_n(u_n - u_{n-1} + 2u_{n-2}).
 \end{aligned} \tag{6.1}$$

We make a linear transformation

$$v_1 = u_1, \quad v_i = 2u_i, \quad i = 2, \dots, n-2, \quad v_{n-1} = u_{n-1}, \quad v_n = u_n, \tag{6.2}$$

to obtain the equivalent system

$$\begin{aligned}
 \dot{v}_1 &= v_1(v_1 + v_2), \\
 \dot{v}_i &= v_i(v_{i+1} - v_{i-1}), \quad i = 2, \dots, n-3, \\
 \dot{v}_{n-2} &= v_{n-2}(v_n + v_{n-1} - v_{n-3}), \\
 \dot{v}_{n-1} &= v_{n-1}(v_n - v_{n-1} - v_{n-2}), \\
 \dot{v}_n &= -v_n(v_n - v_{n-1} + v_{n-2}).
 \end{aligned} \tag{6.3}$$

We consider again the 2×2 matrices which were defined in (4.4) and we also set

$$\begin{aligned}
 X &= \begin{pmatrix} \sqrt{v_n} & i\sqrt{v_n} \\ -\sqrt{v_{n-1}} & i\sqrt{v_{n-1}} \end{pmatrix}, \quad Y = \frac{1}{2} \begin{pmatrix} \sqrt{v_{n-2}v_n} & \sqrt{v_{n-2}v_{n-1}} \\ -\sqrt{v_{n-2}v_{n-1}} & \sqrt{v_{n-2}v_{n-1}} \end{pmatrix}, \\
 W &= \frac{i}{2} \begin{pmatrix} 0 & v_{n-1} - v_n \\ v_n - v_{n-1} & 0 \end{pmatrix}.
 \end{aligned} \tag{6.4}$$

Equations (6.3) can be written in a Lax Pair form $\dot{L} = [L, B]$, where

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & \sqrt{v_1} & i\sqrt{v_1} \\ 0 & O & X & O & \cdots & O \\ \vdots & X^t & O & X_{n-2} & \ddots & \vdots \\ 0 & O & X_{n-2} & \ddots & \ddots & O \\ \sqrt{v_1} & \vdots & \ddots & \ddots & O & X_2 \\ i\sqrt{v_1} & O & \cdots & O & X_2 & O \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -\frac{1}{2}\sqrt{v_1 v_2} & -\frac{1}{2}\sqrt{v_1 v_2} & 0 & 0 \\ \vdots & O & O & Y & O & \cdots & \cdots & O \\ \vdots & O & W & O & Y_{n-3} & \ddots & & \vdots \\ 0 & -Y^t & O & O & \ddots & \ddots & O & \vdots \\ \frac{1}{2}\sqrt{v_1 v_2} & O & -Y_{n-3} & \ddots & \ddots & O & Y_3 & O \\ \frac{1}{2}\sqrt{v_1 v_2} & \vdots & \ddots & \ddots & O & O & O & Y_2 \\ 0 & \vdots & & O & -Y_3 & O & O & O \\ 0 & O & \cdots & \cdots & O & -Y_2 & O & Y_0 \end{bmatrix}. \quad (6.5)$$

The invariant polynomials of this system are given by the functions

$$\begin{aligned} H_2, H_4, \dots, H_{n-1} & \quad \text{when } n \text{ is odd,} \\ H_2, H_4, \dots, H_{n-2}, H_{n-1} & \quad \text{when } n \text{ is even,} \end{aligned}$$

where $H_k = \frac{1}{k} \text{Tr}(L^{2k})$.

As in the case of the BV B_{n+1} system we use the variables b_j , $1 \leq j \leq n+1$ of the equations (3.5) in order to find a cubic bracket π_3 of the BV D_{n+1} system. The dynamical system (3.5) in the case of the Lie algebra of type D_{n+1} can be written in Hamiltonian form $\dot{b}_j = \{b_j, H\}$, with Hamiltonian

$$H = \log b_1 + 2 \sum_{j=2}^{n-1} \log b_j + \log b_n + \log b_{n+1}, \quad (6.6)$$

and Poisson bracket

$$\begin{aligned} \{b_j, b_{j+1}\} &= -\{b_{j+1}, b_j\} = 1 \quad \text{for } j = 1, 2, \dots, n-1, \\ \{b_{n-1}, b_{n+1}\} &= -\{b_{n+1}, b_{n-1}\} = 1, \end{aligned} \quad (6.7)$$

all other brackets are zero. In the new variables v_j ($v_1 = b_1^{-1}b_2^{-1}$, $v_k = 2b_k^{-1}b_{k+1}^{-1}$, $k = 2, \dots, n-2$, $v_{n-1} = b_{n-1}^{-1}b_n^{-1}$, $v_n = b_{n-1}^{-1}b_{n+1}^{-1}$) the above skew-symmetric bracket, which we denote by π_3 , is given by

$$\begin{aligned} \{v_1, v_2\} &= v_1 v_2 (2v_1 + v_2), \\ \{v_i, v_{i+1}\} &= v_i v_{i+1} (v_i + v_{i+1}), \quad i = 2, \dots, n-3, \\ \{v_{n-2}, v_{n-1}\} &= v_{n-2} v_{n-1} (2v_{n-1} + v_{n-2}), \\ \{v_{n-1}, v_n\} &= 2v_{n-1} v_n (v_n - v_{n-1}), \\ \{v_i, v_{i+2}\} &= v_i v_{i+1} v_{i+2}, \quad i = 1, \dots, n-3, \\ \{v_{n-2}, v_n\} &= v_{n-2} v_n (v_{n-2} + 2v_n), \\ \{v_{n-3}, v_n\} &= v_{n-3} v_{n-2} v_n. \end{aligned} \quad (6.8)$$

All other brackets are zero. As in the case of KM system we suppose that n is odd ($n = 2l + 1$) and we look again for a bracket π_1 which satisfies $\pi_3 \nabla H_2 = \pi_1 \nabla H_4$.

We define

$$\tau_{ij} = -\tau_{ji} = v_{2i-1} \prod_{k=i}^{j-1} \frac{v_{2k+1}}{v_{2k}} \quad \text{for } i < j, \quad \tau_{ii} = v_{2i-1}, \quad (6.9)$$

and we let π_1 be the bracket which is defined as follows:

$$\begin{aligned} \{v_i, v_j\} &= (-1)^{i+j-1} \tau_{[\frac{i}{2}]+1, [\frac{j+1}{2}]} \quad \text{for } 1 \leq i < j \leq n-2, \\ \{v_i, v_{n-1}\} &= \{v_i, v_n\} = \frac{(-1)^{i+n}}{2} \tau_{[\frac{i}{2}]+1, [\frac{n}{2}]} \quad \text{for } i = 1, \dots, n-2, \\ \{v_{n-1}, v_n\} &= -\{v_n, v_{n-1}\} = \frac{1}{2} (v_n - v_{n-1}). \end{aligned} \quad (6.10)$$

To illustrate, we give the Poisson matrix of the bracket π_1 in the case $n = 7$

$$\begin{aligned} \pi_1 &= \begin{bmatrix} 0 & \tau_{11} & -\tau_{12} & \tau_{12} & -\tau_{13} & \frac{1}{2}\tau_{13} & \frac{1}{2}\tau_{13} \\ -\tau_{11} & 0 & \tau_{22} & -\tau_{22} & \tau_{23} & -\frac{1}{2}\tau_{23} & -\frac{1}{2}\tau_{23} \\ \tau_{12} & -\tau_{22} & 0 & \tau_{22} & -\tau_{23} & \frac{1}{2}\tau_{23} & \frac{1}{2}\tau_{23} \\ -\tau_{12} & \tau_{22} & -\tau_{22} & 0 & \tau_{33} & -\frac{1}{2}\tau_{33} & -\frac{1}{2}\tau_{33} \\ \tau_{13} & -\tau_{23} & \tau_{23} & -\tau_{33} & 0 & \frac{1}{2}\tau_{33} & \frac{1}{2}\tau_{33} \\ -\frac{1}{2}\tau_{13} & \frac{1}{2}\tau_{23} & -\frac{1}{2}\tau_{23} & \frac{1}{2}\tau_{33} & -\frac{1}{2}\tau_{33} & 0 & \frac{1}{2}(v_7 - v_6) \\ -\frac{1}{2}\tau_{13} & \frac{1}{2}\tau_{23} & -\frac{1}{2}\tau_{23} & \frac{1}{2}\tau_{33} & -\frac{1}{2}\tau_{33} & -\frac{1}{2}(v_7 - v_6) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & v_1 & -\frac{v_1 v_3}{v_2} & \frac{v_1 v_3}{v_2} & -\frac{v_1 v_3 v_5}{v_2 v_4} & \frac{v_1 v_3 v_5}{2v_2 v_4} & \frac{v_1 v_3 v_5}{2v_2 v_4} \\ -v_1 & 0 & v_3 & -v_3 & \frac{v_3 v_5}{v_4} & -\frac{v_3 v_5}{2v_4} & -\frac{v_3 v_5}{2v_4} \\ \frac{v_1 v_3}{v_2} & -v_3 & 0 & v_3 & -\frac{v_3 v_5}{v_4} & \frac{v_3 v_5}{2v_4} & \frac{v_3 v_5}{2v_4} \\ -\frac{v_1 v_3}{v_2} & v_3 & -v_3 & 0 & v_5 & -\frac{v_5}{2} & -\frac{v_5}{2} \\ \frac{v_1 v_3 v_5}{2v_2 v_4} & -\frac{v_3 v_5}{v_4} & \frac{v_3 v_5}{v_4} & -v_5 & 0 & \frac{v_5}{2} & \frac{v_5}{2} \\ -\frac{v_1 v_3 v_5}{2v_2 v_4} & \frac{v_3 v_5}{2v_4} & -\frac{v_3 v_5}{2v_4} & \frac{v_5}{2} & -\frac{v_5}{2} & 0 & \frac{v_7 - v_6}{2} \\ -\frac{v_1 v_3 v_5}{2v_2 v_4} & \frac{v_3 v_5}{2v_4} & -\frac{v_3 v_5}{2v_4} & \frac{v_5}{2} & -\frac{v_5}{2} & -\frac{v_7 - v_6}{2} & 0 \end{bmatrix}. \end{aligned}$$

We obtain the following Theorem:

Theorem 4. (i) π_1, π_3 are Poisson.

(ii) The function

$$\frac{1}{4}H_2 = \frac{1}{8} \text{Tr} (L^4) = v_{n-2}v_n + 2v_{n-1}v_n + \sum_{i=1}^{n-2} v_i v_{i+1} + \frac{1}{2} \sum_{i=2}^{n-2} v_i^2,$$

is the Hamiltonian of the BV D_{n+1} system with respect to the bracket π_1 .

(iii) The function

$$H = (v_n - v_{n-1}) \prod_{i=1}^{n-2} v_i,$$

is the Casimir of the BV D_{n+1} system in the bracket π_1 .

(iv) π_1, π_3 are compatible.

Proof. (i) We denote $\{ \}_d$ the bracket π_1 of D_{n+1} and $\{ \}_b$ the Poisson bracket π_1 of B_n ($n = 2l + 1$). Then $\{ \}_d$ can be defined as follows:

$$\begin{aligned} \{v_i, v_j\}_d &= \{v_i, v_j\}_b, & 1 \leq i, j \leq n-2, \\ \{v_i, v_{n-1}\}_d &= \{v_i, v_n\}_d = \frac{1}{2} \{v_i, v_{n-1}\}_b, & 1 \leq i \leq n-2, \\ \{v_{n-1}, v_n\}_d &= \frac{1}{2} (v_n - v_{n-1}). \end{aligned}$$

We set

$$[v_i, v_j, v_k] = \{v_i, \{v_j, v_k\}\} + \{v_j, \{v_k, v_i\}\} + \{v_k, \{v_i, v_j\}\}.$$

For $i, j, k = 1, 2, \dots, n-2$

$$[v_i, v_j, v_k]_d = [v_i, v_j, v_k]_b = 0.$$

For $i, j = 1, 2, \dots, n-2$

$$[v_i, v_j, v_{n-1}]_d = [v_i, v_j, v_n]_d = \frac{1}{2} [v_i, v_j, v_{n-1}]_b = 0.$$

For $i = 1, 2, \dots, n-2$

$$\begin{aligned} [v_i, v_{n-1}, v_n]_d &= \{v_i, \{v_{n-1}, v_n\}_d\}_d + \{v_{n-1}, \{v_n, v_i\}_d\}_d + \{v_n, \{v_i, v_{n-1}\}_d\}_d \\ &= \frac{1}{2} \{v_i, v_n - v_{n-1}\}_d + \frac{1}{2} \{v_{n-1}, \{v_{n-1}, v_i\}_b\}_d + \frac{1}{2} \{v_n, \{v_i, v_{n-1}\}_b\}_d \\ &= \frac{1}{2} \{v_i, v_{n-1} - v_{n-1}\}_b - \frac{1}{4} \{v_{n-1}, \{v_i, v_{n-1}\}_b\}_b + \frac{1}{4} \{v_{n-1}, \{v_i, v_{n-1}\}_b\}_b \\ &= 0. \end{aligned}$$

Therefore, $\{ \}_d$ is Poisson. Relation (6.7) implies that π_3 is Poisson as well.

(ii), (iii) follow from simple calculations.

(iv) The proof that the bracket $\pi_1 + \pi_3$ is Poisson is similar to the above proof that the $\{ \}_d$ is Poisson. ■

Remark. Since the functions $H_2, H_4, \dots, H_{2l}, H$ are independent and in involution the BV D_{2l+2} system is integrable.

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