where $\mathbf{x} \in \Re^m$ and $\mathbf{f}(\mathbf{x},t)$ is continuous differentiable function such that, $\mathbf{f}:(m\times 1)$ vector function. The main definition and lemma of contraction are taken from[13] and represented here same for the clarity purpose.

Definition 1. A region of the state space is called a contraction region with respect to a uniformly positive definite metric $M(\mathbf{x},t)=\Theta^T(\mathbf{x},t)\Theta(\mathbf{x},t)$ where Θ stands for a differential coordinate transformation matrix, if equivalently

$$\mathbf{F}=\left(\dot{\Theta}+\Theta\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)\Theta^{-1} \tag{2}$$

is uniformly negative definite (UND). This leads to the following convergence result.

Lemma 1. *For system in "Eq. (1)", system trajectory for any initial condition which starts in a ball of constant radius centered about a given trajectory and contained at all a time in a contraction region, remains in that ball and converges exponentially to given trajectory.*

Global exponential convergence to given trajectory is guaranteed if whole state space region is contracting.

2.1 *Proportional-derivative observer*

Co-ordinate transformation are used to simplify the observer design problem based on contraction analysis. Consider a system

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\mathbf{t}). \tag{3}$$

and output is given by

$$\mathbf{y}=\mathbf{h}(\mathbf{x},\mathbf{t}). \tag{4}$$

where $\mathbf{x}$ is the state vector and $\mathbf{y}$ is the measurement vector. Now defining the PD observer for the above system

$$\left.\begin{aligned}\dot{\bar{\mathbf{x}}}&=\mathbf{f}(\hat{\mathbf{x}},\mathbf{t})-\mathbf{k}_P(\hat{\mathbf{y}}-\mathbf{y})-\mathbf{k}_D\hat{\mathbf{y}}\\ \hat{\mathbf{x}}&=\bar{\mathbf{x}}+\mathbf{k}_D\mathbf{y}\end{aligned}\right\} \tag{5}$$

where $\hat{\mathbf{y}}=\mathbf{h}(\hat{\mathbf{x}},\mathbf{t})$ and $\dot{\hat{\mathbf{y}}}=\frac{\partial\hat{\mathbf{y}}}{\partial\hat{\mathbf{x}}}\mathbf{f}(\hat{\mathbf{x}},\mathbf{t})+\frac{\partial\mathbf{h}}{\partial t}$

Differentiation of "Eq. (5)", leads to the following observer dynamics

$$\dot{\hat{\mathbf{x}}}=\mathbf{f}(\hat{\mathbf{x}},\mathbf{t})-\mathbf{k}_P(\hat{\mathbf{y}}-\mathbf{y})-\mathbf{k}_D(\dot{\hat{\mathbf{y}}}-\dot{\mathbf{y}}) \tag{6}$$

Since "Eq. (6)" contain $\dot{\mathbf{y}}$ and $\dot{\mathbf{y}}$ is generally not available in the output. So "Eq. (5)" is used to obtain $\hat{\mathbf{x}}$ and $\dot{\mathbf{y}}$ is not explicitly used.

3. Problem Formulation

Consider a Sprott Chaotic system

$$\left.\begin{aligned}\dot{x}_1&=2x_3\\ \dot{x}_2&=-2x_2+x_3\\ \dot{x}_3&=-x_1+x_2+x_2{}^2\end{aligned}\right\}. \tag{7}$$

and the output is given by

$$\mathbf{y}=\mathbf{c}\mathbf{x}=[1\ 0\ 0]\mathbf{x}. \tag{8}$$

where $\mathbf{x}$ is the state vector and $\mathbf{y}$ is the measurement vector.

The open loop response of the system with simulation period of 100 seconds and initial conditions, $[2\ \ 1\ \ \ 3]^T$ is shown in Fig.1. Three dimensional phase portrait is shown in Fig.1 (a). The open loop response of the system to initial condition is shown in Figs.1 (b)-(d).

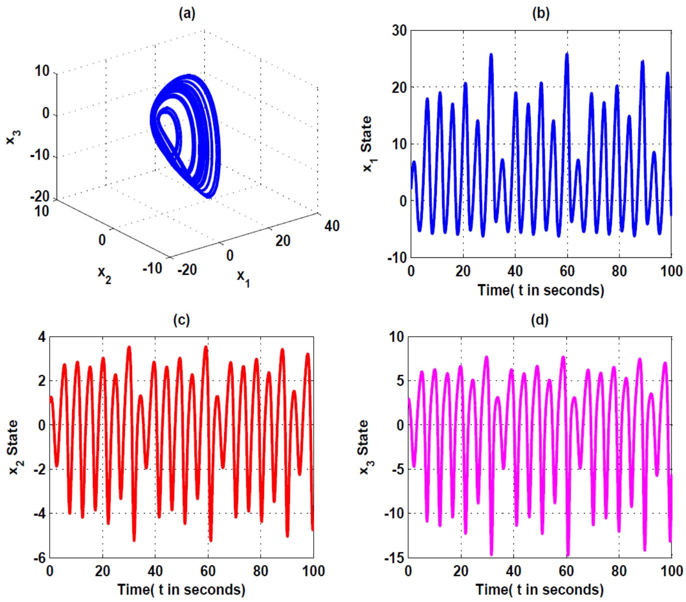

Fig. 1. System without controller: (a) Phase portrait in three dimensions and (b) – (d) time variation of state Trajectories

3.1 *Proportional-derivative observer for sprott chaotic system*

Now defining the PD observer for the above system dynamics,

$$\left.\begin{aligned}\dot{\bar{x}}_1 &= 2\hat{x}_3 - k_{P1}(\hat{x}_1 - x_1) - 2k_{D1}\hat{x}_3\\ \dot{\bar{x}}_2 &= -2\hat{x}_2 + \hat{x}_3 - k_{P2}(\hat{x}_1 - x_1) - 2k_{D2}\hat{x}_3\\ \dot{\bar{x}}_3 &= -\hat{x}_1 + \hat{x}_2 + \hat{x}_2^{\,2} - k_{P3}(\hat{x}_1 - x_1) - 2k_{D3}\hat{x}_3\end{aligned}\right\} \quad (9)$$

and output is given by

$$\hat{\mathbf{y}} = \mathbf{c}\hat{\mathbf{x}} = [1\ 0\ 0]\hat{\mathbf{x}}. \quad (10)$$

Now using co-ordinate transformation

$$\left.\begin{aligned}\hat{x}_1 &= \bar{x}_1 + k_{D1}x_1\\ \hat{x}_2 &= \bar{x}_2 + k_{D2}x_1\\ \hat{x}_3 &= \bar{x}_3 + k_{D3}x_1\end{aligned}\right\} \quad (11)$$

where $\hat{\mathbf{x}}$ represents the observer states and $k_{Pi}\ \&\ k_{Di}$, i=1,2,3 represents PD observer gains.
Now computing the virtual displacement

$$\begin{bmatrix}\delta\dot{\bar{x}}_1\\ \delta\dot{\bar{x}}_2\\ \delta\dot{\bar{x}}_3\end{bmatrix} = \begin{bmatrix}-k_{P1} & 0 & 2-2k_{D1}\\ -k_{P2} & -2 & 1-2k_{D2}\\ -1-k_{P3} & 1+2\bar{x}_2+2k_{D2}x_1 & -2k_{D3}\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{bmatrix}$$

The symmetric part of the Jacobian is $\mathbf{J}_s = \frac{1}{2}\left(\mathbf{J} + \mathbf{J}^T\right)$

$$\mathbf{J}_s = \begin{bmatrix}-k_{P1} & \frac{(-k_{P2})}{2} & \frac{(1-2k_{D1}-k_{P3})}{2}\\ \frac{(-k_{P2})}{2} & -2 & \frac{(2-2k_{D2}+2\bar{x}_2+2k_{D2}x_1)}{2}\\ \frac{(1-2k_{D1}-k_{P3})}{2} & \frac{(2-2k_{D2}+2\bar{x}_2+2k_{D2}x_1)}{2} & -2k_{D3}\end{bmatrix}$$

To ensure $\mathbf{J}_s$ to be UND, the following conditions should be satisfied.

(I) $k_{P1} > 0$

(II) $8k_{P1} > k_{P2}^{\,2}$

(III) $$\frac{k_{P1}(2-2k_{D2}+2\bar{x}_2+2k_{D2}x_1)^2}{4} + \frac{k_{P2}^{\,2}k_{D3}}{2} + \frac{(1-2k_{D1}-k_{P3})^2}{2} < 4k_{P1}k_{D3} + k_{P2}\frac{(1-2k_{D1}-k_{P3})}{2}\frac{(2-2k_{D2}+2\bar{x}_2+2k_{D2}x_1)}{2}$$

4. Simulations

Simulations is run for 100 seconds with initial conditions $[2\ 1\ 3\ 0\ 0\ 0]^T$. The observer gains are kept zero for time period t=15 sec. and after that gains are switched on. The values of the observer gain are $k_{D1} = 5;\ k_{D2} = 3;\ k_{D3} = 20$ $k_{P1} = 2;\ k_{P2} = 1;\ k_{P3} = 3$.

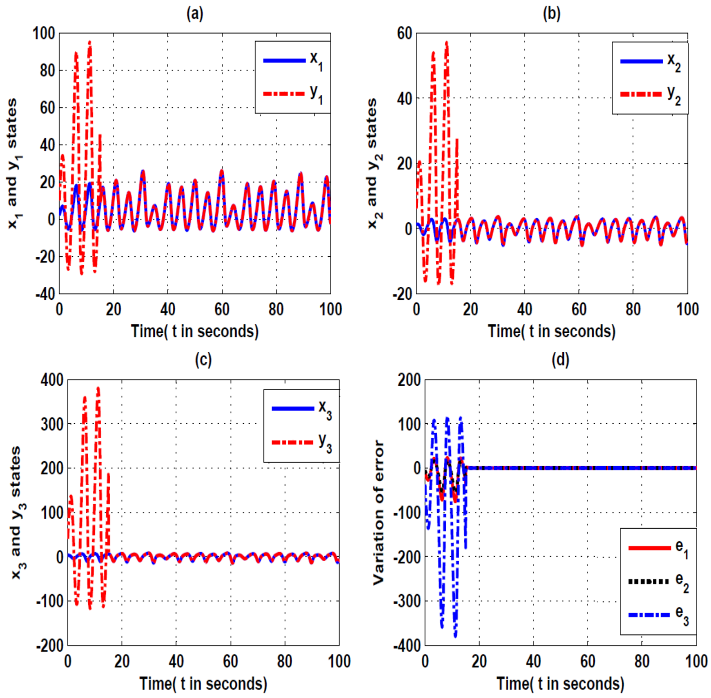

Fig. 2. Convergence behavior & variation of error

5. Conclusion

Co-ordinate transformation are used to design the observer that helps to avoid the derivative of output in the observer dynamics. Results are numerically verified for Sprott chaotic system. Simulations are given to validate the proposed approach.

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