Pseudo-chaotic sequence generation by Piece Wise Linear (PWL) maps and dependence of its properties on characteristics of PWL map

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Abstract

Chaos, which used to be one of the worst enemies of electrical engineers, is nowadays deliberately employed for secure communication. Such systems employing chaotic sequences are called as Chaotic Communication Systems. For ease of implementation pseudo chaotic sequences, generated by Piece-Wise Linear maps, are used in such systems. Properties such as autocorrelation function and spectrum of the chaotic sequence used are very important to achieve the desired performance from the system. This paper discusses that, maps with higher magnitude of slope and having both positive and negative slope portions in equal proportion have better white noise-like autocorrelation and spectral properties.

Keywords: Chaos; map; MATLAB; Pseudo chaotic sequence; PWL map.

1. Introduction

For a long time chaos or randomness was thought to be harmful to most of engineering applications, until recently it was productively employed for secure communication. For ease of reproducibility pseudo chaotic sequences are mainly employed. The properties of pseudo chaotic sequence used in a digital chaotic communication system is very critical to proper working of the system in terms of its security against interception and synchronization between the receiver and transmitter. The simplest way of generating pseudo chaotic sequences is by Piece-Wise Linear (PWL) maps. Several PWL maps which are generated by proper transformation on modulo-2 map (used to generate m-sequence) can be easily implemented on hardware by using a reconfigurable Linear Feedback Shift Register (LFSR).[1]

As it is known, that the sequence generated by a map is chaotic, only if the Lyapunov Exponent, λ for the same is positive.[2] That is, the mean of magnitude of the slope of continuous pieces of the map must be over unity (it can be illustrated by cobwebbing). So, the effect of slope of the continuous portions of map over its chaotic characteristics is evident. This paper discusses the effect of slope of the continuous segments of PWL maps on their characteristics such as spectrum of the sequence generated and their autocorrelation properties.

2. Background theory

2.1. *Chaos from Maps*

Chaotic signals are random-like aperiodic signals. Chaotic behavior is associated with nonlinear systems i.e. systems described by nonlinear differential equations. Similarly, discrete time chaotic sequences are described by difference equations or recursion relations or iterated maps or simply maps.

Consider $X_{n+1} = \cos(X_n)$.

This is an example of a 1-dimensional map - the sequence X_0; X_1; X_2….is called the orbit starting from X_0. Such sequences aren't aperiodic but have a large

period, due to which such sequences are called as pseudo chaotic sequences. For a detailed discussion refer.[2]

Following are some PWL maps which can be used to generate pseudo-chaotic sequences:

(i) Slope 2 maps
  (a) Modulo 2 map
  (b) Tent map
  (c) Twisted map
  (d) Ship map
(ii) Slope 4 maps
  (a) Twisted map
  (b) Ship map

Fig 1.a and fig 1.b illustrate cobwebbing on a slope-2 twisted map.

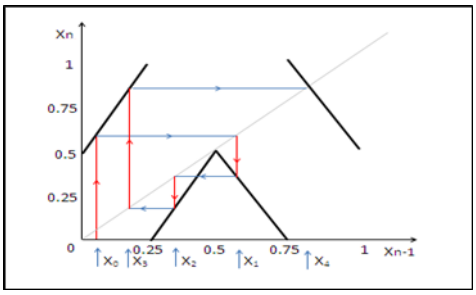

Fig. 1.a. Plot after 4 iterations of cobwebbing

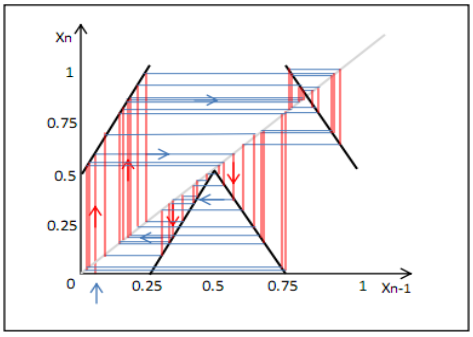

Fig. 1.b. Plot after many iterations of cobwebbing.

2.2. *Effect of slope on Chaotic Characteristics*

There are two constitutive parameters which together define the slope of continuous segments of PWL map and its effect on properties of the sequence so generated. They are:

2.2.1. *Magnitude of slope*

As mentioned previously, sequence generated by a map is chaotic only if the Lyapunov Exponent, λ for the same is positive.[2] That is, the mean of magnitude of the slope of continuous pieces of the map must be over unity (it can be illustrated by cobwebbing), otherwise the sequence generated converges to a stable fixed point. Thus there is a cutoff slope (unity) only above which the map generates chaotic sequence. Larger the slope of the map more rapidly do the successive values of sequence so generated differ from one another. Hence, sequences generated by higher slope maps are more random.

2.2.2. *Sign of slope*

Assuming that $\lambda > 0$ for a map, the fig.2.a and fig.2.b below illustrate the effect of sign of slope on the sequence so generated.

Observe that

a) Positive slope portion of the map (fig 2.a) acts as a unidirectional repeller and directs sequence X_{n+1} to only one direction (either above only or below only) with respect to the unstable fixed point. Thus it contributes to the low frequency contents of the sequence.

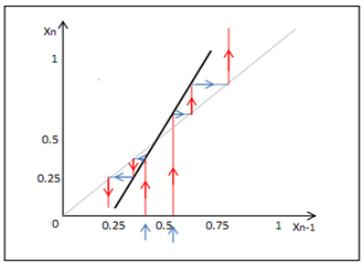

Fig. 2.a Cobwebbing with positive slope portion of map.

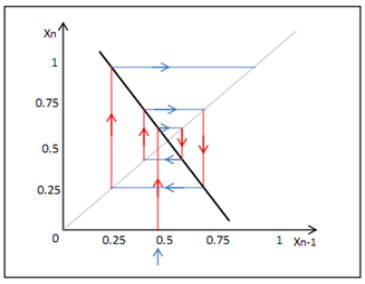

Fig. 2.b Cobwebbing with negative slope portion of map.

b) Unlike this, the negative slope portion of the map (fig 2.b) acts as a bidirectional repeller and directs sequence X_{n+1} to both directions (above and below alternately) with respect to the unstable fixed point. Thus it contributes to the high frequency contents of the sequence.

Hence for a white noise-like PSD, the map must contain both positive and negative slope portions.

3. Simulations and Observations

Modulo-2 map (used to generate PN sequence) and slope 4 twisted tent map were implemented on MATLAB to generate corresponding sequences.

As can be seen in fig. 3.a, modulo-2 map is having only positive slope segments, whereas in fig 3.b, the Slope-4 Twisted Tent map has both positive and negative slope portions in equal proportion.
Sequences generated by both the maps were studied for their respective FFT and autocorrelation properties. They are compared below:

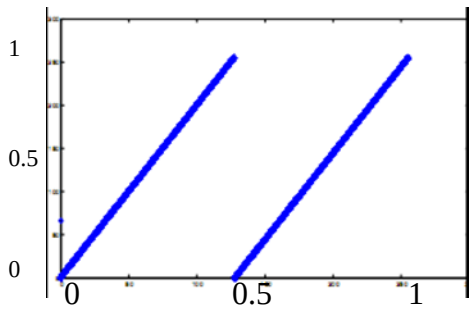

Fig. 3.a. Modulo -2 map.

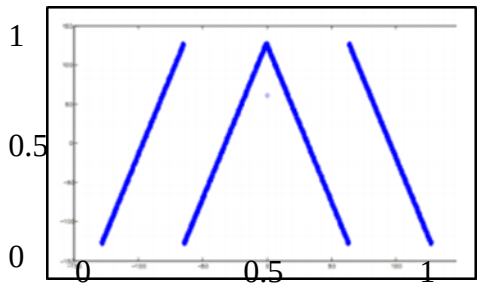

Fig. 3.b. Slope-4 Twisted Tent map.

3.1. *Autocorrelation of the sequence generated*

Fig. 4.a and fig 4.b show the autocorrelation function of the sequence generated by modulo-2 map and slope-4 twisted tent map respectively. As can be seen, the autocorrelation function of sequence generated by modulo-2 map (m-sequence) has a peak at 0 shift, but has many other smaller peaks, which cause error in detection in DSSS communication system.
Similarly, in fig 4.b the autocorrelation function of sequence generated by slope-4 twisted tent map also has a peak at 0 shift, but the rest of smaller peaks aren't as distinct as in the previous case. So, it is in later case, that the autocorrelation is closer to delta function compared to the former one.

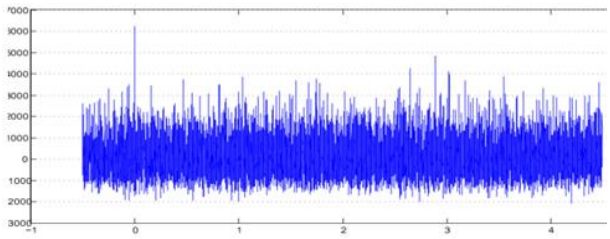

Fig. 4.a. Autocorrelation function of sequence generated by modulo-2 map.

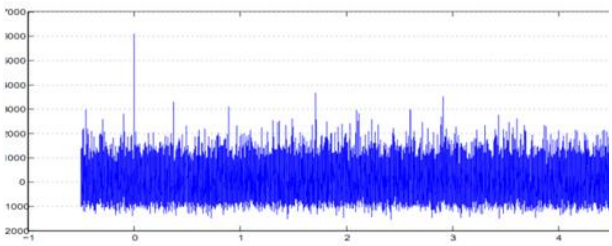

Fig. 4.b. Autocorrelation function of sequence generated by slope-4 twisted tent map.

3.2. *FFT of the sequence generated*

The FFT of the sequence generated by the two maps are compared in fig 5.a and fig 5.b. As is evident from the fig 5.a and fig 5.b that the FFT of m- sequence generated by modulo-2 map has a dominant component at lower frequencies since, it only has positive slope segments. Unlike m-sequence, the FFT of sequence generated by slope-4 twisted tent map is uniform throughout the spectrum, since it contains both positive and negative slope portions in equal proportions in the map.

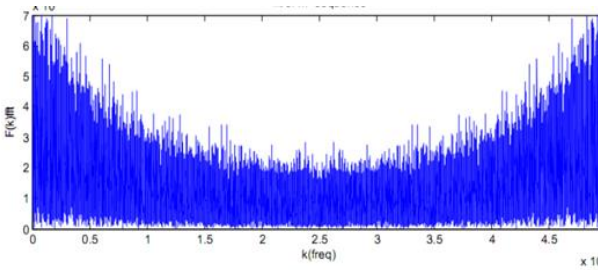

Fig. 5.a. FFT of sequence generated by modulo-2 map.

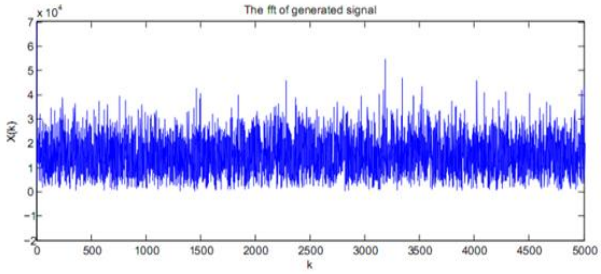

Fig. 5.b. FFT of sequence generated by slope-4 twisted tent map.

4. Conclusion

From the discussion above, it can be concluded that, for a PWL map with $\lambda > 0$,

a) Positive slope portion of map contributes majorly to lower frequencies and

b) Negative slope portion of map contributes majorly to higher frequencies of the spectrum.

Hence the sequences generated by a PWL map comprising of both positive slope and negative slope portions in equal proportion have white noise type of spectrum. Hence, the autocorrelation function for such sequence is closer to the delta function.

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References

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