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# Stability Analysis of Some Integrable Euler Equations for $SO(n)$

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**Abstract**

A family of special cases of the integrable Euler equations on $so(n)$ introduced by Manakov in 1976 is considered. The equilibrium points are found and their stability is studied. Heteroclinic orbits are constructed that connect unstable equilibria and are given by the orbits of certain 1-parameter subgroups of $SO(n)$. The results are complete in the case $n = 4$ and incomplete for $n > 4$.

## 1 Introduction

Suppose that we have a Hamiltonian vector field defined on a Poisson space. A natural problem is to find its equilibrium points and to check whether or not they are stable. Stability is important in mathematical modelling, as it gives an indication which behaviour exhibited by a (mathematical) modelling system is a reliable representative of behaviour in the corresponding (real) modelled system. In the present paper we adopt the following

**Definition.** Let $\mathbf{X}_H$ be the Hamiltonian vector field corresponding to the function $H$. The equilibrium point $x$ of $\mathbf{X}_H$ is *stable* if for any neighbourhood $U$ of $x$, there exists a neighbourhood $V$ of $x$ such that $\phi(0) \in V$ and $\dot{\phi}(t) = \mathbf{X}_H(\phi(t))$ implies that $\phi(t) \in U$ $\forall\, t \geq 0$; otherwise $x$ is an *unstable equilibrium* of $\mathbf{X}_H$.

The aim of the present work is to perform a stability analysis for certain integrable Euler equations associated with the group $SO(n)$, focusing mainly on the case $n = 4$, and in addition to examine the heteroclinic orbits. These equations represent a particularly simple special case of the integrable Hamiltonian systems introduced by Manakov in [1]. In the $n = 3$ case they reduce to the classical Euler equations for the angular momentum of a free rigid body in the moving frame. As explained in several mechanics textbooks, the qualitative behaviour of the solutions of the classical Euler equations is easily visualised in terms of their phase portrait, see for instance the picture on the cover of the book [2].

This system lives on a coadjoint orbit $S^2$ of $SO(3)$, and it has the interesting feature that the unstable equilibria are connected by heteroclinic orbits that are given by great circles on the sphere $S^2$. Let us recall that a heteroclinic orbit in general consists of the points of a nontrivial integral curve of a dynamical system and equilibrium points. Since the great circles on $S^2$ are the orbits of the 1-parameter subgroups of $SO(3)$, we shall in the $SO(n)$ case enquire about the existence of heteroclinic orbits that are orbits of 1-parameter subgroups of $SO(n)$; we consider this to give rise to the most interesting results of the paper.

This work is intended as a step towards a stability analysis of the full set of the Manakov systems [1] which, in addition to the special case studied here, contains for example the $n$-dimensional rigid body of [3], and has many interesting Lie-algebraic generalizations [4, 5]. In fact, most of our results are not difficult to extend. The result for which generalization presents problems is the one described in Section 3.

For convenient reference later, we now recall some standard facts about the linearisation of a Hamiltonian dynamical system and its use in the stability analysis of the original system. Let $M$ be a Poisson space. Let $H \in C^\infty(M)$ and suppose that $x$ is an equilibrium point of the Hamiltonian vector field $\mathbf{X}_H$. The linearisation at $x$ is a flow in $T_xM$ given by

$$\dot{v} = (\mathcal{L}_V \mathbf{X}_H)(x), \tag{1.1}$$

with $V$ any vector field such that $V(x) = v$. By choosing any system of local coordinates in a neighbourhood of $x$, this becomes a system of the form $\dot{v} = \mathbf{L}v$, with $\mathbf{L}$ a square matrix of the same size as the dimension of $M$. For a Hamiltonian system the eigenvalues of the linearisation at $x$ come in groups of four, in the sense that if $\lambda$ is an eigenvalue of $\mathbf{L}$ then $-\lambda$ is an eigenvalue of $\mathbf{L}$ and so is the complex conjugate $\bar{\lambda}$. The following statements are well known, see [6].

**1.** $x$ is unstable if the linearisation at $x$ of the system $\dot{\phi} = \mathbf{X}_H(\phi)$ has an eigenvalue with a positive real part. If no eigenvalues of the linearisation have positive real part, then all eigenvalues have to be imaginary; in this case $x$ may or may not be stable.

**2.** $x$ is stable if there exists $f \in C^\infty(M)$ for which $\{f, H\} = 0$ in a neighbourhood of $x$ and

$$(i)\ \ df(x) = 0, \qquad (ii)\ \ d^2 f(x)\ \text{ is definite.} \tag{1.2}$$

If the rank of the Poisson bracket is constant in some neighbourhood of $x$, then it is sufficient that properties $(i)$ and $(ii)$ in (1.2) be satisfied with respect to vectors tangent to the symplectic leaf through $x$. This is discussed for example in [7]. In this paper we consider such a "regular situation" since the phase spaces of our interest will be generic coadjoint orbits of the Lie group $SO(n)$. We will assume all entities appearing in the definition of the systems studied to be generic, since this would be a reasonable assumption in a physical context and it also simplifies the problem.

The organization of the paper and of our results is as follows. The next section contains the definition of the Hamiltonian systems of interest associated with $SO(n)$ together with a description of their equilibrium points (Proposition 1). In Section 3 we present a complete analysis of the stability of the equilibrium points in the $n = 4$ case. The outcome of our study is given by Proposition 2. In Section 4 we describe a necessary condition

(Proposition 3) for the possibility to construct heteroclinic orbits by means of 1-parameter subgroups for Hamiltonian systems living on a coadjoint orbit, and concretely construct such heteroclinic orbits for the systems associated with $SO(4)$. In Section 5 the main features of the stability analysis are outlined for any $n$. In particular, the construction of the heteroclinic orbits is generalized to the $SO(n)$ case (see Proposition 4). Section 6 contains a brief summary of the results and some open problems.

## 2 A family of integrable Euler equations for $SO(n)$

We define below the Hamiltonian systems to be studied and describe their equilibrium points. As explained at the end of the section, these systems correspond to a special case of the integrable Euler equations introduced in [1].

Consider the Lie algebra $so(n)$ of the real orthogonal group $SO(n)$. An element of $so(n)$ is an $n \times n$ antisymmetric real matrix. The Lie–Poisson bracket of functions on $so(n)^*$ is given by

$$\{\phi, \psi\}(\alpha) = \langle \alpha, [d_\alpha \phi, d_\alpha \psi] \rangle \qquad \forall\, \alpha \in so(n)^*, \tag{2.1}$$

where $d_\alpha \phi \in so(n)$ is defined by

$$\langle \beta, d_\alpha \phi \rangle = \left.\frac{d}{dt}\right|_{t=0} \phi(\alpha + t\beta) \qquad \forall\, \beta \in so(n)^*, \tag{2.2}$$

and $d_\alpha \psi$ similarly. The symplectic leaves in $so(n)^*$ are the coadjoint orbits of $SO(n)$ in $so(n)^*$. It will be convenient to identify $so(n)^*$ with $so(n)$ with the aid of a multiple of the standard trace form for $n \times n$ matrices, so that $\langle \beta, X \rangle := -\frac{1}{2}\,\mathrm{tr}\,(\beta X)$.

Let us define the Cartan subalgebra $\mathbf{h}$ in $so(n)$ to be the set of all matrices $x$ of the form

$$\begin{aligned} x &= \sum_{k=1}^{m} x_k e_{kk} \otimes \mathrm{i}\sigma_2 \qquad \text{if} \quad n = 2m, \\ \text{or} \quad x &= \begin{pmatrix} \sum\limits_{k=1}^{m} x_k e_{kk} \otimes \mathrm{i}\sigma_2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{if} \quad n = 2m+1, \end{aligned} \tag{2.3}$$

where $m$ is any positive integer. Here $e_{ij}$ is the $m \times m$ matrix having 1 for the term in the $i$th row and in the $j$th column and all other terms zero. We use the Pauli matrices

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_0 := \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.4}$$

An element $x$ of $\mathbf{h}$ is generic if $x_k \neq 0\ \forall\, k$ and $x_k^2 \neq x_l^2$ if $k \neq l$. Using the identification of $so(n)^*$ with $so(n)$, a generic symplectic leaf can be written as

$$\mathcal{O}_x = \left\{ g x g^{-1} \,|\, g \in SO(n) \right\} \tag{2.5}$$

with $x$ a generic element in $\mathbf{h}$. The isotropy subalgebra of $x$ in $so(n)$ consists of matrices of the same form as $x$, i.e. given by the same formula as (2.3) with different values of $x_i$. The isotropy subgroup $SO(n)_x$ is the exponential of this algebra.

In this paper we are interested in Hamiltonian systems $(\mathcal{O}_x, \{\ ,\ \}, H)$ on generic coadjoint orbits, where $H$ has the form

$$H(\mu) := -\frac{1}{2}\,\mathrm{tr}\,(J\mu^2), \qquad \mu \in \mathcal{O}_x, \tag{2.6}$$

with some constant matrix $J = \mathrm{diag}\,(J_1, \dots, J_n)$. We assume that $J_i^2 \neq J_j^2$ if $i \neq j$. The generalized Euler equation defined by the Hamiltonian vector field $\mathbf{X}_H$ can be written as follows:

$$\dot{\mu} = \left[J, \mu^2\right]. \tag{2.7}$$

An equilibrium point on $\mathcal{O}_x$, for $x$ given by (2.3), is a point $gxg^{-1}$ such that

$$0 = \left[J, gx^2g^{-1}\right]. \tag{2.8}$$

Let $p$ be an element of the permutation group $S_n$ (the Weyl group of $sl(n)$), and introduce the permutation matrix $\bar{p} \in O(n)$ by

$$\bar{p}_{ij} = \delta_{i,p(j)} \qquad (i, j = 1, \dots, n). \tag{2.9}$$

For any diagonal matrix $D = \mathrm{diag}\,(d_1, \dots, d_n)$, one has

$$p(D) := \mathrm{diag}\,(d_{p^{-1}(1)}, \dots, d_{p^{-1}(n)}) = \bar{p}D\bar{p}^{-1}, \tag{2.10}$$

and the parity of $p$ satisfies $\mathrm{sgn}\,(p) = \det(\bar{p})$. Since $J$ and $x^2$ are diagonal matrices, $\bar{p}x\bar{p}^{-1}$ is clearly an equilibrium point whenever it belongs to $\mathcal{O}_x$. This holds obviously for the even permutations. If $n = (2m+1)$ is odd, then $\bar{p}x\bar{p}^{-1} \in \mathcal{O}_x$ for any $p \in S_n$, since in this case

$$\bar{p}D\bar{p}^{-1} = \hat{p}D\hat{p}^{-1} \qquad \text{with} \quad \hat{p} := \mathrm{sgn}\,(p)\bar{p} \in SO(2m+1). \tag{2.11}$$

We can prove that the equilibrium points associated in this manner with the permutations exhaust all the equilibria on $\mathcal{O}_x$.

**Proposition 1.** *The set of equilibrium points on a generic orbit $\mathcal{O}_x$, for $x$ of the form given in (2.3), consists of the matrices $\bar{p}x\bar{p}^{-1}$, where $p \in S_n$ is an even permutation if $n$ is even, and $p \in S_n$ is an arbitrary permutation if $n$ is odd. The equilibrium points associated with different permutations are different.*

**Proof.** Let us consider the set

$$E_x := \left\{gxg^{-1} \mid \left[J, gx^2g^{-1}\right] = 0,\ g \in O(n)\right\}. \tag{2.12}$$

Since $J$ is a regular diagonal matrix by assumption, $gx^2g^{-1}$ must be a diagonal matrix whose entries are obtained by permuting the entries of the diagonal matrix $x^2$. We can choose a set of elements of $S_n$, say $\{p_i\}_{i=1}^N$, for which the matrices $p_i\left(x^2\right)$ are distinct from each other for $i \neq j$ and they contain all matrices that are obtained by permuting the diagonal entries of $x^2$. Note that $N = \frac{n!}{2^m}$ for $n = 2m$ or $n = (2m+1)$, and the $p_i$ are a set of representatives for the coset space $S_n/S_n^{x^2}$, where

$$S_n^{x^2} = \left\{p \in S_n \mid p\left(x^2\right) = x^2\right\}. \tag{2.13}$$

For $n = 2m$ or $n = (2m+1)$, the group $S_n^{x^2}$ is generated by the elements

$$\tau^{1,2}, \tau^{3,4}, \ldots, \tau^{2m-1,2m}, \tag{2.14}$$

where $\tau^{k,l} \in S_n$ denotes the transposition that exchanges $k$ with $l$.

Since any $g \in O(n)$ that appears in (2.12) satisfies $gx^2g^{-1} = \bar{p}_i x^2 \bar{p}_i^{-1}$ with some $1 \leq i \leq N$, it follows that the most general such $g$ can be written as

$$g = \bar{p}_i \gamma \qquad \text{with some} \quad \gamma \in O(n)_{x^2}, \tag{2.15}$$

$$O(n)_{x^2} := \left\{\gamma \in O(n) \mid \gamma x^2 \gamma^{-1} = x^2\right\}. \tag{2.16}$$

The isotropy group $O(n)_{x^2}$ consists of block-diagonal matrices with arbitrary elements of $O(2)$ in the $2 \times 2$ blocks. It is useful to consider also

$$O(n)_x := \left\{q \in O(n) \mid qxq^{-1} = x\right\}, \tag{2.17}$$

which consists of block-diagonal matrices with each $2 \times 2$ block containing an arbitrary element of $SO(2)$. The point to notice is that any $\gamma \in O(n)_{x^2}$ can be uniquely written in the form[1]

$$\gamma = \bar{\Gamma} q, \qquad \Gamma \in S_n^{x^2}, \quad q \in O(n)_x. \tag{2.18}$$

This follows from the fact that $O(2)/SO(2)$ can be identified with the group generated by the transposition matrix $\sigma_1$. By using these observations, we see that any element $gxg^{-1} \in E_x$ has the form

$$gxg^{-1} = \bar{p}_i \bar{\Gamma} x (\bar{p}_i \bar{\Gamma})^{-1} \qquad \left(1 \leq i \leq N,\ \Gamma \in S_n^{x^2}\right). \tag{2.19}$$

As $\bar{p} = \bar{p}_i \bar{\Gamma}$ for $p = p_i \Gamma$, this implies that all elements of $E_x$ are given by $\bar{p} x \bar{p}^{-1}$ with some $p \in S_n$. It is clear from the definitions that any $p \in S_n$ can be decomposed as $p = p_i \Gamma$ with a unique $p_i$ and a unique element of $S_n^{x^2}$, and one can check directly that different permutations are associated with different points of $E_x$.

If $n = (2m+1)$, the statement of the proposition follows immediately from the above-established results (the coadjoint orbits of $O(2m+1)$ and $SO(2m+1)$ coincide). The proof is completed by noting that in the $n = 2m$ case only those elements $\bar{p} x \bar{p}^{-1}$ lie on the coadjoint orbit of $SO(2m)$ through $x$ for which $p$ is an even permutation. This fact can be verified, for example, by performing an analogous analysis as above in the case for which $g$ in (2.12) is restricted to $SO(n)$ from the beginning. ■

**Remark 1.** Suppose that we study the nature of an equilibrium point $\bar{p} x \bar{p}^{-1} \in \mathcal{O}_x$. We may then choose a different basis in which this point is represented by the matrix of $x$, and $J$ is replaced by the matrix $\bar{p}^{-1} J \bar{p}$. We may thus assume without loss of generality that the equilibrium point of interest is always represented by the same matrix $x \in \mathbf{h}$ in (2.3).

[1] In fact, $O(n)_x$ is a normal subgroup of $O(n)_{x^2}$ and $S_n^{x^2}$ is the corresponding factor group.

**Remark 2.** It follows from (2.7) that in our case the linearised system at the point $x$ is the flow in $T_x\mathcal{O}_x$ defined by

$$\dot{v} = [J, vx + xv]. \tag{2.20}$$

**Remark 3.** In the terminology of generalized rigid bodies [7] the quantity $\mu$ in the Euler equation (2.7) is the angular momentum relative to the body. Correspondingly, the inverse of the moment of inertia operator maps $\mu$ to the angular velocity $\omega$ relative to the body according to $\mu \mapsto \omega = -(J\mu + \mu J)$. Indeed, then (2.7) takes the classical form $\dot{\mu} = [\mu, \omega]$. This is a special case of the integrable rigid body systems introduced in [1] by the relation $\mu_{ij} = \frac{a_i - a_j}{b_i - b_j}\omega_{ij}$ with arbitrary constants $a_i$, $b_i$. The case (2.7) arises by setting $b_i = a_i^2$ with $a_i = -J_i$, while the $n$-dimensional rigid body of [3] is obtained by setting $a_i = b_i^2$.

## 3 Stability analysis in the $n = 4$ case

The Lie algebra $so(4)$ is the same as the direct sum $so(3) \oplus so(3)$. This can be seen by identifying $so(3)$ with $su(2)$ and then finding two commuting copies of $su(2)$ in $so(4)$. In terms of the Pauli matrices, we have $su(2) = \operatorname{span}\{\mathrm{i}\sigma_1, \mathrm{i}\sigma_2, \mathrm{i}\sigma_3\}$, and two commuting $su(2)$ subalgebras that together span $so(4)$ are

$$su(2) \cong \operatorname{span}\{\sigma_1 \otimes \mathrm{i}\sigma_2, \mathrm{i}\sigma_2 \otimes \sigma_0, \sigma_3 \otimes \mathrm{i}\sigma_2\} \cong \operatorname{span}\{i\sigma_2 \otimes \sigma_1, \sigma_0 \otimes \mathrm{i}\sigma_2, \mathrm{i}\sigma_2 \otimes \sigma_3\}. \tag{3.1}$$

In the coordinates $(l, m)$ on $so(4) = so(3) \oplus so(3)$ given by

$$\begin{aligned} \mu = {} & \big[l_1\sigma_1 \otimes \mathrm{i}\sigma_2 + l_2 \mathrm{i}\sigma_2 \otimes \sigma_0 + l_3\sigma_3 \otimes \mathrm{i}\sigma_2\big] \\ & - \big[m_1 \mathrm{i}\sigma_2 \otimes \sigma_1 + m_2 \mathrm{i}\sigma_2 \otimes \sigma_3 + m_3\sigma_0 \otimes \mathrm{i}\sigma_2\big], \end{aligned} \tag{3.2}$$

the Poisson bracket is

$$\{l_i, l_j\} = \epsilon_{ijk} l_k, \qquad \{l_i, m_j\} = 0, \qquad \{m_i, m_j\} = \epsilon_{ijk} m_k, \tag{3.3}$$

and $|l|^2$ and $|m|^2$ are Casimir functions. The rigid body Hamiltonian $H$ and an independent commuting integral $K$ are now given by

$$H(l, m) = l^T \Lambda m, \qquad K(l, m) = \frac{1}{2} l^T \Lambda^2 l + \frac{1}{2} m^T \Lambda^2 m - l^T \Theta m, \tag{3.4}$$

where $\Lambda$ and $\Theta$ are constant diagonal matrices

$$\Lambda = \operatorname{diag}(\Lambda_1, \Lambda_2, \Lambda_3), \qquad \Theta = \operatorname{diag}(\Lambda_2\Lambda_3, \Lambda_1\Lambda_3, \Lambda_1\Lambda_2). \tag{3.5}$$

The equations of motion corresponding to $H$ are

$$\dot{l} = (\Lambda m) \wedge l \qquad \dot{m} = (\Lambda l) \wedge m. \tag{3.6}$$

We apply the usual identification of $so(3)$ with $\mathbb{R}^3$ equipped with the vector-product, denoted by $\wedge$. The formulae in (3.4) may be recovered by a technique due to Manakov [1]: define $\mathcal{L} = \lambda J + \mu$, then the set of coefficients of $\lambda$ amongst all traces of powers of $\mathcal{L}$ forms a commuting family. Here $K \sim (\operatorname{tr} \mathcal{L}^4)|_{\lambda^2}$, $H \sim (\operatorname{tr} \mathcal{L}^3)|_{\lambda^1}$ up to Casimirs and (from (2.6))

$\Lambda$ is related to $J$ by $\Lambda_1 = -J_1+J_2+J_3-J_4$, $\Lambda_2 = -J_1+J_2-J_3+J_4$, $\Lambda_3 = -J_1-J_2+J_3+J_4$. From now on we make the *genericity assumption* that

$$\Lambda_i^2 \neq \Lambda_j^2 \quad \text{if} \quad i \neq j \qquad \text{and} \quad \Lambda_i \neq 0 \quad \forall\, i. \tag{3.7}$$

The first part of these conditions follows from the assumption that $J_p^2 \neq J_q^2$ for $p \neq q$.

Let $e_k$ ($k = 1, 2, 3$) denote the standard basis of $\mathbb{R}^3$. The equilibrium points of (3.6) that lie on generic coadjoint orbits are in fact given by

$$(l, m) = b(ae_k, e_k) \qquad \text{with} \quad \mathbb{R} \ni a, b \neq 0, \quad k = 1, 2, 3. \tag{3.8}$$

We next study the stability of an equilibrium point of the form

$$(l, m) = (ae_3, e_3) \qquad \text{with} \quad e_3 = (0, 0, 1)^T \quad \text{and} \quad \mathbb{R} \ni a \neq 0, \tag{3.9}$$

and then the general case (3.8) will be reduced to this one.

The elements of $T_{(ae_3,e_3)}\mathcal{O}_{(ae_3,e_3)}$ can be parametrized as $(a\xi \wedge e_3, \eta \wedge e_3)$ with $\xi = \xi_1 e_1 + \xi_2 e_2$ and $\eta = \eta_1 e_1 + \eta_2 e_2$. By putting $v = (\xi_1, \eta_1, \xi_2, \eta_2)^T$, the linearised system at $(ae_3, e_3)$ is given explicitly by $\dot{v} = Lv$ with

$$L = \begin{pmatrix} 0 & 0 & -\Lambda_3 & \Lambda_1 \\ 0 & 0 & a\Lambda_1 & -a\Lambda_3 \\ \Lambda_3 & -\Lambda_2 & 0 & 0 \\ -a\Lambda_2 & a\Lambda_3 & 0 & 0 \end{pmatrix}. \tag{3.10}$$

The eigenvalues $\zeta$ of $L$ satisfy

$$\zeta^4 + \left[\left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2\right]\zeta^2 + a^2\left(\Lambda_3^2 - \Lambda_1^2\right)\left(\Lambda_3^2 - \Lambda_2^2\right) = \det(\zeta I - L) = 0. \tag{3.11}$$

Let $\mathcal{D}$ be defined by

$$\begin{aligned} \mathcal{D} &:= \left[\left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2\right]^2 - 4a^2\left(\Lambda_3^2 - \Lambda_1^2\right)\left(\Lambda_3^2 - \Lambda_2^2\right) \\ &= \Lambda_3^2\left[\left(a^2-1\right)^2\Lambda_3^2 + 4a^2\left(\Lambda_1^2 + \Lambda_2^2\right) + 4a\left(a^2+1\right)\Lambda_1\Lambda_2\right]. \end{aligned} \tag{3.12}$$

Stability of the equilibrium point $(ae_3, e_3)$ requires all roots of (3.8) to be imaginary. Hence all three of the following conditions must be fulfilled:

$$\begin{aligned} &(i) && \left(\Lambda_3^2 - \Lambda_1^2\right)\left(\Lambda_3^2 - \Lambda_2^2\right) > 0, \\ &(ii) && \left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2 > 0, \\ &(iii) && a)\ \mathcal{D} > 0, \quad \text{or} \quad b)\ \mathcal{D} = 0. \end{aligned} \tag{3.13}$$

If any one of the conditions of (3.13) is not satisfied then (3.11) has roots of the form $\zeta = \pm\alpha \pm \mathrm{i}\beta$, with $\alpha \neq 0$, and the equilibrium point (3.9) is unstable.

Suppose that $\mathcal{D} = 0$. Every neighbourhood of $(ae_3, e_3)$ contains points of the form $((a \pm \epsilon)e_3, e_3)$ with $\epsilon > 0$. As $\mathcal{D} < 0$ at one of these two points, it follows that there are unstable equilibrium points arbitrarily close to $(ae_3, e_3)$ and hence $(ae_3, e_3)$ is unstable. We have instability then if (i), (ii), (iiib) of (3.13) are satisfied despite all the eigenvalues of the linearised system being pure imaginary.

We shall prove stability in the case (i), (ii), (iiia) of (3.13) by exhibiting a constant of motion for which (1.2) holds. As a preparation let us introduce

$$F := \Lambda_1\Lambda_2 H + \Lambda_3 K \tag{3.14}$$

and denote by $\tilde{H}$ and $\tilde{F}$ the restrictions of $H$ and $F$ to the orbit through the equilibrium point (3.9). One can check that $d\tilde{F} = 0$ at $(ae_3, e_3)$ and, up to a common constant of proportionality, the Hessians of $\tilde{H}$ and $\tilde{F}$ at this critical point are found to be

$$d^2\tilde{H} \sim \begin{pmatrix} \mathbf{H}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 \end{pmatrix} \tag{3.15}$$

and

$$d^2\tilde{F} \sim \begin{pmatrix} \mathbf{F}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_2 \end{pmatrix} \tag{3.16}$$

where $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{F}_1$, $\mathbf{F}_2$ are the following $2 \times 2$ matrices:

$$\mathbf{H}_1 = \begin{pmatrix} \Lambda_3 - \Lambda_2 & 0 \\ 0 & \Lambda_3 + \Lambda_2 \end{pmatrix}, \qquad \mathbf{H}_2 = \begin{pmatrix} \Lambda_3 - \Lambda_1 & 0 \\ 0 & \Lambda_3 + \Lambda_1 \end{pmatrix}, \tag{3.17}$$

$$\mathbf{F}_1 = \left(\Lambda_3^2 - \Lambda_2^2\right)\begin{pmatrix} a+1 & a-1 \\ a-1 & a+1 \end{pmatrix}\begin{pmatrix} \Lambda_3 + \Lambda_1 & 0 \\ 0 & \Lambda_3 - \Lambda_1 \end{pmatrix}\begin{pmatrix} a+1 & a-1 \\ a-1 & a+1 \end{pmatrix}, \tag{3.18}$$

$$\mathbf{F}_2 = \left(\Lambda_3^2 - \Lambda_1^2\right)\begin{pmatrix} a+1 & a-1 \\ a-1 & a+1 \end{pmatrix}\begin{pmatrix} \Lambda_3 + \Lambda_2 & 0 \\ 0 & \Lambda_3 - \Lambda_2 \end{pmatrix}\begin{pmatrix} a+1 & a-1 \\ a-1 & a+1 \end{pmatrix}. \tag{3.19}$$

**Lemma 1.** *If (i), (ii), (iiia) of (3.13) are all satisfied then the equilibrium point (3.9) is stable.*

**Proof.** There are two cases to consider.

*Case one:* $\Lambda_3^2 - \Lambda_1^2 > 0$ and $\Lambda_3^2 - \Lambda_2^2 > 0$. In this case (i) clearly holds. It is obvious that $d^2\tilde{H}$ is either positive or negative definite at $(ae_3, e_3)$ and the same applies to $d^2\tilde{F}$. Of course it can be shown that (ii) and (iiia) also hold.

*Case two:* $\Lambda_3^2 - \Lambda_1^2 < 0$ and $\Lambda_3^2 - \Lambda_2^2 < 0$. In this case again (i) clearly holds. Let us additionally suppose that (ii) and (iiia) both hold. We can show that there exists a $z \in \mathbb{R}$ such that $d^2(4z\tilde{H} + \tilde{F})$ is definite at $(ae_3, e_3)$.

The details of the proof in case two are as follows. Let us write

$$d^2(4z\tilde{H} + \tilde{F}) \sim \begin{pmatrix} \mathbf{Q}_1 & 0 \\ 0 & \mathbf{Q}_2 \end{pmatrix} := \mathbf{Q}, \tag{3.20}$$

with the $2 \times 2$ matrices $\mathbf{Q}_i = 4z\mathbf{H}_i + \mathbf{F}_i$. Now $\mathbf{Q}$ is a positive or negative definite matrix if and only if

$$\det \mathbf{Q}_1 > 0, \qquad \det \mathbf{Q}_2 > 0, \qquad \text{and} \qquad \operatorname{tr} \mathbf{Q}_1 \operatorname{tr} \mathbf{Q}_2 > 0. \tag{3.21}$$

The first and second conditions of (3.21) require

$$z^2 + \left[\left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2\right] z + a^2\left(\Lambda_3^2 - \Lambda_1^2\right)\left(\Lambda_3^2 - \Lambda_2^2\right) < 0. \tag{3.22}$$

Notice that (3.22) is similar to (3.11). Now (i), (ii), (iiia) together are equivalent to (3.11) having four distinct, imaginary eigenvalues, and this is obviously equivalent to the solvability of (3.22) for $z \in \mathbb{R}$. Let us write a solution $z$ in the form

$$z = -\frac{1}{2}\left[\left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2\right] + \frac{1}{2}\beta. \tag{3.23}$$

Then (3.22) implies

$$\beta^2 < \mathcal{D} \tag{3.24}$$

and because of (i),

$$\beta < \left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2. \tag{3.25}$$

Using (3.23) we obtain

$$\begin{aligned}\operatorname{tr}\mathbf{Q}_1 \operatorname{tr}\mathbf{Q}_2 &= \left(8z\Lambda_3 + 4\left(\Lambda_3^2-\Lambda_2^2\right)\Lambda_3\left(a^2+1\right)\right)\left(8z\Lambda_3 + 4\left(\Lambda_3^2-\Lambda_1^2\right)\Lambda_3\left(a^2+1\right)\right)\\ &= 16\Lambda_3^2 XY,\end{aligned} \tag{3.26}$$

with

$$\begin{aligned}X &= \left(\beta - \left[\left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2\right] + \left(a^2+1\right)\left(\Lambda_3^2-\Lambda_2^2\right)\right),\\ Y &= \left(\beta - \left[\left(a^2+1\right)\Lambda_3^2 + 2a\Lambda_1\Lambda_2\right] + \left(a^2+1\right)\left(\Lambda_3^2-\Lambda_1^2\right)\right).\end{aligned} \tag{3.27}$$

Equation (3.25) with the assumptions $\Lambda_3^2-\Lambda_1^2<0$ and $\Lambda_3^2-\Lambda_2^2<0$ imply that $X<0$ and $Y<0$, and hence $\operatorname{tr}\mathbf{Q}_1 \operatorname{tr}\mathbf{Q}_2 > 0$. Since all three conditions (3.21) for the definiteness of $\mathbf{Q}$ are satisfied, $f := (4z\tilde{H} + \tilde{F})$ satisfies (1.2) at the equilibrium point $(ae_3, e_3)$, whereby the proof is complete. ■

The results proven above imply the following proposition, which provides a characterization of the stability of the equilibrium points of (3.6) on generic coadjoint orbits.

**Proposition 2.** *The equilibrium point $b(ae_k, e_k)$ in (3.8) is stable if and only if (i), (ii), (iiia) of (3.13) hold for the constant $a$ and the matrix $\Lambda$ replaced by the matrix $\Lambda^P :=$ $\operatorname{diag}(\Lambda_{P(1)}, \Lambda_{P(2)}, \Lambda_{P(3)})$ where $P$ is an even permutation of $(1,2,3)$ for which $P(3)=k$.*

The permutation part of the statement follows obviously from (3.6) after checking that the stability of $b(ae_3, e_3)$ is equivalent to the stability of $(ae_3, e_3)$. In general, equation (2.7) has the property that $\mu(t)$ is a solution if and only if $\mu_b(t) := b\mu(bt)$ is a solution for any $b \neq 0$. This implies the required result for $b>0$. The $b=-1$ case is settled by using the facts that the matrix of the linear system (2.20) simply gets multiplied by $-1$ under such a rescaling of the equilibrium point, while the conserved quantities $H$ and $K$ in (3.4), and thus also their second variations, remain unchanged.

## 4 Heteroclinic orbits from 1-parameter subgroups

Consider two equilibrium points, $x_0$ and $x_1$, of a smooth Hamiltonian vector field $\mathbf{X}_H$ on a coadjoint orbit $\mathcal{O}$ of a compact Lie group $G$ with Lie algebra $\mathbf{g}$. Let us look for a 1-parameter subgroup of $G$ that generates a heteroclinic orbit of $\mathbf{X}_H$ connecting these equilibria. For $Y \in \mathbf{g}$, define

$$\gamma(s) = \mathrm{Ad}^*_{\exp(sY)} x_0. \tag{4.1}$$

Then our first requirement is that $\gamma(s_1) = x_1$ for some $s_1 > 0$. Setting $s_0 := 0$, our second requirement is that the curve $\gamma : (s_0, s_1) \to \mathcal{O}$ yields an integral curve of $\mathbf{X}_H$ by a suitable reparametrization. In other words, there should exist an increasing diffeomorphism $T : (s_0, s_1) \to (-\infty, +\infty)$ for which the curve $c(t)$ defined by

$$c(T(s)) = \gamma(s) \qquad \forall\, s \in (s_0, s_1) \tag{4.2}$$

satisfies $\dot{c}(t) = \mathbf{X}_H(c(t))$ for any $t \in \mathbb{R}$. Denoting the derivative with respect to $s$ by prime, it follows that $\forall\, s \in (s_0, s_1)$ we have

$$\chi(s)\gamma'(s) = \mathbf{X}_H(\gamma(s)) \qquad \text{with} \quad \chi(s) = \frac{1}{T'(s)}. \tag{4.3}$$

Because of the smoothness of the right hand side as a function of $s \in \mathbb{R}$, we observe that a unique extension of $\chi$ to $[s_0, s_1]$ must exist. This extended function must clearly satisfy the conditions

$$\chi(s_0) = \chi(s_1) = 0, \qquad \chi'(s_0 + 0) \geq 0, \qquad \chi'(s_1 - 0) \leq 0. \tag{4.4}$$

By using that (4.3) holds on $[s_0, s_1]$ and taking the appropriate derivatives of this equality at the endpoints, one arrives at the following statement.

**Proposition 3.** *If $\gamma(s)$ in (4.1) yields a heteroclinic orbit in the above-described sense, then the vectors $\mathrm{ad}^*_Y x_i \in T_{x_i}\mathcal{O}$ are eigenvectors of the linearisation of $\mathbf{X}_H$ at $x_i$, for $i = 0, 1$, with the respective eigenvalues being $\chi'(s_0 + 0)$ and $\chi'(s_1 - 0)$.*

In particular, notice from the proposition that the existence of a real eigenvalue of the linearisation of $\mathbf{X}_H$ at $x_0$ is a *necessary* condition for the construction of a heteroclinic orbit through $x_0$ by means of a 1-parameter subgroup of $G$. For the rigid body systems described in Section 2, this is in fact also a *sufficient* condition. For $n = 3$ this is a well known result. We verify it below in the $n = 4$ case by using the explicit analysis of the preceding section.

As before we may assume that the equilibrium point of interest is $x_0 = (ae_3, e_3)$ in (3.9). Let $(a\xi \wedge e_3, \eta \wedge e_3) = [Y, x_0]$ be an eigenvector of the linearised flow at $x_0$ with real eigenvalue $z > 0$. Note that $z = 0$ is excluded by (3.11) and that we have $Y = (\xi, \eta)$ by using the identification of the Lie bracket of $so(3) \cong su(2)$ with the vector-product. Then we can check that

$$\xi_1^2 + \xi_2^2 = \eta_1^2 + \eta_2^2. \tag{4.5}$$

This follows from the eigenvector equation $Lv = zv$ with $L$ in (3.10) and $v = (\xi_1, \eta_1, \xi_2, \eta_2)^T$. We set $\Delta := \sqrt{\xi_1^2 + \xi_2^2}$ and consider the curve

$$\gamma(s) = e^{sY} x_0 e^{-sY} = \cos(s\Delta)(ae_3, e_3) + \Delta^{-1}\sin(s\Delta)(a\xi \wedge e_3, \eta \wedge e_3). \tag{4.6}$$

We can verify that this curve yields a heteroclinic orbit that connects $x_0$ with $x_1 := -(ae_3, e_3)$ for $s_0 = 0$ and $s_1 = \frac{\pi}{\Delta}$. Indeed, the functions $\chi$ and $T$ introduced in (4.3) are found as

$$\chi(s) = \frac{z}{\Delta}\sin(s\Delta), \tag{4.7}$$

$$T(s) = \frac{1}{z}\log\tan\frac{s\Delta}{2} \qquad \text{for} \quad s_0 < s < s_1. \tag{4.8}$$

Note that the adjoint and coadjoint actions are the same for any compact Lie group and any orbit $\mathcal{O}_x = G/G_x$ carries a canonical $G$-invariant Riemannian metric induced by the Cartan–Killing form on $\mathbf{g}$. It is well known that the geodesics of this metric coincide with the orbits of the 1-parameter subgroups of $G$. Thus the heteroclinic orbits considered above are proper generalizations of the heteroclinic orbits of the standard rigid body that are great circles on $S^2 = SO(3)/SO(2)$.

# 5 On the stability analysis for $n > 4$

We are able to repeat a large part of the stability analysis performed in the 4-dimensional case. Specifically: we can find the equilibrium points (Proposition 1); we can find the eigenvectors and corresponding eigenvalues of the linearised system at each equilibrium point; we can prove the converse of Proposition 3. However the problem of proving stability (or not) for the equilibrium points having all eigenvalues pure imaginary is more complicated. We present here only an outline of the stability analysis for general $n$.

To find the eigenvalues and eigenvectors of the linearised system (2.20) at $x$ it is useful to decompose $so(n)$ as the vector space direct sum $so(n) = \mathrm{Ker}\,(\mathrm{ad}_x) + \mathrm{Im}\,(\mathrm{ad}_x)$, whereby we can uniquely parametrize $v \in T_x\mathcal{O}_x$ as $v = [Y, x]$ with $Y \in \mathrm{Im}\,(\mathrm{ad}_x)$. The linearised system (2.20) then reads as

$$[\dot{Y}, x] = \left[J, \left[Y, x^2\right]\right] \tag{5.1}$$

and an eigenvector $[Y, x] \in T_x\mathcal{O}_x$ with eigenvalue $z$ satisfies

$$\left[J, \left[Y, x^2\right]\right] = z[Y, x]. \tag{5.2}$$

Let us take $x$ to be of the form (2.3) and choose coordinates on $so(n)$ according to the natural decomposition into blocks. That is, for $n = 2m$ write $Y \in \mathrm{Im}\,(\mathrm{ad}_x)$ as $Y = A - A^T$ with $A = \sum_{i<j} e_{ij} \otimes \xi_{ij}$ and $\xi_{ij}$ a real $2 \times 2$ real matrix. If $n = 2m + 1$, then $\mathrm{Im}(\mathrm{ad}_x) \ni Y = \begin{pmatrix} A - A^T & v \\ -v^T & 0 \end{pmatrix}$ with $A$ as before and $v^T = (v_1^T, v_2^T, \ldots, v_m^T)$ with $v_i$ a real $2 \times 1$ matrix. Writing (5.2) in these coordinates, we see directly that there are several copies of the eigenvector equation for $so(4)$ – each of which has 4 solutions – and in the odd $n$ case also several copies of the eigenvector equation for $so(3)$ – each of which has 2 solutions. In fact we obtain exactly the right number of such decoupled equations to generate all eigenvectors and their eigenvalues. If any eigenvalue is real and nonzero, then we can use either the result described for $so(4)$ or a similar one – which has not been explicitly described here, but which is straightforward – for $so(3)$, to construct heteroclinic orbits by suitable curves of the form in (4.1). This leads to the following converse of Proposition 3.

**Proposition 4.** *Suppose that $z$ is a nonzero, real eigenvalue of the linear system (5.2) at $x$. Then there exists a corresponding eigenvector $[Y, x]$ for which the curve $\gamma(s) = e^{sY} x e^{-sY}$ yields a heteroclinic orbit of the rigid body system (2.7).*

We now sketch the proof of this proposition in the $n = 2m$ case. In this case we can write $J = \sum_{i=1}^{m} e_{ii} \otimes D_i$, where the $D_i$ are 2 by 2 diagonal matrices. By putting

$$Y := \sum_{1 \le i < j \le m} Y_{ij} \qquad \text{with} \quad Y_{ij} := e_{ij} \otimes \xi_{ij} - e_{ji} \otimes \xi_{ij}^T, \tag{5.3}$$

the eigenvector equation (5.2) decouples into separate equations for each pair of indices $i < j$,

$$\left(x_i^2 - x_j^2\right)\left(D_i \xi_{ij} - \xi_{ij} D_j\right) = z(x_j \xi_{ij} S - x_i S \xi_{ij}), \qquad S := \mathrm{i}\sigma_2. \tag{5.4}$$

For any $1 \le i < j \le m$, consider the $so(4)$ subalgebra of $so(2m)$ given by

$$so(4)_{ij} := \operatorname{span}\left\{e_{ii} \otimes S, e_{jj} \otimes S, \left(e_{ij} \otimes Q - e_{ji} \times Q^T\right) \mid \forall\, Q \in gl(2, \mathbb{R})\right\}. \tag{5.5}$$

The point to notice is that (5.4) coincides with the eigenvalue equation for a rigid body system defined on $so(4)_{ij}$ at the corresponding equilibrium point $x_{ij} := x_i e_{ii} \otimes S + x_j e_{jj} \otimes S$. This implies by the $so(4)$ result established in Section 4 that if $Y_{ij}$ is a solution of (5.4) with some real $z \neq 0$, then the curve

$$\gamma_{ij}(s) := e^{sY_{ij}} x_{ij} e^{-sY_{ij}} \tag{5.6}$$

yields a heteroclinic orbit connecting the unstable equilibria $\pm x_{ij}$ of the induced rigid body system on $so(4)_{ij}$. Decomposing $x$ as $x = x_{ij} + x'_{ij}$, we can check the relations

$$\gamma(s) := e^{sY_{ij}} x e^{-sY_{ij}} = \gamma_{ij}(s) + x'_{ij} \tag{5.7}$$

and

$$\left[J, \gamma^2(s)\right] = \left[J_{ij}, \gamma_{ij}^2(s)\right], \qquad J_{ij} := e_{ii} \otimes D_i + e_{jj} \otimes D_j. \tag{5.8}$$

Equation (5.8) relates the Hamiltonian vector fields for the rigid body systems on $so(2m)$ and on $so(4)_{ij}$ along the respective curves $\gamma(s)$ and $\gamma_{ij}(s)$. By collecting the above remarks, we conclude that $\gamma(s)$ in (5.7) yields a heteroclinic orbit that connects the unstable equilibria $x'_{ij} \pm x_{ij}$.

To illustrate what happens for odd $n$, let us look at $n = 5$. Let us assume that the equilibrium point $x$ of interest has the form

$$x = \begin{pmatrix} 0 & a-1 & 0 & 0 & 0 \\ -a+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a-1 & 0 \\ 0 & 0 & a+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.9}$$

and parametrize $Y \in \mathrm{Im}\,(\mathrm{ad}_x)$ according to

$$Y = \frac{1}{2}\begin{pmatrix} 0 & 0 & \xi_2-\eta_2 & \xi_1-\eta_1 & v_1 \\ 0 & 0 & -\xi_1-\eta_1 & \xi_2+\eta_2 & v_2 \\ \eta_2-\xi_2 & \eta_1+\xi_1 & 0 & 0 & w_1 \\ \eta_1-\xi_1 & -\xi_2-\eta_2 & 0 & 0 & w_2 \\ -v_1 & -v_2 & -w_1 & -w_2 & 0 \end{pmatrix}. \tag{5.10}$$

Then consider the eigenvector equation (5.2) with $J = \mathrm{diag}\,(J_1, J_2, J_3, J_4, J_5)$. By setting $v_i = 0 = w_i$ we reduce to the eigenvector condition for $so(4)$; by setting $\xi_i = \eta_i = w_i = 0$ we reduce to the eigenvector condition for $so(3)$ and by setting $\xi_i = \eta_i = v_i = 0$ we reduce to the eigenvector condition for $so(3)$ too. In fact the coordinates have been chosen here so as to agree exactly with those used for the $so(4)$ analysis in Section 3. In this way we find all 8 eigenvalues. The problems of checking if the eigenvalues are real, complex or imaginary reduce to those of the $so(3)$ and $so(4)$ cases. Similarly the construction of heteroclinic orbits as orbits of 1-parameter subgroups reduces to the $so(3)$ and $so(4)$ cases. To check if all eigenvalues of the linearisation being imaginary is sufficient for stability we could try to prove the convexity at $x$ of a function of the form

$$f = \alpha H + \beta H_1 + \gamma H_2 + \delta H_4 \tag{5.11}$$

with $H$, $H_1$, $H_2$, $H_3$ the Hamiltonian together with 3 independent commuting integrals, which can be generated using the Lax matrix of Manakov [1], where $\alpha$, $\beta$, $\gamma$, $\delta$ are expected to depend on the equilibrium point in question. Of course, while even this can be done in principle, there is no strategy telling us how to proceed for general $n$.

# 6 Conclusion

In this paper we studied the equilibrium points for the integrable Euler equations in (2.7). In particular, we described the equilibrium points (Proposition 1) and associated heteroclinic orbits with any nonzero, real eigenvalue of the linearised system for any $n$ (Proposition 4). We also found a complete characterization of the stability of the equilibrium points for $n = 4$ (Proposition 2), but our stability analysis is incomplete for $n > 4$. In this case an open question is to find a criterion for the stability of those equilibrium points for which all eigenvalues of the linearised system are imaginary.

As a final remark, we wish to mention the work of Mishchenko and Fomenko [4] (for a review, see [5]) that contains generalizations of the systems of Manakov [1] to other Lie algebras. Various elements of our results have a general Lie-algebraic nature and thus may be applicable to the systems of [4]. In this respect, it is natural to ask if Proposition 4 is valid only for the special cases (2.7) that we considered here, or can be extended to other systems among those in [1, 4], too. It would also be interesting to find a general criterion of stability that could be used effectively to analyse these systems.

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