

q-Analogues of Classical 6-Periodicity: From Euler to Chebyshev

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Abstract

The sequence of period 6 starting with $1, 1, 0, -1, -1, 0$ appears in many different disguises in mathematics. Various q -versions of this sequence are found, and their relations with Euler's pentagonal numbers theorem and Chebyshev polynomials are discussed.

The motto on Cardinal Newman's tomb ought to be the funeral motto of every Catholic,
Ex umbris et imaginibus in veritatem,
Out of shadows and appearances into the truth.

Ronald Knox, *The Pastoral Sermons*

1 Introduction

The sequence of period 6, starting with

$$1, 1, 0, -1, -1, 0, \tag{1.1}$$

appears as the coefficients in the expansion

$$\frac{1}{1-x+x^2} = 1 + x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + x^{12} + x^{13} - \dots \tag{1.2}$$

This is easy to see ([15, Ch. V]):

$$\begin{aligned} \sum_{s=0}^{\infty} a_s x^s &= \frac{1}{1-x+x^2} = \frac{1+x}{1+x^3} = \frac{1}{1+x^3} + x \frac{1}{1+x^3} \\ &= \sum_{\ell=0}^{\infty} (-x^3)^{\ell} + x \sum_{\ell=0}^{\infty} (-x^3)^{\ell}, \end{aligned} \tag{1.3}$$

so that

$$a_{s+3} = -a_s, \quad s \in \mathbb{Z}_+, \tag{1.4a}$$

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 0 : \tag{1.4b}$$

the 6-periodicity results from 3-antiperiodicity.

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This is so far unremarkable. However,

$$\frac{1}{1-x+x^2} = \frac{1}{1-x(1-x)} = \sum_{m=0}^{\infty} x^m (1-x)^m \quad (1.5a)$$

$$= \sum_{m \geq 0} x^m \sum_{k=0}^m \binom{m}{k} (-1)^k x^k = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k}. \quad (1.5b)$$

Thus, the binomial sums

$$s_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \quad (1.6a)$$

are equal to

$$1, 1, 0, -1, -1, 0 \quad \text{for} \quad n \equiv 0, 1, 2, 3, 4, 5 \pmod{6}. \quad (1.6b)$$

Such periodicity is very rare for binomial sums; see some examples and references in [12, 13]. The problem is: can this periodicity be quantized, in the sense of q -mathematics?

2 Guessing an answer

We need to find a suitable q -analog of the series $\sum_{m=0}^{\infty} x^m (1-x)^m$ (1.5a). After some experimenting and rescaling, the following sum suggests itself for consideration:

$$S(x) = S(x; q) = \sum_{m=0}^{\infty} x^m (1 \dot{-} x)^m \quad (2.1a)$$

$$= \sum_{m=0}^{\infty} x^m (x; q)_m, \quad (2.1b)$$

where

$$(a \dot{+} b)^m = \left[\prod_{i=0}^m (a + q^i b) \right] / (a + q^m b) = \begin{cases} (a+b) \cdots (a + q^{m-1} b), & m \in \mathbb{N}, \\ 1, & m = 0 \end{cases} \quad (2.2)$$

and (see [2, p. 487])

$$(x; q)_m = \left[\prod_{i \geq 0} (1 - q^i x) \right] / (1 - q^m x) = \begin{cases} (1-x) \cdots (1 - q^{m-1} x), & m \in \mathbb{N}, \\ 1, & m = 0. \end{cases} \quad (2.3)$$

By Euler's formula [2, p. 492],

$$(1 \dot{-} x)^m = \sum_{k=0}^m \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} (-x)^k, \quad (2.4)$$

where

$$\begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m] \cdots [m-k+1]}{[1] \cdots [k]}, \quad k \in \mathbb{N}; \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1, \quad (2.5)$$

are the q -binomial coefficients, and

$$[m] = [m]_q = (1 - q^m) / (1 - q) \quad (2.6)$$

is a q -analogue of the classical number (or object) m .

Substituting (2.4) into (2.1), we get

$$S(x) = \sum_{m=0}^{\infty} x^m (1 \dot{-} x)^m = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q q^{\binom{k}{2}}. \quad (2.7)$$

Thus, as a q -analogue of the classical binomial sum (1.6a) we obtain

$$S_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q q^{\binom{k}{2}}. \quad (2.8)$$

Is it any good? Calculating a few terms, we find:

$$S(x) = 1 + x + 0 \cdot x^2 - qx^3 - q^2x^4 + 0 \cdot x^5 + q^5x^6 + q^7x^7 + 0 \cdot x^8 + O(x^9). \quad (2.9)$$

This looks promising, as the 3-antiperiodicity is preserved here, in q -clothes, but there are not enough terms to guess the rule for the q -exponents. We need some sort of a functional equation to determine that rule, assuming it exists.

We proceed as follows. Set

$$S(x, a) = S(x, a; q) = \sum_{m=0}^{\infty} a^m x^m (1 \dot{-} qx)^m. \quad (2.10)$$

Since

$$(1 \dot{-} x)^{m+1} = (1 - x)(1 \dot{-} qx)^m, \quad (2.11)$$

we find that

$$S(x, a) = 1 + ax(1 - x) \sum_{m=0}^{\infty} (ax)^m (1 \dot{-} qx)^m = 1 + ax(1 - x)S(qx, a/q). \quad (2.12a)$$

Therefore, if we write

$$S(x, a) = \sum_{n=0}^{\infty} S_n(a)x^n, \quad (2.12b)$$

the functional equation (2.12a) converts itself into

$$S_{n+2}(a) = aS_{n+1}(a/q)q^{n+1} - aS_n(a/q)q^n, \quad n \in \mathbb{Z}_+, \quad (2.13a)$$

$$S_1(a) = a, \quad S_0(a) = 1. \quad (2.13b)$$

In particular, for $a = q$, we find:

$$S_0(q) = 1, \quad S_1(q) = q, \quad (2.14a)$$

$$S_{n+2}(q) = q^{n+1}(qS_{n+1} - S_n), \quad S_n = S_n(1), \quad n \in \mathbb{Z}_+. \quad (2.14b)$$

Using the expansion (2.9), we obtain then

$$\begin{aligned} S(x, q) &= \sum_{m=0}^{\infty} (qx)^m (1 - x)^m = 1 + qx + (q^2 - q)x^2 - q^2x^3 - q^5x^4 \\ &\quad - (q^2 - q^7)x^5 + q^7x^6 + q^{12}x^7 + (q^{15} - q^{12})x^8 + O(x^9). \end{aligned} \quad (2.15)$$

This is a bit less simple than (2.9) but still looks enticing. Moreover, comparing both expansions we can't fail to notice some coincidences and regularities; the following general ansatz suggests itself:

$$S_{6n}(q) = q^{x(n)}, \quad S_{6n} = q^{x(n)-2n}, \quad (2.16a)$$

$$S_{6n+1}(q) = q^{y(n)}, \quad S_{6n+1} = q^{x(n)}, \quad (2.16b)$$

$$S_{6n+2}(q) = q^{y(n)}(q^{2n+1} - 1), \quad S_{6n+2} = 0, \quad (2.16c)$$

$$S_{6n+3}(q) = -q^{u(n)}, \quad S_{6n+3} = -q^{u(n)-2n-1}, \quad (2.16d)$$

$$S_{6n+4}(q) = -q^{v(n)}, \quad S_{6n+4} = -q^{u(n)}, \quad (2.16e)$$

$$S_{6n+5}(q) = -q^{v(n)}(q^{2n+2} - 1), \quad S_{6n+5} = 0. \quad (2.16f)$$

Making this ansatz compatible with the relations (2.14) leads to the determination of all the unknown exponents $x(n)$, $y(n)$, $u(n)$, $v(n)$ in formulae (2.16):

$$y(n) = x(n) + 4n + 1, \quad (2.17a)$$

$$u(n) = x(n) + 6n + 2, \quad (2.17b)$$

$$v(n) = u(n) + 4n + 3, \quad (2.17c)$$

$$x(n+1) = 6n + 5 + u(n) \quad (2.17d)$$

\Rightarrow

$$x(n) = 6n^2 + n, \quad (2.18a)$$

$$y(n) = 6n^2 + 5n + 1, \quad (2.18b)$$

$$u(n) = 6n^2 + 7n + 2 \quad (2.18c)$$

$$v(n) = 6n^2 + 11n + 5. \quad (2.18d)$$

These are our *two* – conjectured so far – candidates for a quantum versions of the 6-periodic classical binomial sum (1.6). Are there any other candidates? It seems unlikely, at least if we insist on having relatively *compact* answers. For example,

$$S_4(a)/a^2 = q - a[3] + a^2, \quad (2.19a)$$

$$S_5(a)/a^3 = q[3] - a[4] + a^2, \quad (2.19b)$$

and for $a = q^L$ we don't get anything attractive unless $L = 0$ or 1 .

Let us now collect our conjectured formulae into a series form. For $a = 1$, we get

$$\begin{aligned} S(x) &= S(x, 1) = \sum_{n=0}^{\infty} S_n x^n \\ &= \sum_{n=0}^{\infty} \left(x^{6n} q^{6n^2-n} + x^{6n+1} q^{6n^2+n} - x^{6n+3} q^{6n^2+5n+1} - x^{6n+4} q^{6n^2+7n+2} \right) \end{aligned} \quad (2.20a)$$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \left\{ \left(x^{6n} q^{6n^2-n} - x^{6n+3} q^{6n^2+5n+1} \right) + \left(x^{6n+1} q^{6n^2+n} - x^{6n+4} q^{6n^2+7n+2} \right) \right\} \\ &= \sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n-1)/2} + x^{3n+1} q^{n(3n+1)/2} \right\} : \end{aligned} \quad (2.20b)$$

$$\sum_{m=0}^{\infty} x^m (1 - x)^m = \sum_{n=0}^{\infty} (-x)^{3n} q^{n(3n-1)/2} (1 + xq^n). \quad (2.21)$$

Similarly, for $a = q$ we obtain

$$\begin{aligned} S(x, q) &= \sum_{n=0}^{\infty} S_n(q) x^n \\ &= \sum_{n=0}^{\infty} x^{6n} \left\{ q^{6n^2+n} + xq^{6n^2+5n+1} + x^2 q^{6n^2+5n+1} (q^{2n+1} - 1) \right. \\ &\quad \left. - x^3 q^{6n^2+7n+2} - x^4 q^{6n^2+11n+5} - x^5 q^{6n^2+11n+5} (q^{2n+2} - 1) \right\} \end{aligned} \quad (2.22a)$$

$$= \sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n+1)/2} + x^{3n+1} q^{n(3n+5)/2} + x^{3n+2} q^{n(3n+5)/2} (q^{n+1} - 1) \right\} : \quad (2.22b)$$

$$\sum_{m=0}^{\infty} q^m x^m (1 - x)^m = \sum_{n=0}^{\infty} (-x)^{3n} q^{n(3n+1)/2} \{1 + xq^{2n} + x^2 q^{2n} (q^{n+1} - 1)\}. \quad (2.23)$$

In view of the functional equation (2.12), formula (2.21) implies formula (2.23). In the next Section we shall prove the former formula.

Remark 2.24. Formula (2.12a) shows that the parameter

$$y = ax \quad (2.25)$$

remains invariant under iteration. Therefore, let us set

$$R(x, y) = R(x, y; q) = \sum_{m=0}^{\infty} y^m (1 - x)^m. \quad (2.26)$$

Then

$$R(x, y) = 1 + y(1 - x)R(qx, y). \quad (2.27)$$

Setting

$$R(x, y) = \sum_{\ell=0}^{\infty} c_{\ell}(y) x^{\ell}, \quad (2.28)$$

we convert the functional equation (2.27) into

$$c_0 = 1 + yc_0, \quad (2.29a)$$

$$c_{\ell+1} = yq^{\ell+1}c_{\ell+1} - yq^{\ell}c_{\ell} \quad \Rightarrow \quad (2.29b)$$

$$c_{\ell} = \frac{(-y)^{\ell}q^{\binom{\ell}{2}}}{(1-y)^{\ell+1}}, \quad \ell \in \mathbb{Z}_+ \quad \Rightarrow \quad (2.30)$$

$$R(x, y) = \sum_{m=0}^{\infty} y^m (1-yx)^m = \sum_{\ell=0}^{\infty} \frac{(-xy)^{\ell}}{(1-y)^{\ell+1}} q^{\binom{\ell}{2}} \quad \Rightarrow \quad (2.31)$$

$$S(x, a) = \sum_{m=0}^{\infty} (ax)^m (1-yx)^m = \sum_{\ell=0}^{\infty} \frac{(-ax^2)^{\ell}}{(1-ax)^{\ell+1}} q^{\binom{\ell}{2}}. \quad (2.32)$$

Neither of these expansions, however, is helpful for our task of proving formula (2.21). We need something completely different. Notice that for $q = 1$, formula (2.31) yields:

$$\frac{1}{1-y(1-x)} = \sum_{m=0}^{\infty} y^m (1-x)^m = \sum_{\ell=0}^{\infty} \frac{(-xy)^{\ell}}{(1-y)^{\ell+1}} = \frac{1}{1-y} \frac{1}{1 + \frac{xy}{1-y}}. \quad (2.33)$$

3 Euler's tower

We have to prove formula (2.21). In the form (2.20b), it is:

$$\sum_{m=0}^{\infty} x^m (1-yx)^m = \sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n-1)/2} + x^{3n+1} q^{n(3n+1)/2} \right\}. \quad (3.1)$$

This is an identity between formal power series in x , with coefficients that are polynomials in q . We can not therefore let x to be 1 (unlike q). But let's get wild for a moment. The RHS of formula (3.1) for $x = 1$ is:

$$\sum_{n=0}^{\infty} (-1)^n \left\{ q^{n(3n-1)/2} + q^{n(3n+1)/2} \right\} = 1 + \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \quad (3.2)$$

This is essentially the RHS of Euler's famous identity, conjectured by him in 1741:

$$\prod_{j=1}^{\infty} (1 - q^j) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \quad (3.3)$$

(See [15, Ch. 6] for an English translation of Euler's fascinating memoir.) Nowadays, Euler's formula (3.3) is subsumed by the more general Jacobi triple product identity

$$\prod_{n=1}^{\infty} (1 + Q^{2n-1}z) (1 + Q^{2n-1}z^{-1}) (1 - Q^{2n}) = \sum_{n=-\infty}^{\infty} Q^{n^2} z^n \quad (3.4)$$

(see [16, p. 10 and p. 186]). Euler's formula (3.3) results from (3.4) when one makes the substitution

$$Q = q^{3/2}, \quad z = -q^{1/2}. \quad (3.5)$$

Neither Euler's formula (3.3) nor Jacobi's formula (3.4) seem of any help to our identity (3.1). However, Euler's ingenious *Proof* of his identity (see [24, p. 281]) has enough ingredients in it to establish (3.1). Weil describes Euler's Proof as "another dazzling display of algebraic virtuosity but quite elementary ...".

Euler sets

$$P_0 = \prod_{j=1}^{\infty} (1 - q^j) \quad (3.6)$$

and notices that

$$P_0 = 1 - q - q^2 P_1, \quad (3.7)$$

where

$$P_1 = \sum_{\nu=0}^{\infty} q^{\nu} (1 - q) \cdots (1 - q^{\nu+1}). \quad (3.8)$$

This follows from the easily proven by induction identity

$$\prod_{j=1}^n (1 - \alpha_j) = 1 - \alpha_1 - \sum_{k=2}^n \alpha_k (1 - \alpha_1) \cdots (1 - \alpha_{k-1}), \quad n \geq 2, \quad (3.9)$$

by setting $\alpha_j = q^j$, replacing k by $\nu + 2$, and then passing to the limit $n \rightarrow \infty$. Euler then introduces the infinite tower of series P_n :

$$P_n = \sum_{\nu=0}^{\infty} q^{\nu n} (1 - q^n) \cdots (1 - q^{n+\nu}), \quad n \in \mathbb{N}, \quad (3.10)$$

and shows by a *different argument* that

$$P_n = 1 - q^{2n+1} - q^{3n+2} P_{n+1}, \quad n \in \mathbb{N}. \quad (3.11)$$

Euler's argument for $n > 0$ is ingenious:

$$\begin{aligned} P_n &= \sum_{\nu=0}^{\infty} q^{\nu n} (1 - q^n) \cdots (1 - q^{n+\nu}) \\ &= (1 - q^n) + q^n (1 - q^n) (1 - q^{n+1}) + \sum_{\mu=2}^{\infty} q^{\mu n} (1 - q^n) \cdots (1 - q^{n+\mu}) \\ &= 1 - q^n + q^n (1 - q^{n+1}) - q^{2n} (1 - q^{n+1}) \\ &\quad + \sum_{\mu=2}^{\infty} q^{\mu n} (1 - q^{n+1}) \cdots (1 - q^{n+\mu}) - q^n \sum_{\mu=2}^{\infty} q^{\mu n} (1 - q^{n+1}) \cdots (1 - q^{n+\mu}) \end{aligned}$$

$$\begin{aligned}
 &= 1 - q^{2n+1} - q^{2n} (1 - q^{n+1}) + \sum_{\nu=0}^{\infty} q^{\nu n+2n} (1 - q^{n+1}) \cdots (1 - q^{n+2+\nu}) \\
 &\quad - \sum_{\mu=1}^{\infty} q^{(\mu+1)n} (1 - q^{n+1}) \cdots (1 - q^{n+\mu}) + q^{2n} (1 - q^{n+1}) \\
 &= 1 - q^{2n+1} + \sum_{\nu=0}^{\infty} q^{\nu(n+1)+2n-\nu} (1 - q^{n+1}) \cdots (1 - q^{n+1+\nu}) (1 - q^{n+2+\nu}) \\
 &\quad - \sum_{\nu=0}^{\infty} q^{(\nu+2)n} (1 - q^{n+1}) \cdots (1 - q^{n+1+\nu}) \\
 &= 1 - q^{2n+1} + \sum_{\nu=0}^{\infty} q^{\nu(n+1)} (1 - q^{n+1}) \cdots (1 - q^{n+1+\nu}) \{q^{2n-\nu} (1 - q^{n+2+\nu}) - q^{2n-\nu}\} \\
 &= 1 - q^{2n+1} - q^{3n+2} \sum_{\nu=0}^{\infty} q^{\nu(n+1)} (1 - q^{n+1}) \cdots (1 - q^{n+1+\nu}) = 1 - q^{2n+1} - q^{3n+2} P_{n+1}.
 \end{aligned}$$

Collecting all the relations (3.11) together, we find

$$\begin{aligned}
 \prod_{j=1}^{\infty} (1 - q^j) &= P_0 = 1 - q - q^2 P_1 = 1 - q - q^2 (1 - q^3 - q^5 P_2) \\
 &= 1 - q - q^2 (1 - q^3) + q^{2+5} (1 - q^5 - q^8 P_3) = \cdots \\
 &= 1 - q + \sum_{n=1}^{\infty} (-1)^n q^{2+5+\cdots+(3n-1)} (1 - q^{2n+1}) \tag{3.12}
 \end{aligned}$$

$$= 1 - q + \sum_{n=1}^{\infty} (-1)^n q^{n(3n+1)/2} (1 - q^{2n+1}) = \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} (1 - q^{2n+1}) \tag{3.13}$$

$$= \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2} + \sum_{n=0}^{\infty} (-1)^{n+1} q^{(3n^2+n+4n+2)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}. \tag{3.14}$$

While we are at it, let's notice that a similar iteration yields

$$P_{\ell} = \sum_{n=0}^{\infty} (-1)^n q^{3\ell n+n(3n+1)/2} (1 - q^{2\ell+2n+1}), \quad \ell \in \mathbb{Z}_+. \tag{3.15}$$

From this, we can readily see that

$$\bar{P}_{\ell} := 1 + q^{\ell} P_{\ell} = \sum_{n=0}^{\infty} (-1)^n \left\{ q^{3\ell n} q^{n(3n-1)/2} + q^{(3n+1)\ell} q^{n(3n+1)/2} \right\}. \tag{3.16}$$

It now only remains to compare \bar{P}_{ℓ} with

$$\bar{S}_{\ell} = S(q^{\ell}; q) = S(x; q)|_{x=q^{\ell}}, \quad \forall \ell \in \mathbb{N} : \tag{3.17}$$

$$\bar{S}_{\ell} = S(q^{\ell}; q) = \sum_{m=0}^{\infty} q^{\ell m} (1 - q^{\ell})^m = 1 + \sum_{m=1}^{\infty} q^{\ell m} (1 - q^{\ell}) \cdots (1 - q^{\ell+m-1}), \tag{3.18a}$$

$$\bar{P}_{\ell} = 1 + q^{\ell} \sum_{\nu=0}^{\infty} q^{\nu \ell} (1 - q^{\ell}) \cdots (1 - q^{\ell+\nu}), \tag{3.18b}$$

and these are identical. Thus,

$$\bar{S}_\ell = \bar{P}_\ell, \quad \ell \in \mathbb{N}. \quad (3.19)$$

Comparing formulae (3.1) and (3.16), we get

$$\begin{aligned} & \left\{ \sum_{m=0}^{\infty} x^m (1 - x)^m \right\} \Big|_{x=q^\ell} \\ &= \left(\sum_{n=0}^{\infty} (-1)^n \left\{ x^{3n} q^{n(3n-1)/2} + x^{3n+1} q^{n(3n+1)/2} \right\} \right) \Big|_{x=q^\ell}, \quad \forall \ell \in \mathbb{N}. \end{aligned} \quad (3.20)$$

This proves formula (3.1), because:

Lemma 3.21. *Suppose $f \in \mathbb{C}[[x, y]]$ is formal power series in x, y , and let $r(1), r(2), \dots$ be an increasing sequence of positive integers. If*

$$f(y^{r(i)}, y) = f(x, y)|_{x=y^{r(i)}} = 0, \quad \forall i \in \mathbb{N}, \quad (3.22)$$

then f is identically zero.

Proof. If $f(y^{r(1)}, y) = 0$ then f is divisible by $x - y^{r(1)}$:

$$f(x, y) = (x - y^{r(1)}) f_1(x, y). \quad (3.23)$$

Continuing on, we find that

$$f(x, y) = \left\{ \prod_{i=1}^k (x - y^{r(i)}) \right\} f_k(x, y) \quad (3.24)$$

Thus, all homogeneous components of f of degrees $< k$ vanish, and k is arbitrary. ■

4 Euler's tower revisited

Woman: Your're a pugilist, aren't you?
World champ boxer Rocky Graziano: Nah,
I'm just a prizefighter.

The Euler relation (3.11)

$$P_n = 1 - q^{2n+1} - q^{3n+2} P_{n+1}, \quad n \in \mathbb{N}, \quad (4.1)$$

for the sequence of series

$$P_n = \sum_{\nu=0}^{\infty} q^{\nu n} (1 - q^n) \cdots (1 - q^{n+\nu}), \quad n \in \mathbb{N}, \quad (4.2)$$

has an implicit gem hidden inside. To make it explicit, notice that in the definition of P_n (4.2) the index n could be treated as a formal parameter and not necessarily as a positive integer. So, set

$$q^n = x \quad (4.3)$$

and define

$$\mathcal{P}(x) = \mathcal{P}(x; q) = \sum_{\nu=0}^{\infty} x^{\nu} (1-x) \cdots (1-xq^{\nu}). \quad (4.4)$$

Applying the Euler argument to $\mathcal{P}(x)$, we get:

$$\begin{aligned} \mathcal{P}(x) &= \sum_{\nu=0}^{\infty} x^{\nu} (1-x) \cdots (1-xq^{\nu}) = (1-x) + x(1-x)(1-xq) \\ &+ \sum_{\mu=2}^{\infty} x^{\mu} (1-x) \cdots (1-xq^{\mu}) = (1-x) + x(1-xq) - x^2(1-xq) \\ &+ \sum_{\mu=2}^{\infty} x^{\mu} (1-xq) \cdots (1-xq^{\mu}) - x \sum_{\mu=2}^{\infty} x^{\mu} (1-xq) \cdots (1-xq^{\mu}) \\ &= 1 - qx^2 - x^2(1-xq) + \sum_{\nu=0}^{\infty} x^{\nu+2} (1-xq) \cdots (1-xq^{\nu+1}) (1-xq^{\nu+2}) \\ &- \sum_{\mu=1}^{\infty} x^{\mu+1} (1-xq) \cdots (1-xq^{\mu}) + x^2(1-xq) \\ &= 1 - qx^2 + x^2 \sum_{\nu=0}^{\infty} x^{\nu} (1-xq) \cdots (1-xq^{\nu+1}) \{ (1-xq^{\nu+2}) - 1 \} \\ &= 1 - qx^2 - x^3 q^2 \sum_{\nu=0}^{\infty} (xq)^{\nu} (1-xq) \cdots (1-xq \cdot q)^{\nu} = 1 - qx^2 - q^2 x^3 \mathcal{P}(xq): \quad (4.5) \end{aligned}$$

$$\mathcal{P}(x; q) = 1 - qx^2 - q^2 x^3 \mathcal{P}(xq; q). \quad (4.6)$$

Writing

$$\mathcal{P}(x; q) = \sum_{\ell=0}^{\infty} c_{\ell}(q) x^{\ell}, \quad (4.7)$$

we can transform the relation (4.6) into

$$c_{\ell+3} = -q^{\ell+2} c_{\ell}, \quad (4.8a)$$

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = -q \quad \Rightarrow \quad (4.8b)$$

$$c_{3\ell} = (-1)^{\ell} q^{\ell(3\ell+1)/2}, \quad (4.9a)$$

$$c_{3\ell+1} = 0, \quad (4.9b)$$

$$c_{3\ell-1} = (-1)^{\ell} q^{\ell(3\ell-1)/2} \quad \Rightarrow \quad (4.9c)$$

$$\mathcal{P}(x) = \sum_{\nu=0}^{\infty} x^{\nu} (1-x) \cdots (1-xq^{\nu}) \quad (4.10a)$$

$$= 1 + \sum_{\ell=1}^{\infty} (-1)^{\ell} \left\{ q^{\ell(3\ell-1)/2} x^{3\ell-1} + q^{\ell(3\ell+1)/2} x^{3\ell} \right\}. \quad (4.10b)$$

If we notice that

$$S(x) = 1 + x\mathcal{P}(x), \quad (4.11)$$

then from formula (4.10b) we obtain at once that

$$S(x) = \sum_{m=0}^{\infty} x^m (1-x)^m \quad (4.12a)$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} \left\{ x^{3\ell} q^{\ell(3\ell-1)/2} + x^{3\ell+1} q^{\ell(3\ell+1)/2} \right\}, \quad (4.12b)$$

which is our conjectured formula (2.20b). The functional equation (4.6) in the S -language becomes:

$$S(x) = 1 + x - qx^3 S(qx); \quad (4.13)$$

this equation by itself implies formula (4.12b) at once.

Remark 4.14. The identity (4.10) is essentially due to Euler, but it was repeatedly rediscovered many times since. See [1, p. 282] for more on that.

Remark 4.15. What is the origin of number 6, and are there interesting periodic sequences of period other than 6? I don't yet have a definite answer to these questions but suspect that our 6-periodic sequence is a camouflaged group of units of the field $\mathbb{Q}[\sqrt{-3}]$; this suggests that there exists a 4-periodic sequence attached to the units $\pm 1, \pm i$ of $\mathbb{Q}[\sqrt{-1}]$, and if a number field $K \supset \mathbb{Q}$ has a finite group of units then these can be organized into a periodic (vector) sequence.

5 Recurrence relations of second order

The space of 3-antiperiodic sequences is 3-dimensional. It contains a special 2-dimensional subspace consisting of 2^{nd} -order recurrent sequences $\{u_n\}$ satisfying the relation

$$u_{n+2} = u_{n+1} - u_n. \quad (5.1)$$

Indeed, for $u_0 = a, u_1 = b$, the sequence u_n starts as

$$a, b, b-a, -a, -b, -b+a \mid a, b, \dots \quad (5.2)$$

(In the language of characteristic polynomials: $\lambda^3 + 1$ is divisible by $\lambda^2 - \lambda + 1$).

Our sequence of binomial sums s_n (1.6a)

$$s_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \quad (5.3)$$

is of this special form, with $a = b = 1$. This is true a posteriori, from comparing formulae (5.2) and (1.6b), but can be easily seen directly from the definition of s_n (5.3):

$$\begin{aligned} s_{n+2} - s_{n+1} &= \sum_{k \geq 0} (-1)^k \left\{ \binom{n+2-k}{k} - \binom{n+1-k}{k} \right\} = \sum (-1)^k \binom{n+1-k}{k-1} \\ &= \sum (-1)^{k+1} \binom{n-k}{k} = -s_n. \end{aligned} \quad (5.4)$$

In quantizing s_n in § 2, we went along the route of generating functions, quantizing the latter for the sequence s_n :

$$\sum_{n=0}^{\infty} s_n x^n = \sum_{m=0}^{\infty} x^m (1-x)^m. \quad (5.5)$$

Generating functions make a very useful and powerful device (see [25].) However, when one attempts to quantize a classical object, property, relation, etc., the generating function methods may unnecessarily restrict one's choice. A cautionary tale follows.

Let's try to quantize the *calculation* (5.4). Set

$$f_n = f_{n|L} = f_{n|L}(q) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-q^L)^k q^{\theta(n,k)} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \quad (5.6)$$

with some unknown function $\theta(n, k)$ to be specified later on. We shall use the formulae

$$\begin{bmatrix} \alpha+1 \\ r \end{bmatrix} = q^r \begin{bmatrix} \alpha \\ r \end{bmatrix} + \begin{bmatrix} \alpha \\ r-1 \end{bmatrix} \quad (5.7a)$$

$$= \begin{bmatrix} \alpha \\ r \end{bmatrix} + q^{\alpha+1-r} \begin{bmatrix} \alpha \\ r-1 \end{bmatrix}. \quad (5.7b)$$

First, by formula (5.7a)

$$\begin{aligned} f_{n+2|L} &= \sum_k (-q^L)^k q^{\theta(n+2,k)} \begin{bmatrix} n+2-k \\ k \end{bmatrix} \\ &= \sum_k (-q^L)^k q^{\theta(n+2,k)} \left\{ q^k \begin{bmatrix} n+1-k \\ k \end{bmatrix} + \begin{bmatrix} n+1-k \\ k-1 \end{bmatrix} \right\} \\ &= \sum_k (-q^{L+1})^k q^{\theta(n+2,k)} \begin{bmatrix} n+1-k \\ k \end{bmatrix} - q^L \sum_k (-q^L)^k q^{\theta(n+2,k+1)} \begin{bmatrix} n-k \\ k \end{bmatrix}. \end{aligned} \quad (5.8)$$

This is an opaque expression. If we simplify it by having $L = 0$, demanding that

$$\theta(n+2, k) + k = \theta(n+1, k) \quad (5.9)$$

modulo rescaling of f_n by a function of n , and then requiring that

$$\{\theta(n+2, k+1) - \theta(n, k)\} \text{ is } k\text{-independent}, \quad (5.10)$$

we find that

$$\theta(n, k) = k(k-n), \quad (5.11)$$

and formula (5.8) yields:

$$f_{n+2} = f_{n+1} - q^{-(n+1)} f_n. \quad (5.12)$$

Changing q into q^{-1} , we get:

$$f_{n+2} = f_{n+1} - q^{n+1} f_n, \quad f_0 = f_1 = 1, \quad (5.13)$$

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q \quad (5.14)$$

because

$$[n]!_{q^{-1}} = [n]!q^{-\binom{n}{2}}, \quad (5.15a)$$

$$\begin{bmatrix} n \\ r \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ r \end{bmatrix}_q q^{r(r-n)}. \quad (5.15b)$$

The sequence f_n is:

$$\begin{aligned} &1, 1, 1 - q, 1 - q - q^2, 1 - q - q^2 - q^3 + q^4, 1 - q - q^2 - q^3 + q^5 + q^6, \\ &1 - q - q^2 - q^3 + 2q^6 + q^7 + q^8 + q^9, \dots \end{aligned} \quad (5.16a)$$

and it doesn't appear interesting; however, its limit f_∞ certainly is: from formula (5.14) with $n = \infty$, we get

$$f_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2}}{(1 - q)^k}. \quad (5.16b)$$

Secondly, if we use formula (5.7b) instead of (5.7a), we find (with $L = 0$):

$$\begin{aligned} f_{n+2} &= \sum (-1)^k q^{\theta(n+2,k)} \left\{ \begin{bmatrix} n+1-k \\ k \end{bmatrix} + q^{n+2-2k} \begin{bmatrix} n+1-k \\ k-1 \end{bmatrix} \right\} \\ &= \sum (-1)^k q^{\theta(n+2,k)} \begin{bmatrix} n+1-k \\ k \end{bmatrix} - q^n \sum (-1)^k q^{\theta(n+2,k+1)-2k} \begin{bmatrix} n-k \\ k \end{bmatrix}. \end{aligned} \quad (5.17)$$

The first summand in (5.17) implies that $\theta(n, k)$ is n -independent, and the second summand yields

$$\theta(n, k) = k^2. \quad (5.18)$$

Thus,

$$f_{n+2} = f_{n+1} - q^{n+1} f_n, \quad f_0 = f_1 = 1, \quad (5.19)$$

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}_q, \quad (5.20)$$

which is the same as formula (5.14).

If we set

$$F(x) = \sum_{n=0}^{\infty} f_n x^n, \quad (5.21)$$

then formula (5.19) converts itself into

$$F(x)(1-x) = 1 - qx^2 F(qx), \quad (5.22)$$

whence, by repeated iteration,

$$F(x) = \sum_{k=0}^{\infty} \frac{(-x^2)^k q^{k^2}}{(1-x)^{k+1}}. \quad (5.23)$$

Our original sequence S_n (2.8),

$$S_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} q^{\binom{k}{2}} \quad (5.24)$$

satisfies, as can be directly checked, the following q -analog of the relation (5.1):

$$S_n = q^{\lfloor n/3 \rfloor} S_{n-1} - q^{\lfloor 2n/3 \rfloor - 1} S_{n-2}; \quad (5.25)$$

the more general sequence $S_n(a)$ (2.12):

$$S_n(a) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \begin{bmatrix} n-k \\ k \end{bmatrix} q^{\binom{k}{2}} a^{n-k}, \quad (5.26)$$

for $a = q$ satisfies the easily verified relation

$$S_n(q) = q^{\lfloor (n+2)/3 \rfloor} S_{n-1}(q) - q^{\lfloor (2n+1)/3 \rfloor} S_{n-2}(q). \quad (5.27)$$

6 Chebyshev polynomials and series

The special 3-antiperiodic relation (5.1)

$$u_{n+2} = u_{n+1} - u_n \quad (6.1)$$

is satisfied by the values of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ at $x = 1/2$. Let us recall the salient facts.

Originally, Chebyshev polynomials appeared as the least deviant monic polynomials of a fixed degree, n say, on the interval $-1 \leq x \leq 1$. For the norm

$$\|f\| = \max_{-1 \leq x \leq 1} |f(x)|,$$

the answer is $T_n(x)/2^{n-1}$,

$$T_n(x) = \cos(n\theta), \quad x = \cos \theta, \quad (6.2)$$

being the Chebyshev polynomial of the 1st kind; for the norm

$$\|f\| = \int_{-1}^1 |f(x)| dx,$$

the answer is $U_n(x)/2^n$,

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin \theta}, \quad x = \cos \theta, \quad (6.3)$$

being the Chebyshev polynomial of the 2nd kind.

The following table is extracted from [11, p. 287]:

$$\begin{aligned}
 T_0(x) &= 1, \\
 T_1(x) &= x, \\
 T_2(x) &= 2x^2 - 1, \\
 T_3(x) &= 4x^3 - 3x, \\
 T_4(x) &= 8x^4 - 8x^2 + 1, \\
 T_5(x) &= 16x^5 - 20x^3 + 5x, \\
 T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1, \\
 T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x, \\
 T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1, \\
 T_9(x) &= 256x^9 - 576x^7 + 232x^5 - 120x^3 + 9x;
 \end{aligned} \tag{6.4}$$

$$\begin{aligned}
 U_0(x) &= 1, \\
 U_1(x) &= 2x, \\
 U_2(x) &= 4x^2 - 1, \\
 U_3(x) &= 8x^3 - 4x, \\
 U_4(x) &= 16x^4 - 12x^2 + 1, \\
 U_5(x) &= 32x^5 - 32x^3 + 6x, \\
 U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1, \\
 U_7(x) &= 128x^7 - 192x^5 + 80x^3 - 8x, \\
 U_8(x) &= 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1, \\
 U_9(x) &= 512x^9 - 1024x^7 + 672x^5 - 106x^3 + 10x.
 \end{aligned} \tag{6.5}$$

The table (6.4) suggests that

$$T_p(x) \equiv x^p \pmod{p}, \quad p \text{ an odd prime}, \tag{6.6}$$

which is true because of the easily verifiable formula

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} (x^2 - 1)^k, \tag{6.7}$$

and this interesting congruence is listed as an exercise in [2, p. 117]; there are more exercises on that page, about other interesting arithmetic properties of the Chebyshev polynomials of the 1st and 2nd kind. Some other properties not on that page are:

$$T_{p^2}(x) \equiv T_p(x^p) \pmod{p^2}, \quad p \text{ an odd prime}, \tag{6.8a}$$

$$T_{2^{n+1}}(x) \equiv 1 \pmod{2^{2^{n+1}}}, \quad n \in \mathbb{Z}_+, \tag{6.8b}$$

which follow from formula (6.6) and formula (6.17) below. No doubt there exist many more interesting arithmetic properties of the Chebyshev polynomials.

Such as. Define the sequence $\{\gamma_n\}$ by the rule:

$$\gamma_{n+2} = 2\gamma_{n+1} + \gamma_n, \quad \gamma_0 = 0, \quad \gamma_1 = 1. \quad (6.9a)$$

Then

$$T_{2n}(\pm\sqrt{-1}) = 1 + (-1)^n \gamma_n^2, \quad n \in \mathbb{Z}_+, \quad (6.9b)$$

$$T_{2n+1}(\pm\sqrt{-1}) = \pm\sqrt{-1} \{-1 + (-1)^n (\gamma_{n+1}^2 - \gamma_n^2)/2\}, \quad n \in \mathbb{Z}_+. \quad (6.9c)$$

Most trigonometric formulae translate into formulae for the Chebyshev polynomials in view of the representations (6.2) and (6.3). For example, formula

$$\cos u + \cos v = 2 \cos \frac{u+v}{2} \cos \frac{u-v}{2},$$

for $u = (n+2)\theta$, $v = n\theta$; yields

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x); \quad (6.10a)$$

formula

$$\sin u + \sin v = 2 \sin \frac{u+v}{2} \cos \frac{u-v}{2}$$

yields

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad (6.10b)$$

for $u = (n+2)\theta$, $v = n\theta$; and

$$U_n(x) - U_{n-2}(x) = 2T_n(x) \quad (6.11)$$

for $u = (n+1)\theta$, $v = -(n-1)\theta$. (Many more formulae involving Chebyshev polynomials can be found in [3, § 10.11], [17, 7], and [14, § 5.7].)

Formulae (6.10) for $x = 1/2$ show that $\{T_n(1/2)\}$ and $\{U_n(1/2)\}$ are 3-antiperiodic sequences of the form (6.1), with

$$T_0(1/2) = 1, \quad T_1(1/2) = 1/2, \quad (6.12a)$$

$$U_0(1/2) = 1, \quad U_1(1/2) = 1. \quad (6.12b)$$

The preceding Sections thus have dealt with quantum aspects of Chebyshev polynomials $U_n(x)$'s at just *one point* $x = 1/2$. We won't attempt anything quantum in this Section, as this would be a rather formidable undertaking that is better left to the interested reader. Instead, we shall explore the problem of *interpolating* the function $u(n) = u_n$ from integers to real and complex numbers.

We need a few more standard formulae.

Set temporarily $\tilde{U}_n(x) = (n+1)^{-1}dT_{n+1}/dx$. Differentiating formula (6.10a), we get:

$$(n+1)(\tilde{U}_{n+1} - 2x\tilde{U}_n + \tilde{U}_{n-1}) + (\tilde{U}_{n+1} - \tilde{U}_{n-1} - 2T_n) = 0. \quad (6.13)$$

This equation defines \tilde{U}_n 's recursively. From the table (6.4),

$$\tilde{U}_0 = T'_1 = 1 = U_0, \quad (6.14a)$$

$$\tilde{U}_1 = T'_2/2 = 2x = U_1, \quad (6.14b)$$

and the U_n 's also satisfy the equation (6.13) by formulae (6.10b) and (6.11). Therefore, $\tilde{U}_n = U_n$:

$$U_n(x) = \frac{1}{n+1} \frac{dT_{n+1}}{dx}. \quad (6.15)$$

(Alternatively, one gets formula (6.15) by differentiating formula (6.2) with respect to x .)
Formulae (6.10) and (6.15) yield:

$$T_n(1) = 1, \quad (6.16a)$$

$$U_n(1) = n+1, \quad (6.16b)$$

$$T'_n(1) = n^2. \quad (6.16c)$$

Formula (6.2) implies that ([19, p. 45], [21, 27])

$$T_n(T_m(x)) = T_{nm}(x), \quad n, m \in \mathbb{N}. \quad (6.17)$$

Thus, the polynomials T_n 's form a commutative semigroup; below we shall see that this semigroup is in fact a discrete part of a commutative 1-dimensional *group* considered as either a formal group or as a group of holomorphic automorphisms of a neighborhood of the complex plane around $z = 1$. Formula (6.15) implies that $\{T_n$'s} are a more fundamental object than $\{U_n$'s}.

Now, from the definition (6.2),

$$\frac{d^2 T_n}{d\theta^2} = -n^2 T_n. \quad (6.18a)$$

Since

$$\frac{d}{d\theta} = -\sin \theta \frac{d}{dx}, \quad \frac{d^2}{d\theta^2} = (1-x^2) \frac{d^2}{dx^2} - x \frac{d}{dx}, \quad (6.18b)$$

we get the classical equation

$$(1-x^2) T''_n - x T'_n + n^2 T_n = 0. \quad (6.19)$$

The T_n is the unique regular solution of the equation

$$(1-x^2) y'' - xy' + n^2 y = 0 \quad (6.20a)$$

satisfying the initial condition

$$y(1) = 1. \quad (6.20b)$$

In the variable

$$z = \frac{1-x}{2}, \quad (6.21)$$

the equation (6.20a) takes the hypergeometric form

$$z(1-z)\frac{d^2y}{dz^2} + \left(\frac{1}{2} - z\right)\frac{dy}{dz} + n^2y = 0, \quad (6.22)$$

and we recover another classical result:

$$T_n(x) = F\left(-n, n, \frac{1}{2}; \frac{1-x}{2}\right). \quad (6.23)$$

(The point $x = \frac{1}{2}$ directs us to the continuous 3-antiperiodic family $u(\alpha) = F(-\alpha, \alpha, \frac{1}{2}; \frac{1}{4})$.)
Formulae (6.4), (6.5), (6.10) imply that

$$T_n(0) = U_n(0) = 0, \quad n \text{ odd}; \quad (6.24a)$$

$$T_{2n}(0) = U_{2n}(0) = (-1)^n. \quad (6.24b)$$

The differential equation (6.19) then yields

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}, \quad n \in \mathbb{N}, \quad (6.25)$$

and then formula (6.15) yields

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}, \quad n \in \mathbb{Z}_+; \quad (6.26)$$

another classical pair of very useful formulae.

We now extend the index n in $T_n(x)$ by allowing n to be an arbitrary complex number or a formal parameter: we set

$$T_\alpha(x) = \sum_{k=0}^{\infty} c_{\alpha|k} (x-1)^k, \quad c_{\alpha|0} = 1, \quad (6.27)$$

and require T_α to satisfy the differential equation

$$(1-x^2)T_\alpha'' - xT_\alpha' + \alpha^2T_\alpha = 0, \quad (6.28)$$

together with the boundary condition

$$T_\alpha(1) = 1. \quad (6.29)$$

This is equivalent to the relations

$$c_{\alpha|k+1} = \frac{\alpha^2 - k^2}{(k+1)(2k+1)} c_{\alpha|k}, \quad k \in \mathbb{Z}_+ \Rightarrow \quad (6.30)$$

$$c_{\alpha|k+1} = \frac{1}{(k+1)!(2k+1)!!} \prod_{i=0}^k (\alpha^2 - i^2), \quad c_{\alpha|0} = 1. \quad (6.31)$$

Since

$$\lim_{k \rightarrow \infty} |c_{\alpha|k+1}/c_{\alpha|k}| = 1/2, \quad (6.32)$$

the series $T_\alpha(x)$ (6.27) converges for

$$|x - 1| < 2. \quad (6.33)$$

The series terminates only when α is an integer. It is readily verified that

$$T_{\alpha+2} - 2(1 + (x - 1))T_{\alpha+1} + T_\alpha = 0, \quad \forall \alpha, \quad (6.34)$$

and that

$$U_\alpha - U_{\alpha-2} = 2T_\alpha, \quad \forall \alpha, \quad (6.35)$$

where

$$U_\alpha(x) = \frac{1}{\alpha + 1} \frac{dT_\alpha}{dx}, \quad \forall \alpha. \quad (6.36)$$

From this it follows that

$$U_{\alpha+2} - 2((1 + (x - 1))U_{\alpha+1} + U_\alpha = 0, \quad \forall \alpha. \quad (6.37)$$

Also, formula (6.30) shows that

$$T_{-\alpha} = T_\alpha, \quad \forall \alpha. \quad (6.38)$$

Finally,

$$T_\alpha(T_\beta(x)) = T_{\alpha\beta}(x), \quad \forall \alpha, \beta. \quad (6.39a)$$

Indeed, the difference

$$T_\alpha(T_\beta(x)) - T_{\alpha\beta}(x),$$

for each fixed degree k of $(x - 1)^k$, is a *polynomial* in α, β which vanishes, by (6.17), for all $\alpha, \beta \in \mathbb{N} \times \mathbb{N}$. Therefore, this polynomial is identically zero.

In particular, since $T_1(x) = x = 1 + (x - 1)$ plays the role of the identity in the group $\{T_\alpha\}$, we have:

$$(T_\alpha)^{-1} = T_{1/\alpha}. \quad (6.39b)$$

Most of the classical formulae for Chebyshev polynomials survive the constructed extension – either by the argument given above to prove formula (6.39a), or by observing that

$$T_\alpha(x) = F\left(-\alpha, \alpha, \frac{1}{2}, \frac{1-x}{2}\right), \quad \forall \alpha. \quad (6.40)$$

Remark 6.41. The theme of parameter extension is quite common in mathematics. The *methods* for constructing such extensions vary wildly. I chose in this Section the group

property of T_n 's. A more direct approach is to extend the parameters' meaning in the hypergeometric representations, as in formulae (6.23) and (6.40). Then there are various *relations* one can extend. For example, the Legendre polynomials $P_n(x)$ enter into the following relations with the $T_n(x)$'s ([14, p. 261]):

$$\left(n + \frac{1}{2}\right) (1+x)^{1/2} \int_{-1}^x P_n(t)(x-t)^{-1/2} dt = T_n(x) + T_{n-1}(x), \quad (6.42a)$$

$$\left(n + \frac{1}{2}\right) (1-x)^{1/2} \int_x^1 P_n(t)(t-x)^{-1/2} dt = T_n(x) - T_{n+1}(x), \quad (6.42b)$$

where $|x| < 1$, $n \in \mathbb{N}$, and the integrals are taken as Cauchy principal values.

Remark 6.43. Treatises and collections of formulae on special functions should always be read with a pen, pencil, and salt shaker ready. For example, the book [9] lists on p. 206 the following formula:

$$U_{n+1}(x) = (1-x)^{1/2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (n-k)!}{k!(n-2k)!} (2x)^{n-2k}. \quad (6.44)$$

Misprints are widespread, common, and not unexpected. Some errors are more subtle, and propagate from one book to another for *many* years. (See an example of 186 years old errors in [20]).

Remark 6.45. The Chebyshev polynomials $\{T_n(x)\}$ and the system $\{x^n\}$ are essentially the only two *polynomial* systems satisfying the semigroup/group law (see [18, 4]). More general systems, if any exist, could be found only in *series*. For example, if one restricts oneself to the hypergeometric (classical or basic) groups, one looks for triples of functions

$$a(x) = {}_p\mathcal{F}_q(\dots; \omega_1(x-1)), \quad (6.46a)$$

$$b(x) = {}_r\mathcal{F}_s(\dots; \omega_2(x-1)), \quad (6.46b)$$

$$c(x) = {}_p\mathcal{F}_Q(\dots; \Omega(x-1)), \quad (6.46c)$$

such that

$$a(b(x)) = c(x). \quad (6.46d)$$

Remark 6.47. Suppose one quantizes the Chebyshev polynomials $U_n(x)$ by any of the standard methods of q -hypergeometric (= basic) series (see [2, 5, 6, 8].) Will then the sequences S_n and $S_n(q)$ constructed in § 2 appear as “values at $x = 1/2$ ” of the quantized polynomials? I don't see how, and think it is very unlikely, as one glance at formulae (5.25) and (5.27) will show. There is a deep mystery hidden here.

Remark 6.48. The proper meaning of the Chebyshev polynomials, as the generators of a semigroup of polynomial maps of a vector space, has been discovered by Veselov [22, 23] and by Hoffman and Withers [26, 10], with such semigroups parametrized by affine Weyl groups. It would be very interesting to find the analogs of the one-dimensional point $x = 1/2$ and periodic sequences, both classical and quantum, for this more general set-up.

I hope that posterity will judge me kindly,
not only as to the things which I have explained,
but also to those which I have intentionally
omitted so as to leave to others the pleasure
of discovery.

Descartes

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