On a q-Difference Painlevé III Equation: II. Rational Solutions

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Abstract

Rational solutions for a q-difference analogue of the Painlevé III equation are considered. A Determinant formula of Jacobi–Trudi type for the solutions is constructed.

1 Introduction

This paper is the second half of the work on a q-difference Painlevé III equation (q-$\mathrm{P_{III}}$),

$$
\begin{aligned}
\overline{f_1} &= \frac{q^{2N+1}c^2}{f_0 f_1} \frac{1 + a_0 q^n f_0}{a_0 q^n + f_0}, \\
\underline{f_0} &= \frac{q^{2N+1}c^2}{f_0 f_1} \frac{a_1 q^{-n+\nu} + f_1}{1 + a_1 q^{-n+\nu} f_1},
\end{aligned}
\tag{1.1}
$$

where $f_i = f_i(n;\nu,N)$ ($i = 0,1$) are dependent variables, $n \in \mathbb{Z}$ is the independent variable, $\nu, N \in \mathbb{Z}$ are parameters, and q, a_0, a_1, c are constants. Moreover, $\overline{f_i}$ and $\underline{f_i}$ denote $f_i(n+1;\nu,N)$ and $f_i(n-1;\nu,N)$, respectively.

In the previous paper [6], we have discussed the derivations, symmetry and particular solutions of Riccati type of q-$\mathrm{P_{III}}$ (1.1). In this paper, we consider the rational solutions.

We remark that q-$\mathrm{P_{III}}$ (1.1) appears as a dynamical system on certain rational surface ("Mul.6 surface" in Sakai's classification [27]). Moreover, q-$\mathrm{P_{III}}$ admits a symmetry of (extended) affine Weyl group of type $A_1^{(1)} \times A_1^{(1)}$ as a group of Bäcklund transformations, which is the same type as that for the Painlevé III equation ($\mathrm{P_{III}}$).

The six Painlevé equations ($\mathrm{P_J}$, $\mathrm{J = I, II, \ldots, VI}$), except for $\mathrm{P_I}$, admit two classes of classical solutions for special values of parameters; one is so-called transcendental classical solutions, namely, one-parameter family of particular solutions expressible in terms of classical special functions of hypergeometric type. Another class is algebraic or rational solutions.

There are two classes among algebraic or rational solutions. One class consists of such solutions that can be regarded as special case of transcendental classical solutions. Another class of solutions are expressed in terms of log derivative or ratio of certain characteristic

polynomials (after some change of independent variable), and satisfy recurrence relations of Toda type. We call such polynomials "special polynomials" for the Painlevé equations. They are sometimes referred as Yablonskii–Vorob'ev polynomials for P_{II}, Okamoto polynomials for P_{IV}, Umemura polynomials for P_{III}, P_V and P_{VI}.

It may be natural then to ask a question: what are those special polynomials? One answer is that the special polynomials are specialization of the Schur function or its generalizations. Such description has been established through the Jacobi–Trudi type determinant formulas for the special polynomials. Moreover, the entries are expressed in terms of classical special polynomials [7, 10, 11, 16, 18, 20, 21, 29].

Compared to the continuous case, the amount of known results for discrete Painlevé equations is not so much. Although it is not difficult to construct rational solutions if we have Bäcklund transformations of given discrete Painlevé equation (a systematic method to construct Bäcklund transformations is given in [5]), construction of determinant formulas and identifying the special polynomials as classical object sometimes contain technical difficulty. So far the rational solutions are systematically discussed for the standard discrete Painlevé II equation [12], a *q*-difference Painlevé IV equation [8, 9] and a *q*-difference Painlevé V equation [17] (determinant formula for rational solutions of standard discrete Painlevé IV equation is conjectured in [4]).

The purpose of this paper is to construct a class of rational solutions for q-P_{III} (1.1) and present a determinant formula of Jacobi–Trudi type.

This paper is organized as follows. In Section 2 we give a brief review for rational solutions of P_{III} and show that all the rational solutions admit determinant formula of Jacobi–Trudi type. In Section 3 we construct a class of rational solutions for q-P_{III} and present the Jacobi–Trudi type determinant formula, which is the main result of this paper. We give the proof in Section 4.

2 Rational solutions of P_{III}

It is known that the Painlevé III equation (P_{III}),

$$\frac{d^2v}{dx^2} = \frac{1}{v}\left(\frac{dv}{dx}\right)^2 - \frac{1}{x}\frac{dv}{dx} + \frac{1}{x}\left(\alpha v^2 + \beta\right) + \gamma v^3 + \frac{\delta}{v}, \qquad \gamma = -\delta = 4, \tag{2.1}$$

admits a class of rational solutions expressible in terms of determinant of Jacobi–Trudi type associated with the two-core partition $\lambda = (N, N-1, \ldots, 1)$.

Theorem 1 ([7]). *Let $p_k(x,s)$, $k \in \mathbb{Z}$ be polynomials in x defined by*

$$\sum_{k=0}^{\infty} p_k(x,s)\lambda^k = (1+\lambda)^{s+1/2}e^{2x\lambda}, \quad k \geq 0, \qquad p_k(x,s) = 0, \quad k < 0. \tag{2.2}$$

For each $N \in \mathbb{Z}_{\geq 0}$, we define a polynomial $U_N(x,s)$ by

$$U_N(x,s) = \begin{vmatrix} p_N(x,s) & p_{N+1}(x,s) & \cdots & p_{2N-1}(x,s) \\ p_{N-2}(x,s) & p_{N-1}(x,s) & \cdots & p_{2N-3}(x,s) \\ \vdots & \vdots & \ddots & \vdots \\ p_{-N+2}(x,s) & p_{-N+3}(x,s) & \cdots & p_1(x,s) \end{vmatrix}, \qquad U_0 = 1. \tag{2.3}$$

Then,

$$v = \frac{U_N(x, s-1)U_{N-1}(x, s)}{U_N(x, s)U_{N-1}(x, s-1)}, \tag{2.4}$$

satisfies P_{III} *(2.1) with* $\alpha = 4(s + N)$, $\beta = 4(-s + N)$.

This theorem asserts that a class of rational solutions of P_{III} is given by ratio of specialization of the Schur functions associated with partition $\lambda = (N, N-1, \ldots, 1)$. We note that $p_k(x, s)$ is nothing but the Laguerre polynomial $L_k^{(s+1/2-k)}(-2x)$. The polynomials $U_N(x, s)$ (after some rescaling) are sometimes called the Umemura polynomials for P_{III}.

It may not be meaningless to state here that the above solutions essentially cover all the rational solutions of P_{III}. Classification of rational solutions for P_{III} is discussed in [3, 19]. According to [19], the result is summarized as follows.

Theorem 2 ([19]). *1. Let* $\alpha = -4\theta_\infty$ *and* $\beta = 4(\theta_0 + 1)$ *in* P_{III} *(2.1). Then* P_{III} *admits rational solutions if and only if there exists an integer* I *such that (i)* $\theta_\infty + \theta_0 + 1 = 2I$ *or (ii)* $\theta_\infty - \theta_0 - 1 = 2I$.
2. If P_{III} *has rational solutions, then the number of rational solutions for each* $(\theta_\infty, \theta_0)$ *is two or four. Furthermore,* P_{III} *admit four rational solutions if and only if there exists two integers* I *and* J *such that* $\theta_\infty + \theta_0 + 1 = 2I$ *and* $\theta_\infty - \theta_0 - 1 = 2J$.

Obviously, the solutions in Theorem 1 correspond to the case (ii) with $I \leq 0$, since in this case $(\theta_\infty, \theta_0)$ is given by $(\theta_\infty, \theta_0) = (-s - N, -s + N - 1)$ with $N \in \mathbb{Z}_{\geq 0}$. We next remark that if we define U_N $(N < 0)$ by

$$U_N = (-1)^{N(N+1)/2} U_{-N-1}, \tag{2.5}$$

then Theorem 1 is extended to all $N \in \mathbb{Z}$. In fact, Theorem 1 was proved based on the fact that if U_N satisfies the Toda equation,

$$(2N + 1)U_{N+1}U_{N-1} = -\frac{x}{2}\left[\frac{d^2U_N}{dx}^2 U_N - \left(\frac{dU_N}{dx}\right)^2\right]$$
$$- \frac{1}{2}\frac{dU_N}{dx}U_N + \left(2x + s + \frac{1}{2}\right)U_N^2, \tag{2.6}$$

with the initial condition,

$$U_{-1} = U_0 = 1, \tag{2.7}$$

then $v = \frac{U_N(x,s-1)U_{N-1}(x,s)}{U_N(x,s)U_{N-1}(x,s-1)}$ satisfies P_{III} for $N \in \mathbb{Z}_{>0}$. This fact is easily generalized to $N \in \mathbb{Z}$ by extending U_N according to the Toda equation (2.6) to $N \in \mathbb{Z}_{<0}$. Therefore, the rational solutions in Theorem 1 with the extension (2.5) cover one class of rational solutions of the case (ii).

According to Theorem 2, there must be another class of rational solutions for the case (ii). However, noticing that P_{III} (2.1) admits a Bäcklund transformation,

$$(\theta_\infty, \theta_0) \mapsto (\overline{\theta}_\infty, \overline{\theta}_0) = (-\theta_0 - 1, -\theta_\infty - 1), \qquad (x, v) \mapsto (u, V) = (x, -1/v), \tag{2.8}$$

we see that $V = -\frac{U_N(x,-s)U_{N-1}(x,-s-1)}{U_N(x,-s-1)U_{N-1}(x,-s)}$ also satisfies P$_{\text{III}}$ with $(\theta_\infty, \theta_0) = (-s - N, -s + N - 1)$. Since it is clear that v and V give different functions, we obtain two rational solutions for each $(\theta_\infty, \theta_0) = (-s - N, -s + N - 1)$ which cover the case (ii).

Rational solutions for the case (i) are obtained from those for the case (ii) by the following Bäcklund transformation,

$$(\theta_\infty, \theta_0) \mapsto (\overline{\theta}_\infty, \overline{\theta}_0) = (-\theta_\infty, \theta_0), \qquad (x, v) \mapsto (u, V) = (\xi x, \xi v), \qquad \xi = \pm i. \quad (2.9)$$

This implies that for each integer N, the functions

$$v = i \frac{U_N(x/i, s - 1)U_{N-1}(x/i, s)}{U_N(x/i, s)U_{N-1}(x/i, s - 1)}, \qquad v = -i \frac{U_N(x/i, -s)U_{N-1}(x/i, -s - 1)}{U_N(x/i, -s - 1)U_{N-1}(x/i, -s)}, \quad (2.10)$$

give two rational solutions for $(\theta_\infty, \theta_0) = (-s + N - 1, s + N)$, which cover the case (i).

Summarizing the discussions above, we arrive at the following conclusion:

Theorem 3. *Let $U_N(x, s)$ $(N \in \mathbb{Z})$ be polynomials in x defined by equations (2.2), (2.3) and (2.5). Then, all the rational solutions for P$_{\text{III}}$ are given as follows:*
   1. For any integer N,

$$v = \frac{U_N(x, s - 1)U_{N-1}(x, s)}{U_N(x, s)U_{N-1}(x, s - 1)}, \qquad v = -\frac{U_N(x, -s)U_{N-1}(x, -s - 1)}{U_N(x, -s - 1)U_{N-1}(x, -s)}, \quad (2.11)$$

give two rational solutions of P$_{\text{III}}$ with $(\theta_\infty, \theta_0) = (-s - N, -s + N - 1)$.
   2. For any integer N,

$$v = i \frac{U_N(x/i, s - 1)U_{N-1}(x/i, s)}{U_N(x/i, s)U_{N-1}(x/i, s - 1)},$$
$$v = -i \frac{U_N(x/i, -s)U_{N-1}(x/i, -s - 1)}{U_N(x/i, -s - 1)U_{N-1}(x/i, -s)}, \quad (2.12)$$

give two rational solutions of P$_{\text{III}}$ with $(\theta_\infty, \theta_0) = (s + N, -s + N - 1)$.
   3. For any integers M and N,

$$v = \frac{U_N(x, M - 1)U_{N-1}(x, M)}{U_N(x, M)U_{N-1}(x, M - 1)},$$
$$v = -\frac{U_N(x, -M)U_{N-1}(x, -M - 1)}{U_N(x, -M - 1)U_{N-1}(x, -M)},$$
$$v = i \frac{U_M(x/i, -N)U_{M-1}(x/i, -N - 1)}{U_M(x/i, -N - 1)U_{M-1}(x/i, -N)},$$
$$v = -i \frac{U_M(x/i, N - 1)U_{M-1}(x/i, N)}{U_M(x/i, N)U_{M-1}(x/i, N - 1)}, \quad (2.13)$$

give four rational solutions P$_{\text{III}}$ with $(\theta_\infty, \theta_0) = (-M - N, -M + N - 1)$.

Remark 1. The case $\gamma\delta = 0$ of P$_{\text{III}}$ (2.1) should be distinguished from other cases because of the essential difference of type of "space of initial conditions" [27], namely, defining manifold of the equation. More precisely, P$_{\text{III}}$ should be divided into four cases;

(i) $\alpha = 0$, $\gamma = 0$ (or $\beta = 0$, $\delta = 0$), (ii) Type D_7 ($\gamma = 0$, $\alpha\delta \neq 0$ or $\delta = 0$, $\beta\gamma \neq 0$), (iii) Type D_8 ($\gamma = \delta = 0$, $\alpha\beta \neq 0$), (iv) Type D_6 (generic case)

First, in the case (i) P_{III} is known to be solvable by quadratures [1, 13, 25]. Second, type D_8 does not admit any classical solutions except for two trivial constant solutions [24]. Also, type D_7 does not admit transcendental classical solutions [24, 28]. Algebraic solutions for type D_7 have been studied by many authors [1, 2, 13, 14, 24], but we do not discuss this class of solutions in this paper.

3   Rational solutions of q-P_{III}

In this section we construct a class of rational solutions of q-P_{III}. For notational simplicity, we introduce

$$z = a_0 q^n, \qquad y = a_0 a_1 q^\nu, \tag{3.1}$$

and write equation (1.1) as

$$
\begin{aligned}
f_1(qz; y, c) &= \frac{c^2}{f_0(z; y, c) f_1(z; y, c)} \frac{1 + z f_0(n; \nu, c)}{z + f_0(n; \nu, c)}, \\
f_0(z/q; y, c) &= \frac{c^2}{f_0(z; y, c) f_1(z; y, c)} \frac{y/z + f_1(z; y, c)}{1 + y/z f_1(z; y, c)}.
\end{aligned}
\tag{3.2}
$$

A Bäcklund transformation is presented in [6, 8] as

$$
\begin{aligned}
f_0(z; y, qc) &= y \, f_1(z; y, c) \, \frac{1 + q/y \, f_2(n; \nu, c) + qz/y \, f_2(z; y, c) f_0(z; y, c)}{1 + z \, f_0(z; y, c) + y \, f_0(z; y, c) f_1(z; y, c)}, \\
f_1(z; y, qc) &= q/z \, f_2(z; y, c) \, \frac{1 + z \, f_0(n; \nu, c) + y \, f_0(z; y, c) f_1(z; y, c)}{1 + y/z \, f_1(z; y, c) + q/z \, f_1(z; y, c) f_2(z; y, c)},
\end{aligned}
\tag{3.3}
$$

where $f_2(z; y, c) = c^2/(f_0(z; y, c) f_1(z; y, c))$.

Now we see that equation (3.2) admits a trivial solution,

$$f_0 = f_1 = 1, \qquad c = 1. \tag{3.4}$$

Applying the Bäcklund transformation (3.3), we observe that

$$
\begin{aligned}
f_0(z; y, q) &= \frac{y + qz + q}{1 + y + z} = q \frac{\psi_1(y/q, z)}{\psi_1(y, z)}, \\
f_1(z; y, q) &= q \frac{1 + y + z}{y + z + q} = \frac{\psi_1(y, z)}{\psi_1(y/q, z/q)},
\end{aligned}
\tag{3.5}
$$

where

$$\psi_1(y, z) = y + z + 1, \tag{3.6}$$

is also a solution for $c = q$. Similarly, one can construct higher order rational solutions as,

$$f_0\left(z; y, q^2\right) = q^2 \frac{\psi_1(y, z)\psi_2(y/q, z)}{\psi_1(y/q, z)\psi_2(y, z)}, \qquad f_1\left(z; y, q^2\right) = \frac{\psi_1(y/q, z/q)\psi_2(y, z)}{\psi_1(y, z)\psi_2(y/q, z/q)}, \tag{3.7}$$

$$f_0\left(z;y,q^3\right) = q^2\frac{\psi_2(y,z)\psi_3(y/q,z)}{\psi_2(y/q,z)\psi_3(y,z)}, \qquad f_1\left(z;y,q^3\right) = \frac{\psi_2(y/q,z/q)\psi_3(y,z)}{\psi_2(y,z)\psi_3(y/q,z/q)}, \qquad (3.8)$$

where $\psi_2(y,z)$ and $\psi_3(y,z)$ are polynomials in y and z given by

$$
\begin{aligned}
\psi_2(y,z) &= y^3 + \left(q+1+q^{-1}\right)y^2z + \left(q+1+q^{-1}\right)yz^2 + z^3 \\
&\quad + \left(q+1+q^{-1}\right)\left(y^2+2yz+z^2\right) + \left(q+1+q^{-1}\right)(y+z) + 1, \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
\psi_3(y,z) &= y^6 + y^5z\left(q+1+q^{-1}\right)\left(q+q^{-1}\right) \\
&\quad + y^4z^2\left(q^2+q+1+q^{-1}+q^{-2}\right)\left(q+1+q^{-1}\right) \\
&\quad + 2y^3z^3\left(q^2+q+1+q^{-1}+q^{-2}\right)\left(q+q^{-1}\right) \\
&\quad + y^2z^4\left(q^2+q+1+q^{-1}+q^{-2}\right)\left(q+1+q^{-1}\right) \\
&\quad + yz^5\left(q+1+q^{-1}\right)\left(q+q^{-1}\right) + z^6 \\
&\quad + \left(q+1+q^{-1}\right)\Big\{\left(y^5(q+q^{-1}) + 2y^4z\left(q^2+q+1+q^{-1}+q^{-2}\right)\right. \\
&\qquad + y^3z^2\left(q^2+q+1+q^{-1}+q^{-2}\right)\left(q^{1/2}+q^{-1/2}\right)^2 \\
&\qquad + y^2z^3\left(q^2+q+1+q^{-1}+q^{-2}\right)\left(q^{1/2}+q^{-1/2}\right)^2 \\
&\qquad \left.+ 2yz^4\left(q^2+q+1+q^{-1}+q^{-2}\right) + z^5\left(q+q^{-1}\right)\right)\Big\} \\
&\quad + \left(q^2+q+1+q^{-1}+q^{-2}\right)\left(q+1+q^{-1}\right) \\
&\qquad \times \Big\{y^4 + y^3z\left(q^{1/2}+q^{-1/2}\right)^2 + 2y^2z^2\left(q+1+q^{-1}\right) \\
&\qquad + yz^3\left(q^{1/2}+q^{-1/2}\right)^2 + z^4\Big\} \\
&\quad + \left(q^2+q+1+q^{-1}+q^{-2}\right) \\
&\qquad \times \Big\{2y^3\left(q+q^{-1}\right) + y^2z\left(q+1+q^{-1}\right)\left(q^{1/2}+q^{-1/2}\right)^2 \\
&\qquad + yz^2\left(q+1+q^{-1}\right)\left(q^{1/2}+q^{-1/2}\right)^2 + 2z^3\left(q+q^{-1}\right)\Big\} \\
&\quad + \left(q^2+q+1+q^{-1}+q^{-2}\right)\left(q+1+q^{-1}\right)\left(y^2+2yz+z^2\right) \\
&\quad + \left(q+1+q^{-1}\right)\left(q+q^{-1}\right)(y+z) + 1, \qquad (3.10)
\end{aligned}
$$

respectively. In general, we see that $f_0\left(z;y,q^N\right)$ and $f_1\left(z;y,q^N\right)$ $(N \in \mathbb{Z}_{>0})$ may be factorized as

$$
\begin{aligned}
f_0\left(z;y,q^N\right) &= q^N\frac{\psi_{N-1}(y,z)\psi_N(y/q,z)}{\psi_{N-1}(y/q,z)\psi_N(y,z)}, \\
f_1\left(z;y,q^N\right) &= \frac{\psi_{N-1}(y/q,z/q)\psi_N(y,z)}{\psi_{N-1}(y,z)\psi_N(y/q,z/q)}, \qquad (3.11)
\end{aligned}
$$

respectively, where $\psi_N(y,z)$ is a polynomial in y and z with the following nice properties:
(1) $\psi_N(y,z)$ is a polynomial of degree $N(N+1)/2$; (2) $\psi_N(y,z)$ is symmetric with respect to y and z; (3) $\psi_N(y,z)$ is symmetric with respect to q and q^{-1}.

Now we present a determinant formula of Jacobi–Trudi type for the above polynomials.

Theorem 4. *Let $p_k(y, z)$ $(k \in \mathbb{Z})$ be polynomials in y and z defined by*

$$\sum_{n=0}^{\infty} p_n(y, z) t^n = \frac{(-(1-q)t; q)_\infty}{((1-q)yt; q)_\infty ((1-q)zt; q)_\infty}, \qquad p_k(y, z) = 0, \quad \text{for } k < 0, \quad (3.12)$$

where $(a; q)_\infty$ is given by

$$(a; q)_\infty = \prod_{i=0}^{\infty} \left(1 - aq^i\right). \qquad (3.13)$$

For each $N \in \mathbb{Z}$, we define a polynomial $\phi_N(y, z)$ by

$$\phi_N(y, z) = \begin{cases} \begin{vmatrix} p_N(y, z) & p_{N+1}(y, z) & \cdots & p_{2N-1}(y, z) \\ p_{N-2}(y, z) & p_{N-1}(y, z) & \cdots & p_{2N-3}(y, z) \\ \vdots & \cdots & \ddots & \vdots \\ p_{-N+2}(y, z) & p_{-N+3}(y, z) & \cdots & p_1(y, z) \end{vmatrix}, & N > 0, \\ 1, & N = 0, \\ (-1)^{N(N+1)/2} \phi_{-N-1}, & N < 0. \end{cases} \qquad (3.14)$$

Then,

$$f_0\left(z; y, q^N\right) = q^N \frac{\phi_{N-1}(y, z) \phi_N(y/q, z)}{\phi_{N-1}(y/q, z) \phi_N(y, z)},$$

$$f_1\left(z; y, q^N\right) = \frac{\phi_{N-1}(y/q, z/q) \phi_N(y, z)}{\phi_{N-1}(y, z) \phi_N(y/q, z/q)}, \qquad (3.15)$$

satisfy q-P_{III} (3.2) with $c = q^N$.

Remark 2. 1. Two polynomials $\psi_N(y, z)$ and $\phi_N(y, z)$ are related as

$$\psi_N(y, z) = c_N \, \phi_N(y, z), \qquad c_N = q^{-\frac{(N-1)N(N+1)}{6}} \prod_{k=1}^{N} [2k-1]!!, \qquad (3.16)$$

where

$$[k] = \frac{1 - q^k}{1 - q}, \qquad [2k-1]!! = [2k-1][2k-3] \cdots [3][1]. \qquad (3.17)$$

In terms of the "hook-length" $h_{i,j}$ for the partition $\lambda = (\lambda_1, \lambda_2, \ldots) = (N, N-1, \ldots, 1)$ [15], the constant c_N is also expressed as

$$c_N = q^{-\sum_i (i-1)\lambda_i} \prod_{(i,j) \in \lambda} [h_{i,j}]. \qquad (3.18)$$

2. $\phi_N(y, z)$ is quite similar to the q-Schur function associated with two-core partition $\lambda = (N, N-1, \ldots, 1)$ which appears in the rational solutions for q-KP hierarchy [9].

However, they are different due to the factor $(-(1-q)t; q)_\infty$ in the numerator of the generating function in equation (3.12). The first few $p_k(y, z)$ are given by,

$$p_0(y, z) = 1, \tag{3.19}$$

$$p_1(y, z) = y + z + 1, \tag{3.20}$$

$$p_2(y, z) = \frac{y^2}{1+q} + yz + \frac{z^2}{1+q} + y + z + \frac{q}{1+q}, \tag{3.21}$$

$$p_3(y, z) = \frac{y^3}{(1+q+q^2)(1+q)} + \frac{y^2 z}{1+q} + \frac{yz^2}{1+q} + \frac{z^3}{(1+q+q^2)(1+q)}$$
$$+ \frac{y^2}{1+q} + yz + \frac{z^2}{1+q} + \frac{yq}{1+q} + \frac{zq}{1+q} + \frac{q^3}{(1+q+q^2)(1+q)}, \tag{3.22}$$

$$p_4(y, z) = \frac{y^4}{(1+q+q^2+q^3)(1+q+q^2)(1+q)} + \frac{y^3 z}{(1+q+q^2)(1+q)} + \frac{y^2 z^2}{(1+q)^2}$$
$$+ \frac{yz^3}{(1+q+q^2)(1+q)} + \frac{z^4}{(1+q+q^2+q^3)(1+q+q^2)(1+q)}$$
$$+ \frac{y^3}{(1+q+q^2)(1+q)} + \frac{y^2 z}{1+q} + \frac{yz^2}{1+q} + \frac{z^3}{(1+q+q^2)(1+q)}$$
$$+ \frac{y^2 q}{(1+q)^2} + \frac{yzq}{1+q} + \frac{z^2 q}{(1+q)^2}$$
$$+ \frac{yq^3}{(1+q+q^2)(1+q)} + \frac{zq^3}{(1+q+q^2)(1+q)}$$
$$+ \frac{q^6}{(1+q+q^2+q^3)(1+q+q^2)(1+q)}. \tag{3.23}$$

Moreover, from equation (3.12), $p_k(y, z)$ satisfy the following contiguity relations,

$$\left(1 + q + \cdots + q^{k-1}\right) p_k(y, z)$$
$$= \left(y + z + q^{k-1}\right) p_{k-1}(y, z) - (1-q) yz p_{k-2}(y, z), \tag{3.24}$$

$$p_k(y, qz) - p_k(y, z) = (q-1) z p_{k-1}(y, z), \tag{3.25}$$

$$p_k(qy, z) - p_k(y, z) = (q-1) y p_{k-1}(y, z). \tag{3.26}$$

Theorem 4 is a direct consequence of the following "multiplicative formula".

Proposition 1. *The functions* $f_0\left(z; y, q^N\right)$ *and* $f_1\left(z; y, q^N\right)$ *defined by equation (3.15) satisfy the following equations:*

$$1 + z f_0\left(z; y, q^N\right) = (1+z) \frac{\phi_{N-1}(y/q, z/q) \phi_N(y, qz)}{\phi_{N-1}(y/q, z) \phi_N(y, z)}, \tag{3.27}$$

$$1 + \frac{1}{z} f_0\left(z; y, q^N\right) = q^N \left(1 + \frac{1}{z}\right) \frac{\phi_{N-1}(y, qz) \phi_N(y/q, z/q)}{\phi_{N-1}(y/q, z) \phi_N(y, z)}, \tag{3.28}$$

$$1 + \frac{y}{z} f_1\left(z; y, q^N\right) = \left(1 + \frac{y}{z}\right) \frac{\phi_{N-1}(y/q, z) \phi_N(y, z/q)}{\phi_{N-1}(y, z) \phi_N(y/q, z/q)}, \tag{3.29}$$

$$1 + \frac{z}{y} f_1\left(z; y, q^N\right) = \left(1 + \frac{z}{y}\right) \frac{\phi_{N-1}(y, z/q) \phi_N(y/q, z)}{\phi_{N-1}(y, z) \phi_N(y/q, z/q)}. \tag{3.30}$$

In fact, it is easy to check that q-$\mathrm{P}_{\mathrm{III}}$ (3.2) follows from Proposition 1 and equation (3.15). Moreover, Proposition 1 is derived from the bilinear difference equations satisfied by $\phi_N(x, y)$.

Proposition 2. *The function $\phi_N(y, z)$ defined by equations (3.12) and (3.14) satisfies the following bilinear difference equations:*

$$\phi_{N+1}(y, z)\phi_N(y/q, z) + q^{N+1} z \phi_{N+1}(y/q, z)\phi_N(y, z)$$
$$= (1 + z)\phi_{N+1}(y, qz)\phi_N(y/q, z/q), \tag{3.31}$$
$$q^{-N-1} z \phi_{N+1}(y, z)\phi_N(y/q, z) + \phi_{N+1}(y/q, z)\phi_N(y, z)$$
$$= (1 + z)\phi_{N+1}(y/q, z/q)\phi_N(y, qz), \tag{3.32}$$
$$z \phi_{N+1}(y/q, z/q)\phi_N(y, z) + y \phi_{N+1}(y, z)\phi_N(y/q, z/q)$$
$$= (y + z)\,\phi_{N+1}(y, z/q)\phi_N(y/q, z), \tag{3.33}$$
$$y \phi_{N+1}(y/q, z/q)\phi_N(y, z) + z \phi_{N+1}(y, z)\phi_N(y/q, z/q)$$
$$= (y + z)\,\phi_{N+1}(y/q, z)\phi_N(y, z/q). \tag{3.34}$$

Multiplicative formulas (3.27)–(3.30) are derived from equations (3.31)–(3.34), respectively, by shifting N to $N-1$, and dividing the both sides by the first term of each equation. Therefore, we have to prove Proposition 2 in order to establish Theorem 4, which will be given in the next section.

4 Proof of Proposition 2

In the previous section, we have presented Theorem 4 and shown that it is derived from Proposition 2. In this section, we will prove Proposition 2 by reducing the bilinear equations (3.31)–(3.34) to the Plücker relations, namely, quadratic identities among determinants whose columns are properly shifted. However, since equations (3.31)–(3.34) themselves are not directly reduced to the Plücker relations, we will prove the following set of bilinear difference equations instead.

Proposition 3. *The function $\phi_N(y, z)$ defined by equations (3.12) and (3.14) satisfies the following bilinear difference equations:*

$$y \phi_{N+1}(y/q, z)\phi_N(y, z/q) - z \phi_{N+1}(y, z/q)\phi_N(y/q, z)$$
$$= (y - z)\phi_{N+1}(y/q, z/q)\phi_N(y, z), \tag{4.1}$$
$$y \phi_{N+1}(y, z/q)\phi_N(y/q, z) - z \phi_{N+1}(y/q, z)\phi_N(y, z/q)$$
$$= (y - z)\phi_{N+1}(y, z)\phi_N(y/q, z/q), \tag{4.2}$$
$$q^{-N-1} z \phi_{N+1}(y, z)\phi_N(y/q, z) + \phi_{N+1}(y/q, z)\phi_N(y, z)$$
$$= (1 + z)\phi_{N+1}(y/q, z/q)\phi_N(y, qz), \tag{4.3}$$
$$\phi_{N+1}(y/q, z)\phi_{N-1}(y, z) - \phi_{N+1}(y, z)\phi_{N-1}(y/q, z)$$
$$= (1 - q)y/q\,\phi_N(y/q, z)\phi_N(y, z), \tag{4.4}$$
$$\phi_{N+1}(y, z/q)\phi_{N-1}(y, z) - \phi_{N+1}(y, z)\phi_{N-1}(y, z/q)$$
$$= (1 - q)z/q\,\phi_N(y, z/q)\phi_N(y, z), \tag{4.5}$$

$$q^{2N}y\phi_N(y/q,z)\phi_N(y,z) + q^N(1+z)\phi_N(y,qz)\phi_N(y/q,z/q)$$
$$= \frac{1-q^{2N+1}}{1-q}\phi_{N+1}(y,z)\phi_{N-1}(y/q,z). \tag{4.6}$$

In fact, the bilinear difference equations (3.31)–(3.34) in Proposition 2 are derived from equations (4.1)–(4.6) as follows. Equation (3.33) is derived by adding equation (4.1) multiplied by z to equation (4.2) multiplied by y. Similarly, we get equation (3.34) by adding equation (4.1) multiplied by y to equation (4.2) multiplied by z. Equation (4.3) is the same as equation (3.32). Finally, equation (3.31) is derived from equations (4.3)–(4.6) as follows,

$$\begin{aligned}
\phi_{N+1}&(y,qz)\phi_N(y/q,z/q)\\
&= \frac{\phi_{N+1}(y,z)\phi_{N-1}(y,qz) - (1-q)z\phi_N(y,z)\phi_N(y,qz)}{\phi_{N-1}(y,z)} \times \phi_N(y/q,z/q)\\
&= \frac{\phi_{N+1}(y,z)}{\phi_{N-1}(y,z)} \times \left[\frac{q^{-N}z}{1+z}\phi_N(y,z)\phi_{N-1}(y/q,z) + \frac{1}{1+z}\phi_N(y/q,z)\phi_{N-1}(y,z)\right]\\
&\quad - (1-q)z\frac{\phi_N(y,z)\phi_N(y,qz)\phi_N(y/q,z/q)}{\phi_{N-1}(y,z)}\\
&= \frac{\phi_N(y,z)}{\phi_{N-1}(y,z)} \times \frac{q^{-N}z}{1+z} \times [\phi_{N+1}(y,z)\phi_{N-1}(y/q,z)\\
&\quad - (1-q)q^N(1+z)\phi_N(y,qz)\phi_N(y/q,z/q)] + \frac{1}{1+z}\phi_{N+1}(y,z)\phi_N(y/q,z)\\
&= \frac{\phi_N(y,z)}{\phi_{N-1}(y,z)} \times \frac{q^{-N}z}{1+z} \times \left[(1-q)q^{2N}y\phi_N(y/q,z)\phi_N(y,z)\right.\\
&\quad \left. + q^{2N+1}\phi_{N+1}(y,z)\phi_{N-1}(y/q,z)\right] + \frac{1}{1+z}\phi_{N+1}(y,z)\phi_N(y/q,z)\\
&= \frac{\phi_N(y,z)}{\phi_{N-1}(y,z)} \times \frac{q^{N+1}z}{1+z} \times \phi_{N+1}(y/q,z)\phi_{N-1}(y,z) + \frac{1}{1+z}\phi_{N+1}(y,z)\phi_N(y/q,z)\\
&= \frac{1}{1+z}\left[q^{N+1}z\phi_{N+1}(y/q,z)\phi_N(y,z) + \phi_{N+1}(y,z)\phi_N(y/q,z)\right],
\end{aligned}$$

where we have used equation (4.5) with $q \to qz$ in the first equality, equation (4.3) with $N \to N-1$ in the second equality, equation (4.6) in the fourth equality and equation (4.4) in fifth equality, respectively.

Remark 3. The set of equations equations (4.1)–(4.5) is invariant with respect to the transformation,

$$N \mapsto M = -N, \tag{4.7}$$

from the relation $\phi_N = (-1)^{N(N+1)/2}\phi_{-N-1}$. Equation (4.6) itself is not invariant with respect to this transformation, but if we consider the bilinear equation,

$$q^{-2(N+1)}y\phi_N(y/q,z)\phi_N(y,z) + q^{-(N+1)}(1+z)\phi_N(y,qz)\phi_N(y/q,z/q)$$
$$= -\frac{1-q^{-(2N+1)}}{1-q}\phi_{N-1}(y,z)\phi_{N+1}(y/q,z), \tag{4.8}$$

then the set of equations (4.6) and (4.8) is invariant. Equation (4.8) is derived by combining equations (4.4)–(4.6). From this symmetry of bilinear equations, we only have to prove Proposition 3 for $N \in \mathbb{Z}_{\geq 0}$.

Proposition 3 is proved by using a technique of determinants which was used in the previous paper [6]. Namely, we first prepare "difference formulas" which relate $\phi_N(y, z)$ with some determinants whose columns are properly shifted from original $\phi_N(y, z)$. Then choosing suitable Plücker relations, we obtain bilinear difference equations with the aid of the difference formulas. For simpler application of this technique, we refer [22, 23] where it is used to construct solutions for discrete KP and relativistic Toda lattice equations, respectively.

In this section, we demonstrate the derivation of equation (4.1) as an example. Since other bilinear equations in Proposition 3 are derived by using similar technique, we give the complete proof in the appendix.

Let us first introduce a notation,

$$
\phi_N(y, z) = \begin{vmatrix} p_N(y, z) & p_{N+1}(y, z) & \cdots & p_{2N-1}(y, z) \\ \vdots & \vdots & \ddots & \vdots \\ p_{-N+4}(y, z) & p_{-N+5}(y, z) & \cdots & p_3(y, z) \\ p_{-N+2}(y, z) & p_{-N+3}(y, z) & \cdots & p_1(y, z) \end{vmatrix} \tag{4.9}
$$

$$
= \left| -\mathbf{N} + \mathbf{2}_{\substack{y \\ z}}, -\mathbf{N} + \mathbf{3}_{\substack{y \\ z}}, \cdots, \mathbf{0}_{\substack{y \\ z}}, \mathbf{1}_{\substack{y \\ z}} \right|, \tag{4.10}
$$

where the symbol $\mathbf{k}_{\substack{y \\ z}}$ denotes the column vector which ends with $p_k(y, z)$,

$$
\mathbf{k}_{\substack{y \\ z}} = \begin{pmatrix} \vdots \\ p_{k+2}(y, z) \\ p_k(y, z) \end{pmatrix}. \tag{4.11}
$$

We note that the subscripts are used to describe the shift of y and z, and they will be suppressed when there is no shift. Moreover, although the height of the column vectors are N in equation (4.10), we use the same symbol for the determinant with different size. So the height of $\mathbf{k}$ should be read appropriately case by case.

By using this notation, we present a difference formula.

Lemma 1 (Difference Formula I).

$$
\left| -\mathbf{N} + \mathbf{2}, -\mathbf{N} + \mathbf{3}, \ldots, \mathbf{0}, \mathbf{1} \right| = \phi_N(y, z), \tag{4.12}
$$

$$
\left| -\mathbf{N} + \mathbf{2}_{y/q}, -\mathbf{N} + \mathbf{3}, \ldots, \mathbf{0}, \mathbf{1} \right| = \phi_N(y/q, z), \tag{4.13}
$$

$$
\left| -\mathbf{N} + \mathbf{2}_{z/q}, -\mathbf{N} + \mathbf{3}, \ldots, \mathbf{0}, \mathbf{1} \right| = \phi_N(y, z/q), \tag{4.14}
$$

$$
\left| -\mathbf{N} + \mathbf{3}_{y/q}, -\mathbf{N} + \mathbf{3}, \ldots, \mathbf{0}, \mathbf{1} \right| = (1 - q)y/q\,\phi_N(y/q, z), \tag{4.15}
$$

$$
\left| -\mathbf{N} + \mathbf{3}_{z/q}, -\mathbf{N} + \mathbf{3}, \ldots, \mathbf{0}, \mathbf{1} \right| = (1 - q)z/q\,\phi_N(y, z/q), \tag{4.16}
$$

$$
\left| -\mathbf{N} + \mathbf{3}_{z/q}, -\mathbf{N} + \mathbf{3}_{y/q}, -\mathbf{N} + \mathbf{4} \ldots, \mathbf{0}, \mathbf{1} \right| = (1 - q)/q(z - y)\phi_N(y/q, z/q). \tag{4.17}
$$

Proof. Equation (4.12) is nothing but equation (4.10). We next shift $y \to y/q$ in equation (4.10). Then, subtracting $(k-1)$-st column multiplied by $(1-q)y/q$ from k-th column

for $k = N, N-1, \ldots, 2$ and using the contiguity relation (3.26), we have

$$\begin{aligned}
\phi_N(y/q, z) &= |-\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{3}_{y/q}, \ldots, \boldsymbol{0}_{y/q}, \boldsymbol{1}_{y/q} - (1-q)y/q \cdot \boldsymbol{0}_{y/q}| \\
&= |-\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{3}_{y/q}, \ldots, \boldsymbol{0}_{y/q}, \boldsymbol{1}| \\
&= \cdots \\
&= |-\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{0}, \boldsymbol{1}|,
\end{aligned}$$

which is equation (4.13). Moreover, in the last equality, adding the second column to the first column multiplied by $(1-q)y/q$, and using the contiguity relation (3.26), we have

$$\begin{aligned}
(1-q)y/q\,\phi_N(y/q, z) &= |(1-q)y/q \times (-\boldsymbol{N}+\boldsymbol{2}_{y/q}) + (-\boldsymbol{N}+\boldsymbol{3}), -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{0}, \boldsymbol{1}| \\
&= |-\boldsymbol{N}+\boldsymbol{3}_{y/q}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{0}, \boldsymbol{1}|,
\end{aligned}$$

which is equation (4.14). Equations (4.15) and (4.16) are derived by the same procedure by using the contiguity relation (3.25). Finally, shifting $y \to y/q$ in equation (4.16), subtracting $(k-1)$-st column multiplied by $(1-q)y/q$ from k-th column for $k = N, N-1, \ldots, 3$ and using the contiguity relation (3.26), we have

$$\begin{aligned}
(1-q)z/q\,\phi_N(y/q, z/q) &= |-\boldsymbol{N}+\boldsymbol{3}_{\substack{y/q \\ z/q}}, -\boldsymbol{N}+\boldsymbol{3}_{y/q}, -\boldsymbol{N}+\boldsymbol{4}_{y/q}, \ldots, \boldsymbol{0}_{y/q}, \boldsymbol{1}_{y/q}| \\
&= |-\boldsymbol{N}+\boldsymbol{3}_{\substack{y/q \\ z/q}}, -\boldsymbol{N}+\boldsymbol{3}_{y/q}, -\boldsymbol{N}+\boldsymbol{4}, \ldots, \boldsymbol{0}, \boldsymbol{1}|.
\end{aligned}$$

Multiplying the first column by $z - y$ and using the contiguity relation which follows from equations (3.26) and (3.25),

$$z p_k(y, z/q) - y p_k(y/q, z) = (z-y) p_k(y/q, z/q), \tag{4.18}$$

we obtain

$$\begin{aligned}
(1-q)z/q&(z-y)\phi_N(y/q, z/q) \\
&= |(z-y) \times (-\boldsymbol{N}+\boldsymbol{3}_{\substack{y/q \\ z/q}}), -\boldsymbol{N}+\boldsymbol{3}_{y/q}, -\boldsymbol{N}+\boldsymbol{4}, \ldots, \boldsymbol{0}, \boldsymbol{1}| \\
&= |z \times (-\boldsymbol{N}+\boldsymbol{3}_{z/q}) - y \times (-\boldsymbol{N}+\boldsymbol{3}_{y/q}), -\boldsymbol{N}+\boldsymbol{3}_{y/q}, -\boldsymbol{N}+\boldsymbol{4}, \ldots, \boldsymbol{0}, \boldsymbol{1}| \\
&= z|-\boldsymbol{N}+\boldsymbol{3}_{z/q}, -\boldsymbol{N}+\boldsymbol{3}_{y/q}, -\boldsymbol{N}+\boldsymbol{4}, \ldots, \boldsymbol{0}, \boldsymbol{1}|,
\end{aligned}$$

which is equation (4.17). This completes the proof of Lemma 1. ∎

Consider the Plücker relation,

$$\begin{aligned}
0 = {} & |\boldsymbol{\varphi}, -\boldsymbol{N}+\boldsymbol{2}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{1}| \times |-\boldsymbol{N}+\boldsymbol{2}_{z/q}, -\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{1}| \quad (4.19) \\
& - |-\boldsymbol{N}+\boldsymbol{2}_{z/q}, -\boldsymbol{N}+\boldsymbol{2}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{1}| \times |\boldsymbol{\varphi}, -\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{1}| \\
& + |-\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{2}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{1}| \times |\boldsymbol{\varphi}, -\boldsymbol{N}+\boldsymbol{2}_{z/q}, -\boldsymbol{N}+\boldsymbol{3}, \ldots, \boldsymbol{1}|,
\end{aligned}$$

where φ is the column vector,

$$\varphi = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{4.20}$$

After expanding the determinants according to the column φ, we apply Lemma 1 to equation (4.19). Then we obtain,

$$
\begin{aligned}
0 = {} & \phi_N(y,z) \times (1-q)/q(z-y)\phi_{N+1}(y/q, z/q) \\
& - (1-q)z/q\phi_{N+1}(y/q, z) \times \phi_N(y/q, z) + (1-q)y/q\phi_{N+1}(y, z/q) \times \phi_N(y, z/q),
\end{aligned}
$$

which yields equation (4.1). For the derivation of equations (4.2)–(4.6), see appendix.

5   Concluding remarks

In this and previous [6] papers, we have considered q-P$_{\mathrm{III}}$ and shown that it has the following properties:

- q-P$_{\mathrm{III}}$ is derived from the (generalization of) discrete-time relativistic Toda lattice.

- q-P$_{\mathrm{III}}$ and q-P$_{\mathrm{IV}}$ are realized as the dynamical systems on the root lattice of type $A_1^{(1)} \times A_2^{(1)}$. The former equation describes the Bäcklund (Schlesinger) transformation of the latter, and vice versa.

- q-P$_{\mathrm{III}}$ admits symmetry of affine Weyl group of type $A_1^{(1)} \times A_1^{(1)}$ as the group of Bäcklund transformations.

- q-P$_{\mathrm{III}}$ admits two classes of Riccati type solutions. One class consists of such solutions that are expressed by Jackson's q-modified Bessel functions. These solutions are also the solutions for q-P$_{\mathrm{IV}}$. Another class consists of q-P$_{\mathrm{III}}$ specific solutions. Those classes of solutions admit determinant formula of Hankel or Toeplitz type.

- q-P$_{\mathrm{III}}$ admits rational solutions which are expressed by ratio of some subtraction-free special polynomials. Those special polynomials admit determinant formula of Jacobi–Trudi type associated with two-core partition.

It might be an interesting problem to find connections or applications of q-P$_{\mathrm{III}}$ to other fields of mathematical and physical sciences, such as q-orthogonal polynomials, matrix integrations or discrete geometry. Moreover, there are many discrete Painlevé equations to be studied in Sakai's classification [27]. It might be an important problem to consider their solutions, in particular, solutions for the equations with larger symmetries than q-P$_{\mathrm{VI}}$, which are expected to be beyond the hypergeometric world [26].

Acknowledgements

A   Proof of bilinear equations

In this appendix, we give the data which are necessary for proving the bilinear equations in Proposition 3. Following the procedure mentioned in Section 4, we first give the difference

formulas. We introduce the following notations of column vectors in order to describe them:

$$[\boldsymbol{k}]_y = \begin{pmatrix} \vdots \\ p_5\left(y, q^k z\right) \\ p_3\left(y, q^k z\right) \\ p_1\left(y, q^k z\right) \end{pmatrix}, \qquad [\boldsymbol{k'}]_y = \begin{pmatrix} \vdots \\ q^5 p_5\left(y, q^k z\right) \\ q^3 p_3\left(y, q^k z\right) \\ q p_1\left(y, q^k z\right) \end{pmatrix}, \tag{A.1}$$

$$[\overline{\boldsymbol{k}}]_y = \begin{pmatrix} \vdots \\ p_4\left(y, q^k z\right) \\ p_2\left(y, q^k z\right) \\ p_0\left(y, q^k z\right) \end{pmatrix}, \qquad [\overline{\boldsymbol{k}}']_y = \begin{pmatrix} \vdots \\ q^4 \frac{1-q}{1-q^3} p_4\left(y, q^k z\right) \\ q^2 \frac{1-q}{1-q^2} p_2\left(y, q^k z\right) \\ q^0 \frac{1-q}{1-q^1} p_0\left(y, q^k z\right) \end{pmatrix}. \tag{A.2}$$

Note that the subscript y is used for describing the shift of y and it will be suppressed when there is no shift. Moreover, although heights of the column vectors are not specified in equations (A.1) and (A.2), they should be read appropriately case by case. For example, the symbols $\|[\boldsymbol{N}-\boldsymbol{1}], [\boldsymbol{N}-\boldsymbol{2}], \ldots, [\boldsymbol{1}], [\boldsymbol{0}]\|$ and $\|[\overline{\boldsymbol{N}}], [\overline{\boldsymbol{N}-\boldsymbol{1}}], \ldots, [\overline{\boldsymbol{1}}], [\overline{\boldsymbol{0}}]\|$ denote the following determinants,

$$\|[\boldsymbol{N}-\boldsymbol{1}], [\boldsymbol{N}-\boldsymbol{2}], \ldots, [\boldsymbol{1}], [\boldsymbol{0}]\|$$
$$= \begin{vmatrix} p_{2N-1}\left(y, q^{N-1}z\right) & p_{2N-1}\left(y, q^{N-2}z\right) & \cdots & p_{2N-1}(y, qz) & p_{2N-1}(y, z) \\ p_{2N-3}\left(y, q^{N-1}z\right) & p_{2N-3}\left(y, q^{N-2}z\right) & \cdots & p_{2N-3}(y, qz) & p_{2N-3}(y, z) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_3\left(y, q^{N-1}z\right) & p_3\left(y, q^{N-2}z\right) & \cdots & p_3(y, qz) & p_3(y, z) \\ p_1\left(y, q^{N-1}z\right) & p_1\left(y, q^{N-2}z\right) & \cdots & p_1(y, qz) & p_1(y, z) \end{vmatrix},$$

$$\|[\overline{\boldsymbol{N}}], [\overline{\boldsymbol{N}-\boldsymbol{1}}], \ldots, [\overline{\boldsymbol{1}}], [\overline{\boldsymbol{0}}]\|$$
$$= \begin{vmatrix} p_{2N}\left(y, q^N z\right) & p_{2N}\left(y, q^{N-1}z\right) & \cdots & p_{2N}(y, qz) & p_{2N}(y, z) \\ p_{2N-2}\left(y, q^N z\right) & p_{2N-2}\left(y, q^{N-1}z\right) & \cdots & p_{2N-2}(y, qz) & p_{2N-2}(y, z) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_2\left(y, q^N z\right) & p_2\left(y, q^{N-1}z\right) & \cdots & p_2(y, qz) & p_2(y, z) \\ p_0\left(y, q^N z\right) & p_0\left(y, q^{N-1}z\right) & \cdots & p_0(y, qz) & p_0(y, z) \end{vmatrix},$$

respectively.

By using these notations, we present difference formulas.

Lemma 2 (Difference Formula II).

$$\|[\boldsymbol{N}-\boldsymbol{1}], [\boldsymbol{N}-\boldsymbol{2}], \ldots, [\boldsymbol{1}], [\boldsymbol{0}]\|$$
$$= (q-1)^{\frac{N(N-1)}{2}} z^{\frac{N(N-1)}{2}} q^{\frac{(N-2)(N-1)N}{6}} \phi_N(y, z), \tag{A.3}$$

$$\left\|[\boldsymbol{N}-\boldsymbol{1}], \ldots, [\boldsymbol{2}], [\boldsymbol{1}], [\boldsymbol{0'}]_{y/q}\right\|$$
$$= (-1)^{N-1}(q-1)^{\frac{N(N-1)}{2}} z^{\frac{(N-1)(N-2)}{2}} q^{\frac{N(N^2+5)}{6}} \phi_N(y/q, z), \tag{A.4}$$

$$\left\|[\boldsymbol{N}-\boldsymbol{1}], \ldots, [\boldsymbol{2}], [\boldsymbol{1}], [\boldsymbol{1'}]_{y/q}\right\|$$
$$= (-1)^{N-1}(q-1)^{\frac{N(N-1)}{2}} z^{\frac{(N-1)(N-2)}{2}} q^{\frac{N(N^2+5)}{6}} (1+qz)\phi_N(y/q, z). \tag{A.5}$$

Lemma 3 (Difference Formula III).

$$\left|[\overline{\boldsymbol{N}}],[\overline{\boldsymbol{N}-\boldsymbol{1}}],\ldots,[\overline{\boldsymbol{1}}],[\overline{\boldsymbol{0}}]\right| = \{(q-1)z\}^{\frac{N(N+1)}{2}}\, q^{\frac{(N-1)N(N+1)}{6}}\phi_N(y,z), \tag{A.6}$$

$$\left|[\overline{\boldsymbol{N}}],[\overline{\boldsymbol{N}-\boldsymbol{1}}],\ldots,[\overline{\boldsymbol{1}}],[\overline{\boldsymbol{0}}'_{y/q}]\right|$$

$$= q^{\frac{N(N+1)(N+5)}{6}}y^N\{(q-1)z\}^{\frac{N(N+1)}{2}}\prod_{j=1}^{N+1}\frac{1-q}{1-q^{2j-1}}\phi_N(y/q,z), \tag{A.7}$$

$$\left|[\overline{\boldsymbol{N}}],[\overline{\boldsymbol{N}-\boldsymbol{1}}],\cdots,[\overline{\boldsymbol{1}}],[\overline{\boldsymbol{1}}'_{y/q}]\right|$$

$$= -q^{\frac{N(N+1)(N+5)}{6}}y^{N-1}\{(q-1)z\}^{\frac{N(N+1)}{2}}(1+qz)\times\prod_{j=1}^{N+1}\frac{1-q}{1-q^{2j-1}}\phi_N(y/q,z). \tag{A.8}$$

We prove Lemmas 2 and 3 in the appendix B.

We next give the list of the Plücker relations and difference formulas which are necessary for the derivation of bilinear equations (4.1)–(4.5). In the following, the symbol φ denotes the column vector defined in equation (4.20).

1. Equation (4.1)

Plücker relation

$$0 = |\varphi, -\boldsymbol{N}+\boldsymbol{2}, -\boldsymbol{N}+\boldsymbol{3},\ldots,\boldsymbol{1}|\times\left|-\boldsymbol{N}+\boldsymbol{2}_{z/q}, -\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{3},\ldots,\boldsymbol{1}\right|$$
$$-\left|-\boldsymbol{N}+\boldsymbol{2}_{z/q}, -\boldsymbol{N}+\boldsymbol{2}, -\boldsymbol{N}+\boldsymbol{3},\ldots,\boldsymbol{1}\right|\times\left|\varphi, -\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{3},\ldots,\boldsymbol{1}\right|$$
$$+\left|-\boldsymbol{N}+\boldsymbol{2}_{y/q}, -\boldsymbol{N}+\boldsymbol{2}, -\boldsymbol{N}+\boldsymbol{3},\ldots,\boldsymbol{1}\right|\times\left|\varphi, -\boldsymbol{N}+\boldsymbol{2}_{z/q}, -\boldsymbol{N}+\boldsymbol{3},\ldots,\boldsymbol{1}\right|. \tag{A.9}$$

Difference formula Lemma 1.

2. Equation (4.2)

Plücker relation

$$0 = |-\boldsymbol{N}+\boldsymbol{1}, -\boldsymbol{N}+\boldsymbol{2},\ldots,\boldsymbol{0},\boldsymbol{1}|\times\left|-\boldsymbol{N}+\boldsymbol{1}_{z/q}, -\boldsymbol{N}+\boldsymbol{1}_{y/q}, -\boldsymbol{N}+\boldsymbol{2},\ldots,\boldsymbol{0}\right|$$
$$-\left|-\boldsymbol{N}+\boldsymbol{1}_{z/q}, -\boldsymbol{N}+\boldsymbol{1}, -\boldsymbol{N}+\boldsymbol{2},\ldots,\boldsymbol{0}\right|\times\left|-\boldsymbol{N}+\boldsymbol{1}_{y/q}, -\boldsymbol{N}+\boldsymbol{2},\ldots,\boldsymbol{0},\boldsymbol{1}\right|$$
$$+\left|-\boldsymbol{N}+\boldsymbol{1}_{y/q}, -\boldsymbol{N}+\boldsymbol{1}, -\boldsymbol{N}+\boldsymbol{2},\ldots,\boldsymbol{0}\right|\times\left|-\boldsymbol{N}+\boldsymbol{1}_{z/q}, -\boldsymbol{N}+\boldsymbol{2},\ldots,\boldsymbol{0},\boldsymbol{1}\right|. \tag{A.10}$$

Difference formula Lemma 1.

Remark. A relation

$$\phi_N(y,z) = \begin{vmatrix} p_N(y,z) & p_{N+1}(y,z) & \cdots & p_{2N-1}(y,z) \\ p_{N-2}(y,z) & p_{N-1}(y,z) & \cdots & p_{2N-3}(y,z) \\ \vdots & \cdots & \ddots & \vdots \\ p_{-N+2}(y,z) & \cdots & p_0(y,z) & p_1(y,z) \end{vmatrix}$$

$$= \begin{vmatrix} p_N(y,z) & p_{N+1}(y,z) & \cdots & p_{2N-1}(y,z) & p_{2N}(y,z) \\ p_{N-2}(y,z) & p_{N-1}(y,z) & \cdots & p_{2N-3}(y,z) & p_{2N-2}(y,z) \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ p_{-N+2}(y,z) & \cdots & p_0(y,z) & p_1(y,z) & p_2(y,z) \\ p_{-N}(y,z) & \cdots & p_{-2}(y,z) & p_{-1}(y,z) & p_0(y,z) \end{vmatrix} \tag{A.11}$$

$$= |-\boldsymbol{N}, -\boldsymbol{N}+\boldsymbol{1},\ldots,\boldsymbol{0}|. \tag{A.12}$$

is used for the derivation of equation (4.2).

3. Equation (4.3)

Plücker relation

$$0 = \left|[\boldsymbol{N}], [\boldsymbol{N-1}], \ldots, [\boldsymbol{1}], [\boldsymbol{0'}]_{y/q}\right| \times \left|[\boldsymbol{N-1}], \ldots, [\boldsymbol{1}], [\boldsymbol{0}], \boldsymbol{\varphi}\right|$$
$$- \left|[\boldsymbol{N}], [\boldsymbol{N-1}], \ldots, [\boldsymbol{1}], \boldsymbol{\varphi}\right| \times \left|[\boldsymbol{N-1}], \ldots, [\boldsymbol{1}], [\boldsymbol{0}], [\boldsymbol{0'}]_{y/q}\right|$$
$$- \left|[\boldsymbol{N-1}], \ldots, [\boldsymbol{1}], [\boldsymbol{0'}]_{y/q}, \boldsymbol{\varphi}\right| \times \left|[\boldsymbol{N}], [\boldsymbol{N-1}], \ldots, [\boldsymbol{1}], [\boldsymbol{0}]\right|. \tag{A.13}$$

Difference formula Lemma 2.

4. Equation (4.4)

Plücker relation

$$0 = \left|\boldsymbol{-N+1}_{y/q}, \boldsymbol{-N+1}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}\right| \times \left|\boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{1}, \boldsymbol{\varphi}\right|$$
$$+ \left|\boldsymbol{-N+1}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{1}\right| \times \left|\boldsymbol{-N+1}_{y/q}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{\varphi}\right|$$
$$- \left|\boldsymbol{-N+1}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{\varphi}\right| \times \left|\boldsymbol{-N+1}_{y/q}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{1}\right|. \tag{A.14}$$

Difference formula Lemma 1 with equation (A.12).

5. Equation (4.5)

Plücker relation

$$0 = \left|\boldsymbol{-N+1}_{z/q}, \boldsymbol{-N+1}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}\right| \times \left|\boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{1}, \boldsymbol{\varphi}\right|$$
$$+ \left|\boldsymbol{-N+1}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{1}\right| \times \left|\boldsymbol{-N+1}_{z/q}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{\varphi}\right|$$
$$- \left|\boldsymbol{-N+1}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{\varphi}\right| \times \left|\boldsymbol{-N+1}_{z/q}, \boldsymbol{-N+2}, \ldots, \boldsymbol{0}, \boldsymbol{1}\right|. \tag{A.15}$$

Difference formula Lemma 1 with equation (A.12).

6. Equation (4.6)

Plücker relation

$$0 = \left|[\overline{\boldsymbol{N}}], [\overline{\boldsymbol{N-1}}], \ldots, [\overline{\boldsymbol{1}}], [\overline{\boldsymbol{0}}]\right| \times \left|[\overline{\boldsymbol{N-1}}], \ldots, [\overline{\boldsymbol{1}}], [\overline{\boldsymbol{0}}'_{y/q}], \boldsymbol{\varphi}\right|$$
$$- \left|[\overline{\boldsymbol{N}}], [\overline{\boldsymbol{N-1}}], \ldots, [\overline{\boldsymbol{1}}], [\overline{\boldsymbol{0}}'_{y/q}]\right| \times \left|[\overline{\boldsymbol{N-1}}], \ldots, [\overline{\boldsymbol{1}}], [\overline{\boldsymbol{0}}], \boldsymbol{\varphi}\right|$$
$$+ \left|[\overline{\boldsymbol{N}}], [\overline{\boldsymbol{N-1}}], \ldots, [\overline{\boldsymbol{1}}], \boldsymbol{\varphi}\right| \times \left|[\overline{\boldsymbol{N-1}}], \ldots, [\overline{\boldsymbol{1}}], [\overline{\boldsymbol{0}}], [\overline{\boldsymbol{0}}'_{y/q}]\right|. \tag{A.16}$$

Difference formula Lemma 3.

B   Proof of Lemmas 2 and 3

In this appendix, we give the proofs of difference formulas Lemmas 2 and 3.

We first rewrite the Jacobi–Trudi type determinant expression for $\phi_N(y, z)$ (4.9) in terms of the Casorati determinant in z as follows.

Lemma 4.

$$\phi_N(y, z) = \{(q-1)z\}^{-N(N-1)/2} q^{-(N-2)(N-1)N/6}$$
$$\times \begin{vmatrix} p_{2N-1}\left(y, q^{N-1}z\right) & p_{2N-1}\left(y, q^{N-2}z\right) & \cdots & p_{2N-1}(y, qz) & p_{2N-1}(y, z) \\ p_{2N-3}\left(y, q^{N-1}z\right) & p_{2N-3}\left(y, q^{N-2}z\right) & \cdots & p_{2N-3}(y, qz) & p_{2N-3}(y, z) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_1\left(y, q^{N-1}z\right) & p_1\left(y, q^{N-2}z\right) & \cdots & p_1(y, qz) & p_1(y, z) \end{vmatrix}. \tag{B.1}$$

Proof. In the right hand side of equation (4.9), adding $(k+1)$-st column to k-th column multiplied by $(q-1)z$ for $k = 1, \ldots, N-1$ and using the contiguity relation (3.25), we have

$$\phi_N(y, z) = \{(q-1)z\}^{-(N-1)}$$
$$\times \begin{vmatrix} p_{N+1}(y, qz) & p_{N+2}(y, qz) & \cdots & p_{2N-1}(y, qz) & p_{2N-1}(y, z) \\ p_{N-1}(y, qz) & p_N(y, qz) & \cdots & p_{2N-3}(y, qz) & p_{2N-3}(y, z) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_{-N+3}(y, qz) & p_{-N+4}(y, qz) & \cdots & p_1(y, qz) & p_1(y, z) \end{vmatrix}.$$

Moreover, adding $(k+1)$-st column to k-th column multiplied by $(q-1)qz$ for $k = 1, \ldots, N-2$, and continuing this procedure, we finally obtain,

$$\phi_N(y, z) = \{(q-1)z\}^{-(N-1)} \times \{(q-1)qz\}^{-(N-2)} \times \cdots \times \{(q-1)q^{N-2}z\}^{-1}$$
$$\times \begin{vmatrix} p_{2N-1}\left(y, q^{N-1}z\right) & p_{2N-1}\left(y, q^{N-2}z\right) & \cdots & p_{2N-1}(y, qz) & p_{2N-1}(y, z) \\ p_{2N-3}\left(y, q^{N-1}z\right) & p_{2N-3}\left(y, q^{N-2}z\right) & \cdots & p_{2N-3}(y, qz) & p_{2N-3}(y, z) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_1\left(y, q^{N-1}z\right) & p_1\left(y, q^{N-2}z\right) & \cdots & p_1(y, qz) & p_1(y, z) \end{vmatrix},$$

which is desired result.   ∎

We put

$$\tilde{\phi}_N(y, z) = \begin{vmatrix} p_{2N-1}\left(y, q^{N-1}z\right) & p_{2N-1}\left(y, q^{N-2}z\right) & \cdots & p_{2N-1}(y, qz) & p_{2N-1}(y, z) \\ p_{2N-3}\left(y, q^{N-1}z\right) & p_{2N-3}\left(y, q^{N-2}z\right) & \cdots & p_{2N-3}(y, qz) & p_{2N-3}(y, z) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_1\left(y, q^{N-1}z\right) & p_1\left(y, q^{N-2}z\right) & \cdots & p_1(y, qz) & p_1(y, z) \end{vmatrix}$$
$$= \{(q-1)z\}^{N(N-1)/2} q^{(N-2)(N-1)N/6} \phi_N(y, z). \tag{B.2}$$

Proof of Lemma 2. Equation (B.1) is nothing but equation (A.3). In order to prove equations (A.4) and (A.5), we use the contiguity relation,

$$q^{-k} z p_k(y, z) + p_k(y/q, z) = (1+z) p_k(y/q, z/q), \tag{B.3}$$

which follows from equations (3.24)–(3.26). Using equation (B.3) for the first column of $\tilde{\phi}_N(y/q, z)$, we have

$$\tilde{\phi}_N(y/q, z) = \begin{vmatrix} p_{2N-1}\left(y/q, q^{N-1}z\right) & p_{2N-1}\left(y/q, q^{N-2}z\right) & \cdots & p_{2N-1}(y/q, z) \\ p_{2N-3}\left(y/q, q^{N-1}z\right) & p_{2N-3}\left(y/q, q^{N-2}z\right) & \cdots & p_{2N-3}(y/q, z) \\ \vdots & \vdots & \cdots & \vdots \\ p_1\left(y/q, q^{N-1}z\right) & p_1\left(y/q, q^{N-2}z\right) & \cdots & p_1(y/q, z) \end{vmatrix}$$

$$= \begin{vmatrix} -q^{-2N+1}\left(q^{N-1}z\right) p_{2N-1}\left(y, q^{N-1}z\right) + \left(1+q^{N-1}z\right) p_{2N-1}\left(y/q, q^{N-2}z\right) & \cdots \\ -q^{-2N+3}\left(q^{N-1}z\right) p_{2N-3}\left(y, q^{N-1}z\right) + \left(1+q^{N-1}z\right) p_{2N-3}\left(y/q, q^{N-2}z\right) & \cdots \\ \vdots & \cdots \\ -q^{-1}\left(q^{N-1}z\right) p_1\left(y, q^{N-1}z\right) + \left(1+q^{N-1}z\right) p_1\left(y/q, q^{N-2}z\right) & \cdots \end{vmatrix}$$

$$= -q^{N-1}z \begin{vmatrix} q^{-2N+1}p_{2N-1}\left(y,q^{N-1}z\right) & p_{2N-1}\left(y/q,q^{N-2}z\right) & \cdots & p_{2N-1}(y/q,z) \\ q^{-2N+3}p_{2N-3}\left(y,q^{N-1}z\right) & p_{2N-3}\left(y/q,q^{N-2}z\right) & \cdots & p_{2N-3}(y/q,z) \\ \vdots & \vdots & \ddots & \vdots \\ q^{-1}p_1\left(y,q^{N-1}z\right) & p_1\left(y/q,q^{N-2}z\right) & \cdots & p_1(y/q,z) \end{vmatrix}.$$

Continuing this procedure from the second column to the $(N-1)$-st column, we have,

$$\tilde{\phi}_N(y/q,z) = (-q^{N-1}z)\cdots(-qz)$$

$$\times \begin{vmatrix} q^{-2N+1}p_{2N-1}\left(y,q^{N-1}z\right) & \cdots & q^{-2N+1}p_{2N-1}(y,qz) & p_{2N-1}(y/q,z) \\ q^{-2N+3}p_{2N-3}\left(y,q^{N-1}z\right) & \cdots & q^{-2N+3}p_{2N-3}(y,qz) & p_{2N-3}(y/q,z) \\ \vdots & \vdots & \ddots & \vdots \\ q^{-1}p_1\left(y,q^{N-1}z\right) & \cdots & q^{-1}p_1(y,qz) & p_1(y/q,z) \end{vmatrix} \tag{B.4}$$

$$= (-q^{N-1}z)\cdots(-qz)q^{-N^2}$$

$$\times \begin{vmatrix} p_{2N-1}\left(y,q^{N-1}z\right) & \cdots & p_{2N-1}(y,qz) & q^{2N-1}p_{2N-1}(y/q,z) \\ p_{2N-3}\left(y,q^{N-1}z\right) & \cdots & p_{2N-3}(y,qz) & q^{2N-3}p_{2N-3}(y/q,z) \\ \vdots & \vdots & \ddots & \vdots \\ p_1\left(y,q^{N-1}z\right) & \cdots & p_1(y,qz) & qp_1(y/q,z) \end{vmatrix},$$

which yields equation (A.4) by noticing equation (B.2). In equation (B.4), multiplying the N-th column by $(1+qz)$ and using equation (B.3), we have

$$\tilde{\phi}_N(y/q,z) = \left(-q^{N-1}z\right)\cdots(-qz)\frac{1}{1+qz}$$

$$\times \begin{vmatrix} q^{-2N+1}p_{2N-1}\left(y,q^{N-1}z\right) & \cdots & q^{-2N+1}p_{2N-1}(y,qz) & (1+qz)p_{2N-1}(y/q,z) \\ q^{-2N+3}p_{2N-3}\left(y,q^{N-1}z\right) & \cdots & q^{-2N+3}p_{2N-3}(y,qz) & (1+qz)p_{2N-3}(y/q,z) \\ \vdots & \vdots & \ddots & \vdots \\ q^{-1}p_1\left(y,q^{N-1}z\right) & \cdots & q^{-1}p_1(y,qz) & (1+qz)p_1(y/q,z) \end{vmatrix}$$

$$= \left(-q^{N-1}z\right)\cdots(-qz)\frac{1}{1+qz}$$

$$\times \begin{vmatrix} \cdots & q^{-2N+1}p_{2N-1}(y,qz) & q^{-2N+1}(qz)p_{2N-1}(y,qz) + p_{2N-1}(y/q,qz) \\ \cdots & q^{-2N+3}p_{2N-3}(y,qz) & q^{-2N+3}(qz)p_{2N-3}(y,qz) + p_{2N-3}(y/q,qz) \\ \vdots & \ddots & \vdots \\ \cdots & q^{-1}p_1(y,qz) & q^{-1}(qz)p_1(y,qz) + p_1(y/q,qz) \end{vmatrix}$$

$$= \left(-q^{N-1}z\right)\cdots(-qz)\frac{1}{1+qz}\times \begin{vmatrix} \cdots & q^{-2N+1}p_{2N-1}(y,qz) & p_{2N-1}(y/q,qz) \\ \cdots & q^{-2N+3}p_{2N-3}(y,qz) & p_{2N-3}(y/q,qz) \\ \vdots & \ddots & \vdots \\ \cdots & q^{-1}p_1(y,qz) & p_1(y/q,qz) \end{vmatrix}$$

$$= \left(-q^{N-1}z\right)\cdots(-qz)\frac{1}{1+qz}\,q^{-N^2}$$

$$\times \begin{vmatrix} p_{2N-1}\left(y,q^{N-1}z\right) & \cdots & p_{2N-1}(y,qz) & q^{2N-1}p_{2N-1}(y/q,qz) \\ p_{2N-3}\left(y,q^{N-1}z\right) & \cdots & p_{2N-3}(y,qz) & q^{2N-3}p_{2N-3}(y/q,qz) \\ \vdots & \vdots & \ddots & \vdots \\ p_1\left(y,q^{N-1}z\right) & \cdots & p_1(y,qz) & qp_1(y/q,qz) \end{vmatrix},$$

which yields equation (A.5). This completes the proof of Lemma 2. ∎

We finally prove Lemma 3. Starting from equation (A.11), we rewrite it in terms of Casorati determinant in z. Since this is done by using the same technique as that was used in the proof of Lemma 4, we here give only the result.

Lemma 5.

$$
\phi_N(y,z) = \{(q-1)z\}^{-N(N+1)/2}\, q^{-(N-1)N(N+1)/6}
$$

$$
\times \begin{vmatrix}
p_{2N}\left(y, q^N z\right) & p_{2N}\left(y, q^{N-1}z\right) & \cdots & p_{2N}(y,qz) & p_{2N}(y,z) \\
p_{2N-2}\left(y, q^N z\right) & p_{2N-2}\left(y, q^{N-1}z\right) & \cdots & p_{2N-2}(y,qz) & p_{2N-2}(y,z) \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
p_{2}\left(y, q^N z\right) & p_{2}\left(y, q^{N-1}z\right) & \cdots & p_{2}(y,qz) & p_{2}(y,z) \\
p_{0}\left(y, q^N z\right) & p_{0}\left(y, q^{N-1}z\right) & \cdots & p_{0}(y,qz) & p_{0}(y,z)
\end{vmatrix}. \tag{B.5}
$$

We put

$$
\overline{\phi}_N(y,z) = \begin{vmatrix}
p_{2N}\left(y, q^N z\right) & p_{2N}\left(y, q^{N-1}z\right) & \cdots & p_{2N}(y,qz) & p_{2N}(y,z) \\
p_{2N-2}\left(y, q^N z\right) & p_{2N-2}\left(y, q^{N-1}z\right) & \cdots & p_{2N-2}(y,qz) & p_{2N-2}(y,z) \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
p_{2}\left(y, q^N z\right) & p_{2}\left(y, q^{N-1}z\right) & \cdots & p_{2}(y,qz) & p_{2}(y,z) \\
p_{0}\left(y, q^N z\right) & p_{0}\left(y, q^{N-1}z\right) & \cdots & p_{0}(y,qz) & p_{0}(y,z)
\end{vmatrix} \tag{B.6}
$$

$$
= \{(q-1)z\}^{N(N+1)/2}\, q^{(N-1)N(N+1)/6}\phi_N(y,z). \tag{B.7}
$$

Proof of Lemma 3. Equation (B.5) is nothing but equation (A.6). In order to prove equations (A.7) and (A.8), we use the contiguity relation,

$$
q^{-k}\frac{1-q^{k+1}}{1-q}p_{k+1}(qy,z) = (1+z)p_k(y,z/q) + qy\,p_k(y,z), \tag{B.8}
$$

which follows from equations (3.24)–(3.26). In the right hand side of equation (B.6), we add the second column multiplied by $\left(1+q^N z\right)$ to the first column multiplied by qy. Using equation (B.8), we have

$$
\overline{\phi}_N(y,z) = \begin{vmatrix}
p_{2N}\left(y, q^N z\right) & p_{2N}\left(y, q^{N-1}z\right) & \cdots & p_{2N}(y,qz) & p_{2N}(y,z) \\
p_{2N-2}\left(y, q^N z\right) & p_{2N-2}\left(y, q^{N-1}z\right) & \cdots & p_{2N-2}(y,qz) & p_{2N-2}(y,z) \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
p_{2}\left(y, q^N z\right) & p_{2}\left(y, q^{N-1}z\right) & \cdots & p_{2}(y,qz) & p_{2}(y,z) \\
p_{0}\left(y, q^N z\right) & p_{0}\left(y, q^{N-1}z\right) & \cdots & p_{0}(y,qz) & p_{0}(y,z)
\end{vmatrix}
$$

$$
= \frac{1}{qy}\begin{vmatrix}
qy\,p_{2N}\left(y, q^N z\right) + \left(1+q^N z\right)p_{2N}\left(y, q^{N-1}z\right) & p_{2N}\left(y, q^{N-1}z\right) & \cdots \\
qy\,p_{2N-2}\left(y, q^N z\right) + \left(1+q^N z\right)p_{2N-2}\left(y, q^{N-1}z\right) & p_{2N-2}\left(y, q^{N-1}z\right) & \cdots \\
\vdots & \vdots & \cdots \\
qy\,p_{0}\left(y, q^N z\right) + \left(1+q^N z\right)p_{0}\left(y, q^{N-1}z\right) & p_{0}\left(y, q^{N-1}z\right) & \cdots
\end{vmatrix}
$$

$$
= \frac{1}{qy}\begin{vmatrix}
q^{-2N}\frac{1-q^{2N+1}}{1-q}p_{2N+1}\left(qy, q^N z\right) & p_{2N}\left(y, q^{N-1}z\right) & \cdots \\
q^{-2N+2}\frac{1-q^{2N-1}}{1-q}p_{2N-1}\left(qy, q^N z\right) & p_{2N-2}\left(y, q^{N-1}z\right) & \cdots \\
\vdots & \vdots & \cdots \\
q^{0}\frac{1-q^{1}}{1-q}p_{1}\left(qy, q^N z\right) & p_{0}\left(y, q^{N-1}z\right) & \cdots
\end{vmatrix}.
$$

Continuing this procedure from the second to the *N*-th columns, we obtain

$$
\overline{\phi}_N(y,z) = \left(\frac{1}{qy}\right)^N \times \begin{vmatrix} \cdots & q^{-2N}\frac{1-q^{2N+1}}{1-q}\,p_{2N+1}(qy,qz) & p_{2N}(y,z) \\ \cdots & q^{-2N+2}\frac{1-q^{2N-1}}{1-q}\,p_{2N-1}(qy,qz) & p_{2N-2}(y,z) \\ \vdots & \vdots & \cdots \\ \cdots & q^0\frac{1-q^1}{1-q}\,p_1(qy,qz) & p_0(y,z) \end{vmatrix} \tag{B.9}
$$

$$
= \left(\frac{1}{qy}\right)^N \prod_{j=1}^{N+1}\frac{q^{-2j}(1-q^{2j-1})}{1-q}
$$

$$
\times \begin{vmatrix} p_{2N+1}\left(qy,q^N z\right) & \cdots & p_{2N+1}(qy,qz) & q^{2N}\frac{1-q}{1-q^{2N+1}}\,p_{2N}(y,z) \\ p_{2N-1}\left(qy,q^N z\right) & \cdots & p_{2N-1}(qy,qz) & q^{2N-2}\frac{1-q}{1-q^{2N-1}}\,p_{2N-2}(y,z) \\ \vdots & \vdots & \cdots & \\ p_1\left(qy,q^N z\right) & \cdots & p_1(qy,qz) & q^0\frac{1-q}{1-q^1}\,p_0(y,z) \end{vmatrix}.
$$

which yields equation (A.7) by noticing equation (B.7). Furthermore, in equation (B.9), adding the $(N-1)$-st column to the *N*-th column multiplied by $-(1+qz)$ and using equation (B.8), we have

$$
\overline{\phi}_N(y,z) = \left(\frac{1}{qy}\right)^N \frac{-1}{1+qz}
$$

$$
\times \begin{vmatrix} \cdots & q^{-2N}\frac{1-q^{2N+1}}{1-q}p_{2N+1}(qy,qz) & qyp_{2N}(y,z) \\ \cdots & q^{-2N+2}\frac{1-q^{2N-1}}{1-q}p_{2N-1}(qy,qz) & qyp_{2N-2}(y,z) \\ \vdots & \vdots & \cdots \\ \cdots & q^0\frac{1-q^1}{1-q}p_1(qy,qz) & qyp_0(y,z) \end{vmatrix}
$$

$$
= \left(\frac{1}{qy}\right)^{N-1} \frac{-1}{1+qz}\prod_{j=1}^{N+1}\frac{q^{-2(j-1)}(1-q^{2j-1})}{1-q}
$$

$$
\times \begin{vmatrix} p_{2N+1}\left(qy,q^N z\right) & \cdots & p_{2N+1}(qy,qz) & q^{2N}\frac{1-q}{1-q^{2N+1}}p_{2N}(y,qz) \\ p_{2N-1}\left(qy,q^N z\right) & \cdots & p_{2N-1}(qy,qz) & q^{2N-2}\frac{1-q}{1-q^{2N-1}}p_{2N-2}(y,qz) \\ \vdots & \vdots & \cdots & \\ p_1\left(qy,q^N z\right) & \cdots & p_1(qy,qz) & q^0\frac{1-q}{1-q^1}p_0(y,qz) \end{vmatrix},
$$

which gives equation (A.8). This completes the proof of Lemma 3. ∎

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