

A Unified Description of the Asymmetric q - P_V and d - P_{IV} Equations and their Schlesinger Transformations

B GRAMMATICOS^{†1}, *A RAMANI*^{†2} and *Y OHTA*^{†3}

^{†1} *GMPIB, Université Paris VII, Tour 24-14, 5^e étage, case 7021, 75251 Paris, France*

^{†2} *CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France*

^{†3} *Department of Applied Mathematics, Faculty of Engineering, Hiroshima University, 1-4-1 Kagamiyama, Higashi-Hiroshima 739-8527, Japan*

Received July 3, 2002; Revised September 27, 2002; Accepted October 1, 2002

Abstract

We present a geometric description, based on the affine Weyl group $E_6^{(1)}$, of two discrete analogues of the Painlevé VI equation, known as the asymmetric q - P_V and asymmetric d - P_{IV} . This approach allows us to describe in a unified way the evolution of the mapping along the independent variable and along the various parameters (the latter evolution being the one induced by the Schlesinger transformations). It turns out that both discrete Painlevé equations exhibit the property of self-duality: the same equation governs the evolution along any direction in the space of $E_6^{(1)}$.

1 Introduction

The study of integrable discrete systems has revealed the most interesting fact that these systems present many common points with their continuous counterparts. One of these points was the role played by singularities in discrete integrability [1]. By examining the singularities of a given mapping, and requesting that those which appear spontaneously do not propagate *ad infinitum*, we were able to derive the discrete analogue of the Painlevé equations [2].

Painlevé equations were introduced one century ago in order to extend to the nonlinear domain the notion of special function defined by a differential equation. The Painlevé transcendents were discovered in that way. Discrete forms of the Painlevé equations were discovered as soon as 1939 [3] (and are present in essence if not in precise form in the work of Laguerre [4] which precedes that of Painlevé) but were not recognised as such till recently. However it was only after the discovery of the singularity confinement property that the study of the discrete Painlevé equations received a substantial boost. The principle for their derivation is simple: start from an integrable autonomous mapping (typically one of the QRT [5] family) which contains free parameters and apply the singularity confinement

criterion in order to fix the n -dependence of the parameters. This approach made possible the derivation of the q -analogues of the Painlevé equations. The latter are mappings where the independent variable enters not in an additive but, rather, in a multiplicative way. The first instance of such an equation was [6]:

$$\overline{x}\underline{x} = \frac{cd(x - a\lambda^n)(x - b\lambda^n)}{(x - c)(x - d)}, \quad (1.1)$$

where $x = x(n)$, $\overline{x} = x(n + 1)$, $\underline{x} = x(n - 1)$ and a, b, c, d are constant. This equation was shown to be a q -discrete analogue of P_{III} provided one discards the possible even-odd dependence of the coefficients. However when the full freedom of the coefficients is restored and one rewrites the equation in an asymmetric form (the term ‘asymmetric’ being used here in the QRT sense):

$$y\underline{y} = \frac{cd(x - a\lambda^n)(x - b\lambda^n)}{(x - p)(x - q)}, \quad (1.2a)$$

$$\overline{x}x = \frac{pq(y - r\lambda^n)(y - s\lambda^n)}{(y - c)(y - d)}, \quad (1.2b)$$

where a, b, c, d, p, q, r, s are constants constrained by $pqrs = \lambda abcd$. It can be shown, as was done by Jimbo and Sakai [7], that this equation is a discrete form of P_{VI} . The interesting property of equations where no artificial limitation of the richness of their parameters is imposed is the self-duality, first discovered in [8]. While studying the action of Schlesinger transformations of discrete Painlevé equations it was found that the same equation governs the evolution along the independent variable and the Schlesinger-induced shifts of parameters. We know today that not all discrete Painlevé equations possess the property of self-duality [9], however it was this discovery which made possible a geometrical description [10] of the discrete Painlevé equations and their classification. This geometrical description relies heavily on affine Weyl groups (as the one of the continuous Painlevé equations) and was dubbed in [11] the “Grand Scheme”.

In this paper we shall present the geometrical, affine Weyl group-based, description of two equations. The first is known as asymmetric d- P_{IV} :

$$\begin{aligned} (\overline{x} + y)(y + x) &= \frac{(y - a)(y - b)(y - c)(y - d)}{(y - z - \kappa/2)^2 - e^2}, \\ (y + x)(x + \underline{y}) &= \frac{(x + a)(x + b)(x + c)(x + d)}{(x - z)^2 - f^2}, \end{aligned} \quad (1.3)$$

where $z = \kappa n + \mu$ and the constants a, b, c, d, e, f satisfy the constraint $a + b + c + d = 0$. The second is known as the asymmetric q - P_{V}

$$\begin{aligned} (\overline{xy} - 1)(yx - 1) &= \frac{rs\lambda^{2n+1}(y - a)(y - b)(y - c)(y - d)}{(y - p\lambda^n)(y - q\lambda^n)}, \\ (yx - 1)(x\underline{y} - 1) &= \frac{pq\lambda^{2n-1}(x - 1/a)(x - 1/b)(x - 1/c)(x - 1/d)}{(x - r\lambda^n)(x - s\lambda^n)}, \end{aligned} \quad (1.4)$$

where the constants a, b, c, d, p, q, r, s satisfy the constraint $pq = \lambda abcdrs$. These equations have been first proposed in [12] and further studied in [13, 14] but their geometrical description had not been presented yet.

2 The geometry of the $E_6^{(1)}$ weight lattice

Our basic assumption is that the τ -functions of both the discrete Painlevé equations under study live on the points of the weight lattice of the affine Weyl group $E_6^{(1)}$. It turns out that there is no orthonormal basis invariant under the action of the group. In analogy to what we did in the case of $E_7^{(1)}$ [15] we will choose an orthogonal basis where all vectors are not equivalent. The squared length of the first vector will be chosen equal to $1/2$, while that of the five others will be taken equal to $3/2$. In this basis, these points are such that their coordinates are of the form $(a; b_1, b_2, b_3, b_4, b_5)$, where a (the coordinate along the squared-length- $1/2$ vector) and the coordinates b_i 's (on the five squared-length- $3/2$ vectors) are either *all* integer or *all* half-integers, with the additional constraint that the sum of all coordinates (a and b 's) is even.

The origin satisfies these requirements. It has 54 nearest-neighbours (NN) of the following form. First, $(\pm 2; 0, 0, 0, 0, 0)$, then 20 such that $a = \pm 1$ and *one* nonzero coordinate $b_i = \pm 1$ while the other four vanish and finally 32 where both a and each b_i have absolute value $1/2$ and arbitrary signs, up to the constraint that the total number of minus signs be odd, to ensure that the total sum be even, thus leading to 32 rather than 64 points. The squared distance of each of these points to the origin is always 2, be it $4/2$, $1/2 + 3/2$ or $1/8 + 5(3/8)$. Though in this particular basis these points look very different, they are in fact all equivalent. They define 27 directions along which NV's vectors exist and this notation stands for 'Nearest-neighbour-connecting Vectors'. Contrary to the case of $E_7^{(1)}$ [15] where we could not fix consistently the orientations of the NV's, for the case of $E_6^{(1)}$ if we choose as our oriented NV's $(2; 0, 0, 0, 0, 0)$, $(-1; 0, \dots, \pm 1, \dots)$ and $(1/2; \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ (with odd number of minus signs in the last case), then the scalar products of any two distinct NV's is either -1 or $1/2$ but never 1 nor $-1/2$. Moreover each NV has scalar product -1 with exactly 10 NV's and $1/2$ with the 16 others. This shows again that all NV's are equivalent. Also their total sum is zero, as can be easily checked. (For instance see that the scalar product of the sum with any NV is $2 + 10(-1) + 16(1/2) = 0$).

The sum of two NV's of scalar product -1 has squared length 2, and in fact is just the opposite of some other NV. So to get further away from the origin, to a τ which is next-nearest-neighbour (NNN) of the origin, we have to translate by the sum of two vectors of scalar product $-1/2$, i.e. the *difference* of two NV's of scalar product $1/2$, to get an NNV of squared length 3. Here NNV stands for 'Next-Nearest-neighbour-connecting Vectors'. Contrary to the NV's that can be consistently oriented, the NNV's cannot (the orientation of the NV's does not carry over because we are taking differences). Though there are 216 ($= 27 \times 16/2$) pairs of NV's of scalar product $1/2$, there are only 36 NNV's, since each can be obtained in six different ways. For instance the NNV $(0; 1, 1, 0, 0, 0)$ is the difference of the pairs $\{(-1; 1, 0, 0, 0, 0), (-1; 0, -1, 0, 0, 0)\}$ and $\{(-1; 0, 1, 0, 0, 0), (-1; -1, 0, 0, 0, 0)\}$ but also of four pairs $\{(1/2; 1/2, 1/2, \pm 1/2, \pm 1/2, \pm 1/2), (1/2; -1/2, -1/2, \pm 1/2, \pm 1/2, \pm 1/2)\}$ where the three signs are the same for two vectors in a pair, and are constrained by the even-sum rule to comprise an odd number of minus signs. In the basis we consider there are 20 NNV's of this form, with a and three b_i 's vanishing and the two others of absolute value 1 with arbitrary sign, (but only 20 rather than $40 = 4(5 \times 4/2)$ because we ignore the orientation of the NNV's) and 16 more which have $a = -3/2$ and the same values of the b_i 's as the 16 last NV's, but again, all NNV's are equivalent.

3 The nonlinear variables and the Hirota–Miwa equation

Consider a 2-dimensional plane containing the origin, say, and two τ 's, both NN's of the origin, such that the relevant NV's have scalar product $1/2$, for instance $(-1; 1, 0, 0, 0, 0)$ and $(-1; 0, -1, 0, 0, 0)$. They are in NNN position with respect to each other, since their squared-distance is 3. But what is interesting is to consider the fourth point in the parallelogram, the one obtained in translating the origin by the sum of these two NV's. This point (in our case $(-2; 1, -1, 0, 0, 0)$) is at a squared-distance 5 from the origin, and is in next-next-nearest-neighbour (NNNN) position with respect to it. This means that the center of our parallelogram, midpoint of a pair of NNN τ 's is *also* the midpoint of pair of two NNNN τ 's. Moreover, since there are six different ways to write an NNV as the difference of two NV's, the same point is altogether the midpoint of six different pairs of NNNN τ 's. In our case, the point X , $(-1; 1/2, -1/2, 0, 0, 0)$, midpoint of the NNN pair $\{(-1; 1, 0, 0, 0, 0), (-1; 0, -1, 0, 0, 0)\}$, is also midpoint of the NNNN pairs, not only $\{(0; 0, 0, 0, 0, 0), (-2; 1, -1, 0, 0, 0)\}$ but also $\{(-2; 0, 0, 0, 0, 0), (0; 1, -1, 0, 0, 0)\}$ and four pairs of the form $\{(-1/2; 1/2, -1/2, \pm 1/2, \pm 1/2, \pm 1/2), (-3/2; 1/2, -1/2, \mp 1/2, \mp 1/2, \mp 1/2)\}$, where the three last signs have opposite values in the two points of a given pair, the number of minus signs being odd (resp. even) for the first (resp. second) point in each pair, which guarantees that the even-sum rule always holds. These points, midpoints of one pair of NNN τ 's and of six pairs of NNNN τ 's, are the points where we will define nonlinear variables, X or x for the asymmetric d-P_{IV} and q -P_V equations respectively.

Note that, contrary to the NNV's which cannot be oriented consistently, the NNNV's are sums of NV's and can all be consistently oriented by carrying over the orientation of the NV's. The six NNNV's around the site of a particular nonlinear variable are not independent: they all lie in the same hyperplane orthogonal to the NNV joining the pair of NNN τ 's around the same site. In fact one can easily convince oneself that their sum vanishes: each of them has scalar product -1 with each of the five others (we recall that their squared sum is precisely 5). In the particular case we are considering, the correctly oriented NNNV's are $(-2; 1, -1, 0, 0, 0)$, $(-2; -1, 1, 0, 0, 0)$, and four of the form $(1; 0, 0, \pm 1, \pm 1, \pm 1)$ with an odd number of minus signs (such a vector, with an even number of minus signs, would still be a valid NNNV between *some* τ 's, as we shall see later, but *not* around the point we are considering). Let us choose some point O' , not necessarily the origin of coordinates, and call C_i the scalar products of the vector $\overrightarrow{O'X}$ with these six NNNV's, and introduce $c_i = q^{C_i}$, for some number q . We have of course $\sum_i C_i = 0$ and $\prod_i c_i = 1$.

We are now in a position to express the value of the nonlinear variable at the point we considered. Let ψ be the product of the two NNN τ 's (in our case, τ_x at $(-1; 1, 0, 0, 0, 0)$ and τ'_x at $(-1; 0, -1, 0, 0, 0)$) and ϕ_i the product of two NNNN τ 's at the ends of the vector that defines C_i . Then, for asymmetric d-P_{IV} we have:

$$X = C_i + \frac{\phi_i}{\psi} \tag{3.1}$$

and for asymmetric q -P_V

$$x = c_i + c_i^{1/3} \frac{\phi_i}{\psi}. \tag{3.2}$$

This implies compatibility conditions, which are non-autonomous Hirota–Miwa [16] systems:

$$\phi_i - \phi_j + (C_i - C_j)\psi = 0, \quad (3.3)$$

$$c_i^{1/3}\phi_i - c_j^{1/3}\phi_j + (c_i - c_j)\psi = 0. \quad (3.4)$$

The set of equations (3.3) (resp. (3.4)) around *all* possible sites for nonlinear variables is overdetermined but consistent over the entire lattice and is nothing but the bilinear form of the asymmetric d-P_{IV} (resp. q -P_V) equation.

4 The nonlinear equations

Around each site like X , among the 27 NV's, exactly 12 are used up in constructing, in pairs, the 6 NNNV's around X , (or, equivalently, lead by their differences to the NNV around X). There are 15 NV's left. On the other hand there are exactly 15 ways to choose two among the six NNNV's. It turns out that, for any choice of a pair of NNNV around X , the sum of these vectors is exactly twice the opposite of one of the 15 remaining NV's. Not only that, but if one translates X by half of any of these NV's in either direction, one finds another point Y where a nonlinear variable can be defined. This was by no means obvious: if we translate X by half of any of the first 12 NV's, the resulting point would not be the midpoint of two τ 's in NNN position. To be specific, we easily see that no pair of NV's containing $(2; 0, 0, 0, 0, 0)$ allows to construct, by difference, the NNV $(0; 1, 1, 0, 0, 0)$ around X . Conversely, if we take $(-2; 1, -1, 0, 0, 0)$, $(-2; -1, 1, 0, 0, 0)$ among the 6 NNNV's around X , their sum is twice the opposite of this NV. Thus the point X can be translated by $(\pm 1; 0, 0, 0, 0, 0)$ to lead to new sites Y $(0; 1/2, -1/2, 0, 0, 0)$ and \underline{Y} $(-2; 1/2, -1/2, 0, 0, 0)$, where nonlinear variables can be constructed. Note that the environments of Y , \underline{Y} in terms of τ 's are identical (since the distance between these two points is a full NV) but are *not* the same as that of X . For instance the NNV around Y (and \underline{Y}) is not $(0; 1, 1, 0, 0, 0)$ but $(0; -1, 1, 0, 0, 0)$, and the NNNV's also differ from those at X , being $(-2; 1, 1, 0, 0, 0)$, $(-2; -1, -1, 0, 0, 0)$ and of the form $(1; 0, 0, \pm 1, \pm 1, \pm 1)$ but with an *even* number of minus signs. So, if $(a; b_i)$ are the components of the vector \overrightarrow{OX} the six C_i 's around X come into two groups (this is because we are distinguishing a specific NV, that of the direction XY ; from an absolute point of view all C_i 's are equivalent):

$$\begin{aligned} -a + 3/2(b_1 - b_2) &\equiv -2Z + p, \\ -a - 3/2(b_1 - b_2) &\equiv -2Z - p \end{aligned} \quad (4.1)$$

on the one hand,

$$\begin{aligned} a/2 - 3/2(b_3 + b_4 + b_5) &\equiv Z + \alpha, \\ a/2 + 3/2(-b_3 + b_4 + b_5) &\equiv Z + \beta, \\ a/2 + 3/2(b_3 - b_4 + b_5) &\equiv Z + \gamma, \\ a/2 + 3/2(b_3 + b_4 - b_5) &\equiv Z + \delta \end{aligned} \quad (4.2)$$

on the other, where $Z = a/2$ and obvious notations for p , α , β , γ and δ , with $\alpha + \beta + \gamma + \delta = 0$. The six D_j 's around Y are $-(a + 1) + 3/2(b_1 + b_2) \equiv (-2Z - 1 + r)$,

$-(a+1) - 3/2(b_1 + b_2) \equiv (-2Z - 1 - r)$ and four of the form $Z + 1/2 - \alpha$, etc., the ones around \underline{Y} being the same up to replacing $Z + 1/2$ by $Z - 1/2$.

Consider now one of the pairs of NNNN τ 's around X associated to one of the four last C_i 's, say $Z + \alpha$. The NNNV is $(1; 0, 0, 1, 1, 1)$ and the two relevant τ 's are τ_α at $(-1/2; 1/2, -1/2, -1/2, -1/2, -1/2)$ and τ'_α at $(-3/2; 1/2, -1/2, 1/2, 1/2, 1/2)$. The first one also forms an NNNN pair near Y while the second forms an NNNN pair near \underline{Y} , both corresponding to the D_j involving α . Thus if for instance τ_α vanishes, we know the values of both X and Y (from (3.1) and (3.2) because the corresponding ϕ vanishes). At this point we must start to separate the study of q -P_{IV} from that of d-P_{IV}.

Let us first consider d-P_{IV}. Then, if τ_α vanishes, from (3.1) we have $X = Z + \alpha$ and from its analogue at Y , $Y = Z + 1/2 - \alpha$, because the quantity ϕ in that case is $\tau_\alpha \tau'_\alpha$ for X and $\tau_\alpha \bar{\tau}'_\alpha$ for Y where $\bar{\tau}'_\alpha$ is at the point $(1/2; 1/2, -1/2, 1/2, 1/2, 1/2)$ (translated from the site of τ'_α by the full NV along XY). The quantities at the denominators of X and Y are respectively $\tau_x \tau'_x$ and $\tau_0 \tau_y$ where we recall that τ_x is at point $(-1; 1, 0, 0, 0, 0)$ and τ'_x at $(-1; 0, -1, 0, 0, 0)$, while τ_0 is at the origin $(0; 0, 0, 0, 0, 0)$ and τ_y at $(0; 1, -1, 0, 0, 0)$ symmetric of the origin with respect to Y . Consider now the quantity $X + Y - 2Z - 1/2$. Computing X through (3.1) and the appropriate instance of (4.2) (namely the first one), and similarly Y through their analogues for we have

$$X + Y - 2Z - 1/2 = \frac{\tau_\alpha \tau'_\alpha}{\tau_x \tau'_x} + \frac{\tau_\alpha \bar{\tau}'_\alpha}{\tau_0 \tau_y} = \frac{\tau_\alpha (\tau_0 \tau_y \tau'_\alpha + \tau_x \tau'_x \bar{\tau}'_\alpha)}{\tau_x \tau'_x \tau_0 \tau_y}. \quad (4.3)$$

The four numbers α , β , γ and δ are equivalent, and to each of them one can assign a τ which forms an NNNN pair around both X and Y . Whenever each of them vanishes, both X and Y take the corresponding C_i and D_j value, the sum of which is $2Z + 1/2$ in all four cases. This quantity is the scalar product of the NV $(2; 0, 0, 0, 0, 0)$ in the direction of XY with the vector from O' to the midpoint of XY . The left hand side of (4.3) thus vanishes whenever one of these four τ 's vanish. It follows that the numerator of the right hand side must also vanish, since it is equal to the product of the vanishing quantity $X + Y - 2Z - 1/2$ by a product of τ 's which, being entire functions, may never become infinite. Hence this numerator is proportional not only to τ_α but also to the product of the three others. It follows that:

$$\tau_0 \tau_y \tau'_\alpha + \tau_x \tau'_x \bar{\tau}'_\alpha = K \tau_\beta \tau_\gamma \tau_\delta. \quad (4.4)$$

By homogeneity, K does not depend on any other τ 's. This is thus a trilinear equation satisfied by the τ 's. One does not need to impose it independently of (3.3). In fact one can show, eliminating repeatedly various τ functions between (3.3) and its analogues at appropriate points, that (4.4) is just a consequence of the bilinear equation (3.3) and moreover that K is just -1 . So one gets:

$$\tau_0 \tau_y \tau'_\alpha + \tau_x \tau'_x \bar{\tau}'_\alpha + \tau_\beta \tau_\gamma \tau_\delta = 0. \quad (4.5)$$

Indeed the three products play the same role, each being the product of three τ forming an equilateral triangle the side of which has squared-length 3 (indeed, each side is an NNV), all three triangles having the same center of mass, namely the point G at $(-1/2; 1/2, -1/2, 1/6, 1/6, 1/6)$. These nine τ 's are the only ones at distance 1 from G , and there is no other way to arrange them in three such equilateral triangles. Equation

(4.5) is thus an instance of a very general trilinear equation which is true around every point on the lattice equivalent to G , as a consequence of (3.1). The numerator of the r.h.s. of (4.3) can be replaced by a monomial using (4.5) and we find:

$$X + Y - 2Z - 1/2 = -\frac{\tau_\alpha \tau_\beta \tau_\gamma \tau_\delta}{\tau_x \tau'_x \tau_0 \tau_y}. \quad (4.6)$$

Clearly the same reasoning can be done for X and \underline{Y} . If τ'_α at $(-3/2; 1/2, -1/2, 1/2, 1/2, 1/2)$ vanishes, X has still the value $Z + \alpha$, but while we have no direct information on Y , the value of \underline{Y} is $Z - 1/2 - \alpha$, because τ'_α does form an NNNN pair around \underline{Y} . Thus one has $X + \underline{Y} = 2Z - 1/2$, but this is also true when the analogous τ'_β , etc. vanish. So again:

$$X + \underline{Y} - 2Z + 1/2 = -\frac{\tau'_\alpha \tau'_\beta \tau'_\gamma \tau'_\delta}{\tau_x \tau'_x \underline{\tau}_0 \underline{\tau}_y}, \quad (4.7)$$

where $\underline{\tau}_0 \underline{\tau}_y$ is the ψ corresponding to \underline{Y} , at points $(-2; 0, 0, 0, 0, 0)$ and $(-2; 1, -1, 0, 0, 0)$. Taking the product we find:

$$(X + Y - 2Z - 1/2)(X + \underline{Y} - 2Z + 1/2) = \frac{\tau_\alpha \tau_\beta \tau_\gamma \tau_\delta}{\tau_x \tau'_x \tau_0 \tau_y} \frac{\tau'_\alpha \tau'_\beta \tau'_\gamma \tau'_\delta}{r \tau_x \tau'_x \underline{\tau}_0 \underline{\tau}_y}. \quad (4.8)$$

From (3.1) and (4.2) we recognize at the numerator of the right-hand side the product of the numerators of the quantities $(X - Z - \alpha)$, etc. thus:

$$\begin{aligned} & (X + Y - 2Z - 1/2)(X + \underline{Y} - 2Z + 1/2) \\ &= (X - Z - \alpha)(X - Z - \beta)(X - Z - \gamma)(X - Z - \delta) \frac{\tau_x^2 \tau'_x{}^2}{\tau_0 \tau_y \underline{\tau}_0 \underline{\tau}_y}. \end{aligned} \quad (4.9)$$

But τ_0 and $\underline{\tau}_y$, on the one hand, and $\underline{\tau}_0$ and τ_y , on the other, are precisely the last two pairs of NNNN τ 's around X and from (3.1) and (4.1), the last factor in (4.9) is just the inverse of $(X + 2Z - p)(X + 2Z + p)$. Thus:

$$\begin{aligned} & (X + Y - 2Z - 1/2)(X + \underline{Y} - 2Z + 1/2) \\ &= \frac{(X - Z - \alpha)(X - Z - \beta)(X - Z - \gamma)(X - Z - \delta)}{(X + 2Z - p)(X + 2Z + p)}. \end{aligned} \quad (4.10)$$

This is one of the two equations of the system defining the asymmetric d-P_{IV} equation [14], though in a slightly unusual form. In order to obtain the other one, we need to consider the point \overline{X} at $(1; 1/2, -1/2, 0, 0, 0)$. The couple of points (Y, \overline{X}) is translated from (\underline{Y}, X) by exactly one NV, so the environment is the same and from (4.7) one gets (with obvious notations)

$$\overline{X} + Y - 2Z - 3/2 = -\frac{\overline{\tau}'_\alpha \overline{\tau}'_\beta \overline{\tau}'_\gamma \overline{\tau}'_\delta}{\overline{\tau}_x \overline{\tau}'_x \tau_0 \tau_y}. \quad (4.11)$$

Multiplying with (4.6) we see appearing on the right hand side products of τ 's near Y . Finally what we get is the analogue of (4.10):

$$\begin{aligned} & (X + Y - 2Z - 1/2)(\overline{X} + Y - 2Z - 3/2) \\ &= \frac{(Y - Z - 1/2 + \alpha)(Y - Z - 1/2 + \beta)(Y - Z - 1/2 + \gamma)(Y - Z - 1/2 + \delta)}{(Y + 2Z + 1 - r)(Y + 2Z + 1 + r)}. \end{aligned} \quad (4.12)$$

To recover the usual form, we redefine X , Y , \underline{Y} and \overline{X} by adding to them the relevant value of the independent variable (Z , $Z + 1/2$, $Z - 1/2$ and $Z + 1$, respectively) to get in the translated variables:

$$(X + Y)(X + \underline{Y}) = \frac{(X - \alpha)(X - \beta)(X - \gamma)(X - \delta)}{(X + 3Z - p)(X + 3Z + p)}, \quad (4.13a)$$

$$(X + Y)(\overline{X} + Y) = \frac{(Y + \alpha)(Y + \beta)(Y + \gamma)(Y + \delta)}{(Y + 3Z + 3/2 - r)(Y + 3Z + 3/2 + r)}. \quad (4.13b)$$

Equation (4.13) is exactly the asymmetric d-P_{IV} equation [14] up to a redefinition of the variable Z .

The case of the asymmetric q -P_V equation is similar but slightly more complicated. The positions of the relevant τ 's are exactly the same, so we will keep the same names, but the equations coming from (3.2) are not quite the same. Let us follow the corresponding steps. When τ_α vanishes, we have $x = q^{Z+\alpha}$, $y = q^{Z+1/2-\alpha}$ so $xy = q^{2Z+1/2}$. Let us compute in all generality, the quantity $xyq^{-2Z-1/2}$. To get x we use (3.2) with the instance of (4.2) involving α , and similarly for y . We find

$$\begin{aligned} xyq^{-2Z-1/2} &= 1 + q^{-2(Z+\alpha)/3} \frac{\tau_\alpha \tau'_\alpha}{\tau_x \tau'_x} \\ &+ q^{-2(Z+1/2-\alpha)/3} \frac{\tau_\alpha \overline{\tau}'_\alpha}{\tau_0 \tau_y} + q^{-(4Z+1)/3} \frac{\tau_\alpha^2 \tau'_\alpha \overline{\tau}'_\alpha}{\tau_x \tau'_x \tau_0 \tau_y} \end{aligned} \quad (4.14)$$

so indeed $(xyq^{-2Z-1/2} - 1)$ vanishes when τ_α does. But though we have *expressed* this quantity with emphasis on α , the three other quantities β , γ and δ play the same role and $(xyq^{-2Z-1/2} - 1)$ also vanishes whenever the associated τ does. With the same argument as above, it follows that the numerator of the right hand side of (4.14), after subtracting 1, must be proportional to the product of these τ 's. So we get the analogue of equation (4.4)

$$q^{-2(Z+\alpha)/3} \tau'_\alpha \tau_0 \tau_y + q^{-2(Z+1/2-\alpha)/3} \overline{\tau}'_\alpha \tau_x \tau'_x + q^{-(4Z+1)/3} \tau_\alpha \tau'_\alpha \overline{\tau}'_\alpha = K \tau_\beta \tau_\gamma \tau_\delta. \quad (4.15)$$

As in the case of (4.4), homogeneity shows that K does not depend on any τ 's. But contrary to the previous case, K is not a constant, but a function of the point on the lattice. Let us look at this trilinear equation more closely. Three of the products are the same as in (4.5), but they have a prefactor which is not unity. The last product on the left-hand-side was not present in (4.4). It involves one new τ , namely τ_α , plus one τ from two of the three other products. These three τ 's form an isosceles triangle of sides of squared-length 2, 5 and 5. Its center of mass is still the same point G as that of the three others, but now τ_α is at squared-distance 2 from it. There are thus many such equations that are true around the same point, one for each of the 27 ways to pick one τ out of two of the three three- τ 's products in (4.5), and complete the triangle of center of mass G to a τ at squared-distance 2 from it.

In order to obtain the value of K , one repeatedly applies the analogue of (3.4) around various points. One can actually prove (4.15) and obtain the value of K which is $-q^{-(8Z+2)/3}$. Rewriting (4.15), divided by $(-K)$:

$$\begin{aligned} \tau_\beta \tau_\gamma \tau_\delta + q^{2Z+1/2-2(\alpha-1/4)/3} \tau'_\alpha \tau_0 \tau_y \\ + q^{2Z+1/2+2(\alpha-1/4)/3} \overline{\tau}'_\alpha \tau_x \tau'_x + q^{(4Z+1)/3} \tau_\alpha \tau'_\alpha \overline{\tau}'_\alpha = 0. \end{aligned} \quad (4.16)$$

Subtracting one from the r.h.s. of (4.14) we see that we recover the l.h.s. of (4.15) up to a global multiplicative factor. Multiplying by the same factor the r.h.s. of (4.15) and using the value of K we find that the quantity $(xyq^{-2Z-1/2} - 1)$ becomes:

$$xyq^{-2Z-1/2} - 1 = -q^{-(8Z+2)/3} \frac{\tau_\alpha \tau_\beta \tau_\gamma \tau_\delta}{\tau_x \tau'_x \tau_0 \tau_y} \quad (4.17)$$

and similarly between x and \underline{y} we have

$$\underline{xy}q^{-2Z+1/2} - 1 = -q^{-(8Z-2)/3} \frac{\tau'_\alpha \tau'_\beta \tau'_\gamma \tau'_\delta}{\tau_x \tau'_x \tau_0 \tau_y}. \quad (4.18)$$

Taking the product, we recover the products of τ 's involving the quantities $(x - c_i)$. When we take into account carefully all prefactors we find:

$$\begin{aligned} & (xyq^{-2Z-1/2} - 1) (\underline{xy}q^{-2Z+1/2} - 1) \\ &= q^{-8Z} \frac{(x - q^{Z+\alpha})(x - q^{Z+\beta})(x - q^{Z+\gamma})(x - q^{Z+\delta})}{(x - q^{-2Z+p})(x - q^{-2Z-p})} \end{aligned} \quad (4.19)$$

which is one of the two equations of the system defining the asymmetric q -P_V equation [14], although not in its usual form. Again, in order to obtain the second equation, we need to consider the point \bar{x} . Translating (4.18) by a full NV forwards, we get:

$$\bar{x}yq^{-2Z-3/2} - 1 = -q^{-(8Z+6)/3} \frac{\bar{\tau}'_\alpha \bar{\tau}'_\beta \bar{\tau}'_\gamma \bar{\tau}'_\delta}{\bar{\tau}_x \bar{\tau}'_x \tau_0 \tau_y}. \quad (4.20)$$

Multiplying with (4.17) we find products of τ 's near y , and get the analogue of (4.19):

$$\begin{aligned} & (xyq^{-2Z-1/2} - 1) (\bar{x}yq^{-2Z-3/2} - 1) \\ &= q^{-8Z-4} \frac{(y - q^{Z+1/2-\alpha})(y - q^{Z+1/2-\beta})(y - q^{Z+1/2-\gamma})(y - q^{Z+1/2-\delta})}{(y - q^{-2Z-1+r})(y - q^{-2Z-1-r})}. \end{aligned} \quad (4.21)$$

The equations (4.21) and (4.19) together form the asymmetric q -P_V equation. To recover the usual form, we absorb q^{-Z} into a redefinition of x (and appropriate factors for the other variables) to get:

$$(xy - 1)(\underline{xy} - 1) = \frac{(x - q^\alpha)(x - q^\beta)(x - q^\gamma)(x - q^\delta)}{(1 - xq^{3Z-p})(1 - xq^{3Z+p})}, \quad (4.22a)$$

$$(xy - 1)(\bar{x}y - 1) = \frac{(y - q^{-\alpha})(y - q^{-\beta})(y - q^{-\gamma})(y - q^{-\delta})}{(1 - yq^{3Z+3/2-r})(1 - yq^{3Z+3/2+r})} \quad (4.22b)$$

which is the asymmetric q -P_V equation in its usual form up to a redefinition of Z .

The continuous limits of asymmetric q -P_V and asymmetric d-P_{IV} were presented in [14], where it was shown that both equations have P_{VI} as continuous limit.

5 The contiguity relations and the Miura's

Among the 15 NV's around X which allow to reach sites of nonlinear variables, 8 have scalar product $1/2$ with the NV $(2; 0, 0, 0, 0, 0)$ along XY (corresponding to taking one of the two NNNV with $a = -2$ and one of the four ones with $a = 1$) and 6 with scalar product -1 with this NV (taking 2 among the 4 NNNV's with $a = 1$). Note that the latter six come by pairs, the sum of two NV's of one pair being the opposite of the NV along XY . This is the case, for instance, for the NV $(-1; 0, 0, 1, 0, 0)$ related to α and β , say $-$ half the opposite of the sum of the two NNNV's $(1; 0, 0, -1, -1, -1)$ and $(1; 0, 0, -1, 1, 1)$ $-$ which lead to the two first C_i 's in (4.2) and the one related to γ and δ , namely $(-1; 0, 0, -1, 0, 0)$. The point $W_{\alpha,\beta}$ $(-1/2; 1/2, -1/2, -1/2, 0, 0)$, reached by translating X by half the *opposite* of the first forms an equilateral triangle with X and Y . Similarly, $W_{\alpha,\beta}$ at $(-3/2; 1/2, -1/2, 1/2, 0, 0)$ forms an equilateral triangle with X and \underline{Y} . Contrary to the case of $E_7^{(1)}$ we considered in [15], where points analogous to these two points were the only ones near X in the two-dimensional plane containing them together with X and Y , \bar{Y} , here the two points $W_{\gamma,\delta}$ at $(-3/2; 1/2, -1/2, -1/2, 0, 0)$ and $W_{\gamma,\delta}$ at $(-1/2; 1/2, -1/2, 1/2, 0, 0)$ are also in the same plane and form, with Y , \underline{Y} , $W_{\alpha,\beta}$ and $W_{\alpha,\beta}$, a regular hexagon of center X . Note that the two τ 's in NNV relative position around $W_{\alpha,\beta}$ are τ_α at $(-1/2; 1/2, -1/2, -1/2, -1/2, -1/2)$ and a τ at point $(-1/2; 1/2, -1/2, -1/2, 1/2, 1/2)$ which is what we implicitly called τ_β . We will not consider all six possible pairs of NNNN τ 's around $W_{\alpha,\beta}$, but two of such pairs are precisely $\{\bar{\tau}'_\gamma, \tau'_\delta\}$ and $\{\bar{\tau}'_\delta, \tau'_\gamma\}$, associated to the NNNV $(-2; 0, 0, 0, -1, 1)$ and $(-2; 0, 0, 0, 1, -1)$ respectively. (We are not giving the coordinates of all these τ 's; they can be deduced from those of index α by changing the sign of two of the last three components, the component which does not change sign being the fourth, fifth and sixth one, respectively, for β , γ and δ). Moreover note that the τ_0 at the origin and τ_y each belong to one NNNN pair, around $W_{\alpha,\beta}$, associated to the NNNV's $(1; -1, 1, 1, 0, 0)$ and $(1; 1, -1, 1, 0, 0)$. When τ'_γ , say, vanishes, both X and $W_{\alpha,\beta}$ take specific values, namely $(Z + \gamma)$ and $(-2Z + (\delta - \gamma - 1)/2)$, respectively. So the sum $(X + W_{\alpha,\beta})$ takes the value $-(Z + (\alpha + \beta + 1)/2)$ (we have used the fact that the $\alpha + \beta + \gamma + \delta = 0$), which, as it turns out, is the same when either τ'_δ , τ_0 or τ_y vanish. In fact these four τ 's play, for the pair of points $\{X, W_{\alpha,\beta}\}$ the same role as τ_α , etc., for $\{X, Y\}$. If we were considering the asymmetric q -PV equation, the relevant nonlinear variables x , $w_{\alpha,\beta}$ are just q raised at a power equal to the relevant quantities, and their product now takes the same values whether τ'_γ , τ'_δ , τ'_0 or τ'_y vanish. What happens is that we have another instance of a trilinear equation like (4.5) (resp. 4.16).

Considering first the asymmetric d-P_{IV} equation, and following the same line of reasoning we find that:

$$X + W_{\alpha,\beta} + Z + (\alpha + \beta + 1)/2 = -\frac{\tau'_\gamma \tau'_\delta \tau_0 \tau_y}{\tau_\alpha \tau_\beta \tau_x \tau'_x}. \quad (5.1)$$

One could easily obtain an equation relating $W_{\alpha,\beta}$, X and $W_{\alpha,\beta}$, analogous to (4.10). But we are here interested in the Miura relating $W_{\alpha,\beta}$, X and Y . Let us multiply (5.1) and (4.6). Four τ 's drop out and we are left with:

$$(X + Y - 2Z - 1/2)(X + W_{\alpha,\beta} + Z + (\alpha + \beta + 1)/2) = \frac{\tau_\gamma \tau_\delta \tau'_\gamma \tau'_\delta}{\tau_x^2 \tau_x'^2} \quad (5.2)$$

which, if we compute X in two different ways from (3.1) with the appropriate instances (namely the two last ones) of (4.2) is just:

$$\begin{aligned} & (X + Y - 2Z - 1/2)(X + \tilde{W}_{\alpha,\beta} + Z + (\alpha + \beta + 1)/2) \\ & = (X - Z - \gamma)(X - Z - \delta). \end{aligned} \quad (5.3)$$

The quantity $-(Z + (\alpha + \beta + 1)/2)$ is the scalar product of the NV $(-1; 0, 0, 1, 0, 0)$ from $\tilde{W}_{\alpha,\beta}$ to X , with the vector joining O' to the midpoint of $\tilde{W}_{\alpha,\beta}X$ and plays exactly the same role as $2Z + 1/2$ between X and Y . We could get an analogous equation for any of the six \tilde{W} that form an equilateral triangle with X and Y , with any two among $\alpha, \beta, \gamma, \delta$ as indices of \tilde{W} and the other two appearing in the right hand side. The factor involving X and Y is the same for all six choices. Equation (5.3) on the equilateral triangle $X, Y, \tilde{W}_{\alpha,\beta}$ seems to singularize X but this is not true. If we expand this equation we find:

$$\begin{aligned} & XY + X\tilde{W}_{\alpha,\beta} + Y\tilde{W}_{\alpha,\beta} - (2Z + 1/2)\tilde{W}_{\alpha,\beta} \\ & + (Z + (\alpha + \beta + 1)/2)Y + (Z - (\alpha + \beta)/2)X = (2Z + 1/2)^2/2 \\ & + (Z + (\alpha + \beta + 1)/2)^2/2 + (Z - (\alpha + \beta)/2)^2/2 - (\gamma - \delta)^2/4. \end{aligned} \quad (5.4)$$

The quadratic terms are obviously symmetric, and the coefficients of the three linear terms are just the opposite of the scalar product of the vector joining O' to the relevant point with the NV connecting the two others. As for the right hand side, it is a quantity which treats these three NV's in the same way. This is the Miura written in a symmetric way. The translations of X, Y that were used to simplify (4.10) are not appropriate, but there is still a way to simplify (5.4), by subtracting from each variable its coefficient in this equation. In the new variables, one finds a very elegant result:

$$\mathcal{X}\mathcal{Y} + \mathcal{X}\mathcal{W}_{\alpha,\beta} + \mathcal{Y}\mathcal{W}_{\alpha,\beta} + (\gamma - \delta)^2/4 = 0. \quad (5.5)$$

Unfortunately \mathcal{X} and \mathcal{Y} are not exactly the same variables as the X and Y of (4.13). Equation (5.4) is the contiguity relation on a triangle, the Miura transformation that allows to determine any nonlinear variable in the lattice from two initial data, for the asymmetric d-P_{IV} equation.

The case of the asymmetric q -P_V equation is similar. With the same reasoning we get the relation between x, y and $w_{\alpha,\beta}$, first written in a way that seems to singularize x :

$$\left(xyq^{-2Z-1/2} - 1\right) \left(xw_{\alpha,\beta}q^{Z+(\alpha+\beta+1)/2} - 1\right) = q^{2(-2Z+\alpha+\beta)/3} \frac{\tau_\gamma \tau_\delta \tau'_\gamma \tau'_\delta}{\tau_x^2 \tau_x'^2} \quad (5.6)$$

and thus

$$\left(xyq^{-2Z-1/2} - 1\right) \left(xw_{\alpha,\beta}q^{Z+(\alpha+\beta+1)/2} - 1\right) = (q^{-Z-\gamma}x - 1) \left(q^{-Z-\delta}x - 1\right). \quad (5.7)$$

Again, six similar equations can be obtained by permuting the four quantities α , etc. and (5.6) can also be written in a way that is symmetric in terms of x, y and $w_{\alpha,\beta}$. After some elementary algebra we find:

$$\begin{aligned} xy\tilde{w}_{\alpha,\beta} & = q^{-Z+(\alpha+\beta)/2}x + q^{-Z-(\alpha+\beta+1)/2}y \\ & + q^{2Z+1/2}\tilde{w}_{\alpha+\beta} - q^{(\gamma-\delta)/2} - q^{(\delta-\gamma)/2}. \end{aligned} \quad (5.8)$$

The coefficients of the nonlinear variables are just q raised to the opposite of the coefficients that appear in (5.4). This shows that this equation is also invariant in the exchange of the three variables. Obviously, a redefinition of the variables can put all their coefficients to unity,

$$xy\underline{w}_{\alpha,\beta} = \mathbf{x} + \mathbf{y} + \underline{w}_{\alpha,\beta} - q^{(\gamma-\delta)/2} - q^{(\delta-\gamma)/2} \quad (5.9)$$

but the \mathbf{x} and \mathbf{y} of (5.9) are not the same variables as the x and y of equation (4.22). Equation (5.8) is the contiguity relation on a triangle, the Miura transformation that allows to determine any nonlinear variable in the lattice from two initial data, for the asymmetric q -P_V equation.

In Section 4, we were able to write the equation between X , Y and \underline{Y} (or x , y , \underline{y}) without having to go through the Miura's. But it is interesting to show how to recover it from the Miura's. First, note that the various W obtained by translating X forward by half of six of the NV's form equilateral triangles with X and \underline{Y} . In particular, we could choose $W_{\gamma,\delta}$. Then we have the analogues of (5.3) and (5.7) for asymmetric d-P_{IV} and asymmetric q -P_V respectively.

$$\begin{aligned} &(X + \underline{Y} - 2Z + 1/2)(X + W_{\gamma,\delta} + Z + (\gamma + \delta - 1)/2) \\ &= (X - Z - \alpha)(X - Z - \beta), \end{aligned} \quad (5.10)$$

$$\left(x\underline{y}q^{-2Z+1/2} - 1\right) \left(xw_{\gamma,\delta}q^{Z+(\gamma+\delta-1)/2} - 1\right) = (q^{-Z-\alpha}x - 1) \left(q^{-Z-\beta}x - 1\right). \quad (5.11)$$

But, as we mentioned above, $W_{\gamma,\delta}$ belongs to a regular hexagon around X that contains Y , \underline{Y} , $W_{\alpha,\beta}$ and $W_{\alpha,\beta}$, and in particular, forms an equilateral triangle with X and $W_{\alpha,\beta}$. The analogues of (5.3) and (5.7) can easily be obtained. The factors in the left hand side involving the W 's, w 's are the same as those in the previously written equations, coupling them to Y , y : indeed, they only depend on the NV relating the point under consideration and X . Only the right hand side depends on what we are coupling them to. Carefully checking which are the τ 's that end up in the right hand side, we finally find:

$$\begin{aligned} &(X + \underline{W}_{\alpha,\beta} + Z + (\alpha + \beta + 1)/2)(X + W_{\gamma,\delta} + Z + (\gamma + \delta - 1)/2) \\ &= (X + 2Z - p)(X + 2Z + p), \end{aligned} \quad (5.12)$$

$$\begin{aligned} &\left(x\underline{w}_{\alpha,\beta}q^{Z+(\alpha+\beta+1)/2} - 1\right) \left(xw_{\gamma,\delta}q^{Z+(\gamma+\delta-1)/2} - 1\right) \\ &= (q^{2Z-p}x - 1) (q^{2Z+p}x - 1). \end{aligned} \quad (5.13)$$

From (5.3), (5.10), (5.12) and (5.7), (5.11), (5.13) respectively, one can easily recover (4.10) and (4.19).

6 Conclusion

In this paper we have presented the geometrical description of the discrete Painlevé equations known as asymmetric q -P_V and asymmetric d-P_{IV}. (Despite these names both equations are discrete analogues of P_{VI} as can be assessed through their continuous limits). This geometrical description was performed in the framework of the affine Weyl group $E_6^{(1)}$. It was shown that both discrete \mathbb{P} 's possess the property of self-duality i.e. the

same equation governs the evolution along the individual variable or along the parameters of the d - \mathbb{P} induced by the Schlesinger transformations. This geometrical approach allows to describe all the known d - \mathbb{P} 's in a unified approach. Moreover it makes possible the investigation of all possible equations related to the basic ones (here asymmetric q - P_V and asymmetric d - P_{IV}) by considering various evolution paths within the geometry of $E_6^{(1)}$ (a question we intend to return to in some future work).

References

- [1] Grammaticos B, Ramani A and Papageorgiou V, Do Integrable Mappings Have the Painlevé Property? *Phys. Rev. Lett.* **67** (1991), 1825–1828.
- [2] Grammaticos B, Nijhoff F and Ramani A, Discrete Painlevé Equations, in *The Painlevé Property: One Century Later*, Editor: Conte R, CRM Series in Mathematical Physics, Springer-Verlag, New York, 1999, 413–516.
- [3] Shohat J A, A differential equation for orthogonal polynomials, *Duke Math. J.* **5** (1939), 401–417.
- [4] Laguerre E, Sur la réduction en fractions continues d'une fraction qui satisfait à une équation différentielle du premier ordre dont les coefficients sont rationnels, *J. Math. Pures Appl.* **1** (1885) 135–165.
- [5] Quispel G R M, Roberts J A G and Thompson C J, Integrable Mappings and Soliton Equations. II, *Physica* **D34** (1989), 183–192.
- [6] Jimbo M, Sakai H, Ramani A and Grammaticos B, Bilinear structure and Schlesinger transforms of the q - P_{III} and q - P_{VI} equations, *Phys. Lett.* **A217** (1996), 111–118.
- [7] Jimbo M and Sakai H, A q -Analog of the Sixth Painlevé Equation, *Lett. Math. Phys.* **38** (1996), 145–154.
- [8] Nijhoff F, Satsuma J, Kajiwara K, Grammaticos B and Ramani A, A Study of the Alternate Discrete Painlevé II Equation, *Inv. Probl.* **12** (1996), 697–716.
- [9] Sakai H, Rational Surfaces Associated with Affine Root Systems and Geometry of the Painlevé Equations, PhD thesis, Kyoto University, 1999;
Grammaticos B and Ramani A, The Hunting for the Discrete Painlevé Equations. Sophia Kovalevskaya to the 150th Anniversary, *Reg. and Chaot. Dyn.* **5** (2000), 53–66.
- [10] Ramani A, Ohta Y, Satsuma J and Grammaticos B, Self-Duality and Schlesinger Chains for the Asymmetric d - P_{II} and q - P_{III} Equations, *Comm. Math. Phys.* **192** (1998), 67–76.
- [11] Ramani A and Grammaticos B, The Grand Scheme for Discrete Painlevé Equations, Lecture at the Toda Symposium, 1996.
- [12] Ramani A, Grammaticos B and Hietarinta J, Discrete Versions of the Painlevé Equations, *Phys. Rev. Lett.* **67** (1991), 1829–1832.
- [13] Tamizhmani K M, Grammaticos B and Ramani A, Schlesinger Transforms for the Discrete Painlevé IV Equation, *Lett. Math. Phys.* **29** (1993), 49–54;
Tamizhmani K M, Ramani A, Grammaticos B and Ohta Y, A Study of the discrete P_V Equation: Miura Transformations and Particular Solutions, *Lett. Math. Phys.* **38** (1996), 289–296.

-
- [14] Ramani A, Grammaticos B and Ohta Y, The Painlevé of discrete equations and other stories, Invited talk at the “Atelier sur la théorie des fonctions spéciales non-linéaires: les transcendentes de Painlevé”, Montréal, 1996. CRM Proceedings and Lecture Notes, Vol. 9, Editors: Levi D, Vinet L and Winternitz P, AMS, 1996, 303 –.
- [15] Ramani A, Grammaticos B and Ohta Y, A Geometrical Description of the Painlevé VI and V Equations, *Comm. Math. Phys.*, **217** (2001), 315–329.
- [16] Hirota R, Discrete Analogue of a Generalized Toda Equation, *J. Phys. Soc. Japan* **50** (1981), 3785–3791;
Miwa T, On Hirota’s Difference Equations, *Proc. Japan Acad., Ser. A Math. Sci.* **58** (1982), 9–12.