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Deformations of Modules of Differential Forms

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Abstract

We study non-trivial deformations of the natural action of the Lie algebra $\mathrm{Vect}(\mathbb{R}^n)$ on the space of differential forms on $\mathbb{R}^n$. We calculate abstractions for integrability of infinitesimal multi-parameter deformations and determine the commutative associative algebra corresponding to the miniversal deformation in the sense of [3].

1 Introduction

The Lie algebra $\mathrm{Vect}(M)$ of vector fields on a smooth manifold M naturally acts on the space $\Omega(M)$ of differential forms on M. Our goal is to study deformations of this action.

In this paper we restrict ourselves to the case $M = \mathbb{R}^n$ with $n \geq 2$, and consider only those deformations which are given by differentiable maps. We will use the framework of Fialowski–Fuchs [3] (see also [1]) and consider (multi-parameter) deformations over commutative associative algebras. We will construct the miniversal deformation of Lie derivative and define the commutative associative algebra related to this deformation.

The first step of any approach to the deformation theory consists in the study of infinitesimal deformations. According to Nijenhuis–Richardson [6], infinitesimal deformations are classified by the first cohomology space $H^1(\mathrm{Vect}(M); \mathrm{End}(\Omega(M)))$, while the obstructions for integrability of infinitesimal deformations belong to the second cohomology space $H^2(\mathrm{Vect}(M); \mathrm{End}(\Omega(M)))$. The first space was calculated in [2]; our task, therefore, is to calculate the obstructions.

We will prove that all the multi-parameter deformations of the Lie derivative are, in fact, infinitesimal, that is of first-order in the parameters of deformation. The parameters have to satisfy a system of quadratic relations; this defines interesting commutative algebra. We will give concrete explicit examples of the deformed Lie derivatives.

We mention that a similar problem was considered in [1] for the case of symmetric contravariant tensor fields instead of differential forms. In [7, 8] a slightly different approach was used for deformations of embeddings of Lie algebras.

2 The main definitions

In this section we introduce the algebra $\mathcal{D}(\Omega(\mathbb{R}^n))$ of linear differential operators on the space of differential forms on $\mathbb{R}^n$. The Lie algebra $\mathrm{Vect}(\mathbb{R}^n)$ has a natural embedding into $\mathcal{D}(\Omega(\mathbb{R}^n))$ given by the Lie derivative. We then describe the first cohomology space of $\mathrm{Vect}(\mathbb{R}^n)$ with coefficients in $\mathcal{D}(\Omega(\mathbb{R}^n))$.

2.1 Differential operators acting on differential forms

The space of differential forms has a natural (i.e. $\mathrm{Vect}(\mathbb{R}^n)$-invariant) grading

$$\Omega(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \Omega^k(\mathbb{R}^n),$$

where $\Omega^k(\mathbb{R}^n)$ is the space of differential k-forms. Let $\Omega_0^{n-k}(\mathbb{R}^n)$ be the space of compactly supported $(n-k)$-forms, there is a pairing

$$\Omega^k(\mathbb{R}^n) \otimes \Omega_0^{n-k}(\mathbb{R}^n) \to \mathbb{R} \tag{2.1}$$

so that $\Omega_0^{n-k}(\mathbb{R}^n) \subset \left(\Omega^k(\mathbb{R}^n)\right)^*$.

The space of differential operators on $\Omega(\mathbb{R}^n)$ is an associative algebra. As a $\mathrm{Vect}(\mathbb{R}^n)$-module, it is split into a direct sum of subspaces

$$\mathcal{D}^{k,\ell} = \mathcal{D}(\Omega^k(\mathbb{R}^n), \Omega^\ell(\mathbb{R}^n))$$

with $k \le \ell \le n$. Obviously, one has a natural embedding $\mathcal{D}^{k,\ell} \subset \Omega^\ell(\mathbb{R}^n) \otimes \left(\Omega^k(\mathbb{R}^n)\right)^*$.

2.2 The Lie derivative

We introduce the notations that will be useful for computations. We denote ξ^i the covector dx^i. A differential k-form can be written

$$\omega = \omega_{i_1 \dots i_k}(x)\, \xi^{i_1} \cdots \xi^{i_k}$$

and ξ^i are odd variables, i.e., $\xi^i \xi^j = -\xi^j \xi^i$. The Lie derivative L_X along a vector field X is given by the differential operator on $\Omega(\mathbb{R}^n)$

$$L_X = X^i \partial_{x^i} + \frac{\partial X^i}{\partial x^j} \xi^j \partial_{\xi^i}, \tag{2.2}$$

where $\partial_{x^i} = \partial/\partial x^i$ and $\partial_{\xi^i} = \partial/\partial \xi^i$; here and below summation over repeated indices is understood.

2.3 The first cohomology space

In the case $n \ge 2$, the first cohomology space of $\mathrm{Vect}(\mathbb{R}^n)$ with coefficients in $\mathcal{D}^{k,\ell}$ was calculated in [2]. The result is as follows

$$H^1(\mathrm{Vect}(\mathbb{R}^n); \mathcal{D}^{k,\ell}) = \begin{cases} \mathbb{R}, & \ell = k, \\ \mathbb{R}^2, & \ell = k+1, \\ \mathbb{R}, & \ell = k+2, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

In the case $\ell = k$, this space is generated by the cohomology class of the 1-cocycle

$$C_0(X) = \mathrm{Div}(X), \tag{2.4}$$

where X is a vector field and $\mathrm{Div}(X)$ is the divergence of X with respect to the standard volume form on $\mathbb{R}^n$.

In the case $\ell = k + 1$, there are two non-trivial cohomology classes corresponding to the 1-cocycles

$$C_1(X) = d \circ \mathrm{Div}(X), \qquad \widetilde{C}_1(X) = \mathrm{Div}(X) \circ d, \tag{2.5}$$

where d is the de Rham differential.

Finally, in the case $\ell = k + 2$, the cohomology space is spanned by the class of the 1-cocycle

$$C_2(X) = d \circ \mathrm{Div}(X) \circ d. \tag{2.6}$$

Definition 2.1. We denote $C_i^k \in \mathcal{D}^{k,k+i}$ the cocycle C_i (and $\widetilde{C}_1$) restricted to $\Omega^k(\mathbb{R}^n)$:

$$C_i^k(X) : C_i(X) \mid_{\Omega^k(\mathbb{R}^n)}, \qquad i = 0, 1, 2. \tag{2.7}$$

Note that these elements of $Z^1(\mathrm{Vect}(\mathbb{R}^n); \Omega(\mathbb{R}^n))$ are linearly independent for every $k = 0, \ldots, n - i$.

3 The general framework

In this section we define deformations of Lie algebra homomorphisms and introduce the notion of miniversal deformations over commutative algebras.

3.1 Infinitesimal deformations

Let $\rho_0 : \mathfrak{g} \to \mathrm{End}(V)$ be an action of a Lie algebra $\mathfrak{g}$ on a vector space V. When studying deformations of the $\mathfrak{g}$-action ρ, one usually starts with infinitesimal deformations:

$$\rho = \rho_0 + tC,$$

where $C : \mathfrak{g} \to \mathrm{End}(V)$ is a linear map and t is a formal parameter. The homomorphism condition

$$[\rho(x), \rho(y)] = \rho([x, y]),$$

where $x, y \in \mathfrak{g}$, is satisfied in order 1 in t if and only if C is a 1-cocycle. The first cohomology space $H^1(\mathfrak{g}; \mathrm{End}(V))$ classifies infinitesimal deformations up to equivalence (see, e.g., [4, 6]).

In the case when this space of cohomology is multi-dimensional, it is natural to consider multi-parameter deformations. If $\dim H^1(\mathfrak{g}; \mathrm{End}(V)) = m$, then choose 1-cocycles $C_1, \ldots, C_m$ representing a basis of $H^1(\mathfrak{g}; \mathrm{End}(V))$ and consider the infinitesimal deformation

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i C_i, \tag{3.1}$$

with independent parameters $t_1, \ldots, t_m$.

In our study, the first cohomology space is determined by the formula (2.3). An infinitesimal deformation of the $\mathrm{Vect}(\mathbb{R}^n)$-action on $\Omega(\mathbb{R}^n)$ is then of the form $\mathcal{L}_X = L_X + \mathcal{L}_X^{(1)}$, where L_X is the Lie derivative of differential forms along the vector field X, and

$$\mathcal{L}_X^{(1)} = \sum_{k=0}^{n} t_0^k C_0^k(X) + \sum_{k=0}^{n-1} \left(t_1^k C_1^k(X) + \widetilde{t}_1^k \widetilde{C}_1^k(X) \right) + \sum_{k=0}^{n-2} t_2^k C_2^k(X), \tag{3.2}$$

and where t_0^k, t_1^k, $\widetilde{t}_1^k$ and t_2^k are $4n$ independent parameters and the cocycles C_0^k, C_1^k, $\widetilde{C}_1^k$ and C_2^k are defined by formulae (2.4)–(2.6).

3.2 Integrability conditions and deformations over commutative algebras

Consider the problem of integrability of infinitesimal deformations. Starting with the infinitesimal deformation (3.1), one looks for a formal series

$$\rho = \rho_0 + \sum_{i=1}^{m} t_i C_i + \sum_{i,j} t_i t_j \rho_{ij}^{(2)} + \cdots, \tag{3.3}$$

where the highest-order terms $\rho_{ij}^{(2)}$, $\rho_{ijk}^{(3)}$, … are linear maps from $\mathfrak{g}$ to $\mathrm{End}(V)$ such that

$$\rho : \mathfrak{g} \to \mathrm{End}(V) \otimes \mathbb{C}[[t_1, \dots, t_m]]$$

satisfy the homomorphism condition in any order in $t_1, \dots, t_m$.

However, quite often the above problem has no solution. Following [3] and [1], we will impose extra algebraic relations on the parameters $t_1, \dots, t_m$. Let $\mathcal{R}$ be an ideal in $\mathbb{C}[[t_1, \dots, t_m]]$ generated by some set of relations, the quotient

$$\mathcal{A} = \mathbb{C}[[t_1, \dots, t_m]]/\mathcal{R} \tag{3.4}$$

is a commutative associative algebra with unity, and one can speak of a deformations with base $\mathcal{A}$, see [3] for details. The map (3.3) sends $\mathfrak{g}$ to $\mathrm{End}(V) \otimes \mathcal{A}$.

Example 3.1. Consider the ideal $\mathcal{R}$ generated by all the quadratic monomials $t_i t_j$. In this case

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{C}^m \tag{3.5}$$

and any deformation is of the form (3.1). In this case any infinitesimal deformation becomes a deformation with base $\mathcal{A}$ since $t_i t_j = 0$ in $\mathcal{A}$, for all $i, j = 1, \dots, m$.

Given an infinitesimal deformation (3.1), one can always consider it as a deformation with base (3.5). Our aim is to find $\mathcal{A}$ which is big as possible, or, equivalently, we look for relations on $t_1, \dots, t_m$ which are necessary and sufficient for integrability (cf. [1]).

3.3 Equivalence and the miniversal deformation

The notion of equivalence of deformations over commutative associative algebras has been considered in [3].

Definition 3.2. Two deformations ρ and ρ' with the same base $\mathcal{A}$ are called equivalent if there exists an inner automorphism Ψ of the associative algebra $\mathrm{End}(V) \otimes \mathcal{A}$ such that

$$\Psi \circ \rho = \rho'$$

and such that $\Psi(\mathbb{I}) = \mathbb{I}$, where $\mathbb{I}$ is the unity of the algebra $\mathrm{End}(V) \otimes \mathcal{A}$.

The following notion of miniversal deformation is fundamental. It assigns to a $\mathfrak{g}$-module V a canonical commutative associative algebra $\mathcal{A}$ and a canonical deformation with base $\mathcal{A}$.

Definition 3.3. A deformation (3.3) with base $\mathcal{A}$ is called miniversal, if for any other deformation ρ' with base $\mathcal{A}'$ there exists a unique homomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ satisfying $\psi(1) = 1$, such that

$$\rho' = (\mathrm{Id} \otimes \psi) \circ \rho.$$

This definition does not depend on the choice of the basis $C_1, \ldots, C_m$.

The miniversal deformation corresponds to the smallest ideal $\mathcal{R}$. We refer to [3] for a construction of miniversal deformations of Lie algebras and to [1] for miniversal deformations of $\mathfrak{g}$-modules.

4 The main result

In this section we obtain the integrability conditions for the infinitesimal deformation (3.2). The main result of this paper is the following

Theorem 4.1. *(i) The following $4n-4$ conditions*

$$\begin{aligned}
&R_1^k(t) = t_0^k \widetilde{t}_1^k + t_0^{k+1} t_1^k = 0, && \textit{for} \quad k = 0, \ldots, n-1,\\
&R_2^k(t) = t_0^k t_2^k + t_1^{k+1} t_1^k = 0, && \textit{for} \quad k = 0, \ldots, n-2,\\
&\widetilde{R}_2^k(t) = t_0^{k+2} t_2^k + \widetilde{t}_1^{k+1} \widetilde{t}_1^k = 0, && \textit{for} \quad k = 0, \ldots, n-2,\\
&R_3^k(t) = t_1^{k+2} t_2^k + \widetilde{t}_1^k t_2^{k+1} = 0, && \textit{for} \quad k = 0, \ldots, n-3
\end{aligned} \tag{4.1}$$

are necessary and sufficient for integrability of the infinitesimal deformation (3.2).

(ii) Any formal deformation of the $\mathrm{Vect}(\mathbb{R}^n)$*-action on* $\Omega(\mathbb{R}^n)$ *is equivalent to a deformation of order 1, that is, to a deformation given by (3.2).*

The commutative algebra defined by relations (4.1) corresponds to the miniversal deformation of the Lie derivative L_X. Note that the commutative algebra defined in Theorem 4.1 is infinite-dimensional.

We start the proof of Theorem 4.1. It consists in two steps. First, we compute explicitly the obstructions for existence of the second-order term, this will prove that relations (4.1) are necessary. Second we show that under relations (4.1) the highest-order terms of the deformation can be chosen identically zero, so that relations (4.1) are indeed sufficient.

4.1 Computing the second-order Maurer–Cartan equation

Assume that the infinitesimal deformation (3.2) can be integrated to a formal deformation

$$\mathcal{L}_X = L_X + \mathcal{L}_X^{(1)} + \mathcal{L}_X^{(2)} + \cdots ,$$

where $\mathcal{L}_X^{(1)}$ is given by (3.2) and $\mathcal{L}_X^{(2)}$ is a quadratic polynomial in t with coefficients in $\mathcal{D}(\Omega(\mathbb{R}^n))$. We compute the conditions for the second-order terms $\mathcal{L}^{(2)}$. Consider the quadratic terms of the homomorphism condition

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}. \tag{4.2}$$

The left hand side of (4.2) contains the operators from $\Omega^k(\mathbb{R}^n)$ to $\Omega^{k+i}(\mathbb{R}^n)$, for every $k = 0, \ldots, n$ and $i = 1, 2, 3$. Condition (4.2) must be satisfied in any order of i independently.

Collecting the terms with $i = 1$, one readily gets:

$$R_1^k(t)\gamma_1(X, Y),$$

where γ_1 is the bilinear skew-symmetric map from $\mathrm{Vect}(\mathbb{R}^n)$ to $\mathcal{D}^{k,k+1}$ defined by

$$\gamma_1(X, Y) = (\mathrm{Div}(X)d(\mathrm{Div}(Y)) - \mathrm{Div}(Y)d(\mathrm{Div}(X))) \wedge . \tag{4.3}$$

The terms with $i = 2$ in (4.2) are as follows

$$R_2^k(t)\gamma_2(X, Y) + \widetilde{R}_2^k(t)\,\widetilde{\gamma}_2(X, Y),$$

where γ_2 and $\widetilde{\gamma}_2$ are the bilinear skew-symmetric maps from $\mathrm{Vect}(\mathbb{R}^n)$ to $\mathcal{D}^{k,k+2}$ defined by

$$\gamma_2(X, Y) = (\mathrm{Div}(X)d(\mathrm{Div}(Y)) - \mathrm{Div}(Y)d(\mathrm{Div}(X))) \wedge d \tag{4.4}$$

and

$$\widetilde{\gamma}_2(X, Y) = (d(\mathrm{Div}(X)) \wedge d(\mathrm{Div}(Y)) - d(\mathrm{Div}(Y)) \wedge d(\mathrm{Div}(X))) \wedge . \tag{4.5}$$

Finally, the terms with $i = 3$ in (4.2) are as follows

$$R_3^k(t)\gamma_3(X, Y),$$

where γ_3 is the bilinear skew-symmetric map from $\mathrm{Vect}(\mathbb{R}^n)$ to $\mathcal{D}^{k,k+3}$ defined by

$$\gamma_3(X, Y) = (d(\mathrm{Div}(X)) \wedge d(\mathrm{Div}(Y)) - d(\mathrm{Div}(Y)) \wedge d(\mathrm{Div}(X))) \wedge d, \tag{4.6}$$

where the coefficients $R_1^k(t)$, $R_2^k(t)$, $\widetilde{R}_2^k$ and R_3^k are precisely the quadratic polynomial from (4.1).

The homomorphism condition (4.2) gives for the second-order terms the following (Maurer–Cartan) equation

$$\delta(\mathcal{L}^{(2)}) = -\frac{1}{2}[\![\mathcal{L}^{(1)}, \mathcal{L}^{(1)}]\!],$$

where δ is the Chevalley–Eilenberg differential and $[\![\,,\,]\!]$ stands for the cup-product of 1-cocycles, so that the right hand side of this expression is automatically a 2-cocycle. In our case, we obtain explicitly:

$$\delta(\mathcal{L}^{(2)}) = R_1^k(t)\gamma_1 + R_2^k(t)\gamma_2 + \widetilde{R}_2^k(t)\,\widetilde{\gamma}_2 + R_3^k(t)\gamma_3. \tag{4.7}$$

The bilinear maps γ_1, γ_2, $\widetilde{\gamma}_2$ and γ_3 are 2-cocycles on $\mathrm{Vect}(\mathbb{R}^n)$. For existence of solutions of the equation (4.7) it is necessary and sufficient that the right hand side be a coboundary.

4.2 Obstructions for integrability

We denote γ_1^k, γ_2^k, $\widetilde{\gamma}_2^k$ and γ_3^k the 2-cocycles on $\mathrm{Vect}(\mathbb{R}^n)$ such that

$$\gamma_i^k(X,Y) = \gamma_i(X,Y)\big|_{\Omega^k(\mathbb{R}^n)}$$

for every $X, Y \in \mathrm{Vect}(\mathbb{R}^n)$. In order to solve the equation (4.7), we now have to study the cohomology classes of these 2-cocycles in $H^2(\mathrm{Vect}(\mathbb{R}^n); \mathcal{D}^{k,k+i})$.

Proposition 4.2. *Each of the 2-cocycles:*

$$\begin{array}{lll} \gamma_1^k, & \textit{for} \quad k = 0, \dots, n-1, & \\ \gamma_2^k, \ \widetilde{\gamma}_2^k, & \textit{for} \quad k = 0, \dots, n-2 & \textit{and} \quad n \geq 2, \\ \gamma_3^k, & \textit{for} \quad k = 0, \dots, n-3 & \textit{and} \quad n \geq 3 \end{array}$$

define non-trivial cohomology class. Moreover, these cohomology classes are linearly independent.

Proof. Assume that, for some differential 1-cohain b on $\mathrm{Vect}(\mathbb{R}^n)$ with coefficients in $\mathcal{D}^{k,k+i}$, one has $\gamma_i^k = \delta(b)$. The general form of such a cochain is

$$b(X) = b_{\ell\, a_1 \dots a_r}^{i_1 \dots i_s j_1 \dots j_t m_1 \dots m_r}(x)\, \frac{\partial^s X^\ell}{\partial x^{i_1} \cdots \partial x^{i_s}}\, \xi^{a_1} \cdots \xi^{a_r} \partial_{x^{j_1}} \cdots \partial_{x^{j_t}} \partial_{\xi^{m_1}} \cdots \partial_{\xi^{m_u}}.$$

We will need the following:

Lemma 4.3. *The condition $\delta(b) = \gamma_i^k$ implies that the coefficients $b_{\ell\, a_1 \dots a_r}^{i_1 \dots i_s j_1 \dots j_t m_1 \dots m_r}$ are constants.*

Proof. The condition $\gamma_i^k = \delta(b)$ reads

$$\gamma_i^k(X,Y) = [L_X, b(Y)] - [L_Y, b(X)] - b([X,Y]). \tag{4.8}$$

We choose a constant vector field $Y = \partial_{x^i}$ and prove that $L_Y(b) = 0$. Indeed, one has

$$(L_Y(b))(X) = [L_Y, b(X)] - b([Y,X]).$$

Since $s \geq 2$ in the expression of b, it follows that $b(Y) = 0$, and thus the last equality gives

$$(L_Y(b))(X) = [L_Y, b(X)] - b([Y,X]) - [L_X, b(Y)] = (\delta b)(Y,X).$$

By assumption, $\delta b = \gamma_i^k$, from the explicit formula for γ_i^k, see (4.3)–(4.6) one obtains $\gamma_i^k(Y,X) = 0$ for all X, since $\mathrm{Div}(Y) = 0$. Therefore, $L_Y(b) = 0$.

Lemma 4.3 is proved. ■

It is now easy to check that the equation (4.8) has no solutions: using the formula (2.2), we see that the expressions $\mathrm{Div}(X)$ and $\mathrm{Div}(Y)$ never appear in the right hand side of (4.8). This is a contradiction with the assumption $\gamma_i^k = \delta(b)$.

Proposition 4.2 is proved. ■

Proposition 4.2 implies that the equation (4.7) has solutions if and only if the quadratic polynomials $R_1^k(t)$, $R_2^k(t)$, $\widetilde{R}_2^k(t)$ and $R_3^k(t)$ vanish simultaneously. We thus proved that the conditions (4.1) are, indeed, necessary.

4.3 Integrability conditions are sufficient

To prove that the conditions (4.1) are sufficient, we will find explicitly a deformation of L_X, whenever the conditions (4.1) are satisfied. The solution $\mathcal{L}^{(2)}$ of (4.7) can be chosen identically zero. Choosing the highest-order terms $\mathcal{L}^{(m)}$ with $m \geq 3$, also identically zero, one obviously obtains a deformation (which is of order 1 in t).

Theorem 4.1, part (i) is proved.

The solution $\mathcal{L}^{(2)}$ of (4.7) is defined up to a 1-cocycle and it has been shown in [3, 1] that different choices of solutions of the Maurer–Cartan equation correspond to equivalent deformations. Thus, one can always reduce $\mathcal{L}^{(2)}$ to zero by equivalence. Then, by recurrence, the highest-order terms $\mathcal{L}^{(m)}$ satisfy the equation $\delta(\mathcal{L}^{(m)}) = 0$ and can also be reduced to the identically zero map. This completes the proof of part (ii).

Theorem 4.1 is proved.

5 An open problem

It seems to be an interesting open problem to compute the full cohomology ring $H^*(\mathrm{Vect}(\mathbb{R}^n); \mathcal{D}^{k,\ell})$. The only complete result here concerns the first cohomology space, see [2]. Proposition 4.2 provides a lower bound for the dimension of the second cohomology space. We formulate

Conjecture 5.1. *The space of second cohomology of* $\mathrm{Vect}(\mathbb{R}^n)$ *with coefficients in the space of differential operators on* $\Omega(\mathbb{R}^n)$ *has the following structure*

$$H^2(\mathrm{Vect}(\mathbb{R}^n); \mathcal{D}^{k,\ell}) = \begin{cases} \mathbb{R}, & \ell = k+1, \\ \mathbb{R}^2, & \ell = k+2, \\ \mathbb{R}, & \ell = k+3, \\ 0, & \textit{otherwise} \end{cases}$$

and spanned by the 2*-cocycles* γ_1^k, γ_2^k, $\tilde{\gamma}_2^k$ *and* γ_3^k *given by (4.3)–(4.6).*

6 A few examples of deformations

We give some examples of deformations of the $\mathrm{Vect}(\mathbb{R}^n)$-action (2.2) on $\Omega(\mathbb{R}^n)$ which are simpler than the general case described by Theorem 4.1.

Example 6.1. The first example is a 1-parameter deformation

$$\mathcal{L}_X = L_X + tC_0(X),$$

that is, we put $t_0^k = t$ and $t_1^k = \widetilde{t}_1^k = t_2^k = 0$. This deformation has a geometric meaning. More precisely, we consider the $\mathrm{Vect}(\mathbb{R}^n)$-module of (generalized) tensor fields $\Omega(\mathbb{R}^n) \otimes_{C^\infty(\mathbb{R}^n)} \mathcal{F}_t(\mathbb{R}^n)$, where $\mathcal{F}_t(\mathbb{R}^n)$ is the space of tensor densities of degree t on $\mathbb{R}^n$.

Example 6.2. Another simple example of a 1-parameter deformation is given by

$$\mathcal{L}_X = L_X + tC_2(X).$$

The geometric meaning of this deformation is not clear.

Proposition 6.3. *The above two examples are the only 1-parameter deformations of the form*

$$\mathcal{L}_X = L_X + t\left(\alpha_0 C_0(X) + \alpha_1 C_1(X) + \widetilde{\alpha}_1 \widetilde{C}_1(X) + \alpha_2 C_2(X)\right),$$

that is, with $t_0^k = \alpha_0 t$, $t_1^k = \alpha_1 t$, $\widetilde{t}_1^k = \widetilde{\alpha}_1 t$ *and* $t_2^k = \alpha_2 t$ *for all* k.

Proof. Straightforward from the relations (4.1). ■

Example 6.4. We give an example of 3-parameter deformations in the two-dimensional case, i.e., for $n = 2$

$$\mathcal{L}_X = L_X + t_0^2\left(C_0^2(X) - \widetilde{C}_1^0(X)\right) + t_1^0 C_1^0(X) + \widetilde{t}_1^1\left(\widetilde{C}_1^1(X) + C_2^0(X)\right),$$

where t_0^2, t_1^0 and $\widetilde{t}_1^1$ are independent parameters.

One can construct a great number of examples of deformations with 3 independent parameters in the multi-dimensional case; it would be interesting to understand their geometric meaning.

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