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Superconformal Algebras and Lie Superalgebras of the Hodge Theory

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Abstract

We observe a correspondence between the zero modes of superconformal algebras $S'(2,1)$ and $W(4)$ ([8]) and the Lie superalgebras formed by classical operators appearing in the Kähler and hyper-Kähler geometry.

1 Lie superalgebras of the Hodge theory

1.1 Kähler manifolds

Let $M = (M^{2n}, g, I, \omega)$ be a compact Kähler manifold of real dimension $2n$, where g is a Riemannian (Kähler) metric, I is a complex structure on M, and ω is the corresponding closed 2-form defined by $\omega(x, y) = g(x, I(y))$ for any vector fields x and y [10].

A number of classical operators on the Dolbeault algebra $A^{*,*}(M)$ of complex differential forms on M is well-known [5]: the exterior differential d and its holomorphic and antiholomorphic parts, ∂ and $\bar\partial$, and $d_c = i(\partial - \bar\partial)$, their dual operators, and the associated Laplacians. Recall that

$$\partial : A^{p,q}(M) \to A^{p+1,q}(M), \qquad \bar\partial : A^{p,q}(M) \to A^{p,q+1}(M), \qquad d = \partial + \bar\partial. \tag{1.1}$$

The Hodge operator $\star : A^{p,q}(M) \longrightarrow A^{n-q,n-p}(M)$, satisfies $\star^2 = (-1)^{p+q}$ on $A^{p,q}(M)$. Accordingly, the Hodge inner product is defined on each of $A^{p,q}(M)$: $(\varphi, \psi) = \int_M \varphi \wedge \star\bar\psi$. Recall that $\triangle = dd^* + d^*d = 2\triangle_\partial = 2\triangle_{\bar\partial}$. In addition, $A^{*,*}(M)$ admits an $\mathfrak{sl}(2)$-module structure, where $\mathfrak{sl}(2) = \langle E, H, F\rangle$ and the generators satisfy

$$[E, F] = H, \qquad [H, E] = 2E, \qquad [H, F] = -2F. \tag{1.2}$$

The operator $E : A^{p,q}(M) \to A^{p+1,q+1}(M)$ is defined by $E(\varphi) = \varphi \wedge \omega$. (Clearly, ω is a $(1,1)$-form). Let $F = E^* : A^{p,q}(M) \to A^{p-1,q-1}(M)$ be its dual operator, and $H|_{A^{p,q}(M)} = p + q - n$. According to the Lefschetz theorem, there exists the corresponding action of $\mathfrak{sl}(2)$ on $H^*(M)$ [5]. These operators satisfy a series of identities, known as the *Hodge identities* [5]. Let $\mathcal{K}$ be the Lie superalgebra, whose even part is spanned by $\mathfrak{sl}(2) = \langle E, H, F\rangle$ and the Laplace operator $\triangle$, and the odd part is spanned by the differentials d, d^*, d_c, d_c^*. The non-vanishing commutation relations in $\mathcal{K}$ are (1.2) and the following relations (see [5]):

$$\begin{aligned}&[d,d^*]=[d_c,d_c^*]=\triangle,\\ &[H,d]=d,\qquad [H,d^*]=-d^*,\qquad [H,d_c]=d_c,\qquad [H,d_c^*]=-d_c^*,\\ &[E,d^*]=-d_c,\qquad [E,d_c^*]=d,\qquad [F,d]=d_c^*,\qquad [F,d_c]=-d^*.\end{aligned} \tag{1.3}$$

Thus $\mathcal{K}=\mathfrak{sl}(2)\supset\!\!\!\!+\ \mathfrak{hei}(0|4)$, where $\mathfrak{hei}(0|4)$ is the Heisenberg Lie superalgebra: $\mathfrak{hei}(0|4)_{\bar 1}=\langle d,d_c^*\rangle\oplus\langle d_c,d^*\rangle$ is a direct sum of two isotropic subspaces with respect to the non-degenerate symmetric form given by $(d,d^*)=(d_c,d_c^*)=1$, and $\mathfrak{hei}(0|4)_{\bar 0}=\langle\triangle\rangle$ is the center. The isotropic subspaces are standard $\mathfrak{sl}(2)$-modules. Since $\mathfrak{sl}(2)\simeq\mathfrak{sp}(2)$, the following is a natural generalization.

1.2 Hyper-Kähler manifolds

Let M be a compact hyper-Kähler manifold. By definition, M is a Riemannian manifold endowed with three complex structures I, J, and K, such that $I\circ J=-J\circ I=K$ and M is Kähler with respect to each of the complex structures I, J and K. Let ω_I, ω_J and ω_K be the corresponding closed 2-forms on M. Having fixed one of the complex structures, for example, I, we obtain the Hodge theory as in the Kähler case.

For each of the complex structures I, J and K, the operators E_i, E_j and E_k and their dual operators F_i, F_j and F_k with respect to the Hodge inner product are naturally defined. One can also define differentials, d_c^l and $(d_c^l)^*$, where $l=i,j,k$ for I, J and K. Set

$$d_c^1=d,\qquad (d_c^1)^*=d^*. \tag{1.4}$$

Let $Q=\{1,i,j,k\}$ be the set of indices satisfying the quaternionic identities.

In the hyper-Kähler case there is a natural action of the Lie algebra $\mathfrak{so}(5)\simeq\mathfrak{sp}(4)$ on $H^*(M)$ [13]. The $\mathfrak{sp}(4)$ is spanned by the operators E_i, F_i, K_i, where i runs through the set $Q\setminus\{1\}$, and H. The non-vanishing commutation relations are

$$\begin{aligned}&[E_i,F_i]=H,\qquad [H,E_i]=2E_i,\qquad [H,F_i]=-2F_i,\qquad [E_i,F_j]=K_{ij},\\ &[K_i,K_j]=-2K_{ij},\qquad K_{ij}=-K_{ji},\qquad [K_i,E_j]=-2E_{ij},\qquad E_{ij}=-E_{ji},\\ &[K_i,F_j]=-2F_{ij},\qquad F_{ij}=-F_{ji},\qquad i\neq j.\end{aligned} \tag{1.5}$$

Let $\mathcal{H}$ be a Lie superalgebra, whose even part is spanned by the $\mathfrak{sp}(4)$ and the Laplace operator $\triangle$, and the odd part is spanned by the differentials d_c^l and $(d_c^l)^*$, where $l\in Q$. Thus $\dim\mathcal{H}=(11|8)$. The non-vanishing commutation relations in $\mathcal{H}$ are (1.5) and the following relations (cf. [2, 14, 15]):

$$\begin{aligned}&[d_c^l,(d_c^l)^*]=\triangle,\qquad [H,d_c^l]=d_c^l,\qquad [H,(d_c^l)^*]=-(d_c^l)^*,\\ &[E_i,(d_c^l)^*]=-d_c^{il},\qquad d_c^{-l}=-d_c^l,\qquad [F_i,d_c^l]=(d_c^{il})^*,\\ &(d_c^{-l})^*=-(d_c^l)^*,\qquad [K_i,d_c^l]=-d_c^{il},\qquad [K_i,(d_c^l)^*]=-(d_c^{il})^*.\end{aligned} \tag{1.6}$$

Thus $\mathcal{H}=\mathfrak{sp}(4)\supset\!\!\!\!+\ \mathfrak{hei}(0|8)$, where $\mathfrak{hei}(0|8)$ is the Heisenberg Lie superalgebra: $\mathfrak{hei}(0|8)_{\bar 1}=\langle d_c^l,(d_c^l)^*|\ l\in Q\rangle$, $\mathfrak{hei}(0|8)_{\bar 0}=\langle\triangle\rangle$, where $\langle\triangle\rangle$ is the center. $\mathfrak{hei}(0|8)_{\bar 1}=V_1\oplus V_2$ is a direct sum of two isotropic subspaces with respect to the non-degenerate symmetric form: $(d^a_c, (d^b_c)^*) = \delta_{ab}$ for $a, b \in Q$;

$$\begin{aligned} V_1 = \langle & d^1_c + \sqrt{-2} d^j_c - d^k_c,\ d^1_c - \sqrt{-2} d^i_c + d^k_c, \\ & (d^i_c)^* + (d^j_c)^* + \sqrt{-2}(d^k_c)^*,\ \sqrt{-2}(d^1_c)^* + (d^i_c)^* - (d^j_c)^* \rangle, \\ V_2 = \langle & d^1_c - \sqrt{-2} d^i_c - d^k_c,\ d^1_c - \sqrt{-2} d^j_c + d^k_c, \\ & (d^i_c)^* - (d^j_c)^* - \sqrt{-2}(d^k_c)^*,\ \sqrt{-2}(d^1_c)^* + (d^i_c)^* + (d^j_c)^* \rangle. \end{aligned} \tag{1.7}$$

The subspaces V_1 and V_2 are irreducible $\mathfrak{sp}(4)$-modules.

2 Superconformal algebras

A *superconformal algebra* (SCA [8, 9]) is a complex $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \oplus_i \mathfrak{g}_i$, such that $\mathfrak{g}$ is simple, $\mathfrak{g}$ contains the centerless Virasoro algebra, i.e. the Witt algebra, $L = \oplus_{n\in\mathbb{Z}} \mathbb{C} L_n$ with the commutation relations $[L_m, L_n] = (m - n)L_{m+n}$ as a subalgebra, and adL_0 is diagonalizable with finite-dimensional eigenspaces: $\mathfrak{g}_i = \{x \in \mathfrak{g} \mid [L_0, x] = ix\}$, so that dim $\mathfrak{g}_i < C$, where C is a constant independent of i. (Other definitions of superconformal algebras, embracing central extensions, are also popular, see [4]; for an intrinsic definition see [6]).

In general, a SCA is spanned by a number of fields; the Virasoro field is among them. The basic example of a SCA is $W(N)$. Let $\Lambda(N)$ be the Grassmann algebra in N variables $\theta_1, \ldots, \theta_N$. Let $\Lambda(1, N) = \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$ be a supercommutative superalgebra with natural multiplication and with the following parity of generators: $p(t) = \bar{0}$, $p(\theta_i) = \bar{1}$ for $i = 1, \ldots, N$. By definition, $W(N)$ is the Lie superalgebra of all derivations of $\Lambda(1, N)$. Let ∂_t stand for $\frac{\partial}{\partial t}$ and ∂_i stand for $\frac{\partial}{\partial \theta_i}$.

The superalgebra $W(N)$ contains a one-parameter family of SCAs $S'(N, \alpha)$. By definition,

$$S(N, \alpha) = \{D \in W(N) \mid \mathrm{Div}(t^\alpha D) = 0\} \qquad \text{for} \quad \alpha \in \mathbb{C}, \tag{2.1}$$

where $\mathrm{Div}\left(f\partial_t + \sum\limits_{i=1}^{N} f_i \partial_i\right) = \partial_t f + \sum\limits_{i=1}^{N} (-1)^{p(f_i)} \partial_i f_i$ for $f, f_i \in \Lambda(1, N)$. The derived superalgebra $S'(N, \alpha) = [S(N, \alpha), S(N, \alpha)]$ is simple.

Let $\mathfrak{g} = S'(2, 1)$ or $W(4)$, respectively. Let

$$L_n = -t^{n+1}\partial_t - \frac{1}{2}(n + 2)t^n \sum_{i=1}^{N} \theta_i \partial_i, \tag{2.2}$$

and let $\mathfrak{g}_0$ be singled out by L_0. There exist an isomorphisms $\varphi : \mathcal{K} \longrightarrow S'(2, 1)_0$, and a monomorphism $\psi : \mathcal{H} \longrightarrow W(4)_0$ in the case of a compact Kähler or hyper-Kähler manifold, respectively.

2.1 Kähler manifolds

Let $N = 2$. $S'(2, 1)$ is spanned by 4 bosonic fields L_n, H_n, E_n, F_n, where E_n, F_n and H_n form the loop algebra of $\mathfrak{sl}(2)$, and 4 fermionic fields X^i_n, Y^i_n, $i = 1, 2$:

$$H_n = t^n(\theta_1\partial_1 - \theta_2\partial_2), \qquad E_n = t^n\theta_1\partial_2, \qquad F_n = t^n\theta_2\partial_1,$$

$$X^1_n = t^n\theta_1\partial_t + (n+1)t^{n-1}\theta_1\theta_2\partial_2, \qquad X^2_n = -t^{n+1}\partial_2,$$
$$Y^1_n = -t^{n+1}\partial_1, \qquad Y^2_n = t^n\theta_2\partial_t + (n+1)t^{n-1}\theta_2\theta_1\partial_1. \tag{2.3}$$

The commutation relations between L_n and the fields, defined by (2.3), are

$$[L_n, H_m] = -mH_{n+m}, \qquad [L_n, E_m] = -mE_{n+m}, \qquad [L_n, F_m] = -mF_{n+m}, \tag{2.4}$$
$$[L_n, X^i_m] = \left(\frac{n}{2} - m\right) X^i_{n+m}, \qquad [L_n, Y^i_m] = \left(\frac{n}{2} - m\right) Y^i_{n+m}, \qquad i = 1, 2. \tag{2.5}$$

Clearly, the fields X^i_n and Y^i_n, where $i = 1, 2$, generate $S'(2, 1)$.

The Lie superalgebra $S'(2, 1)_0$ is isomorphic to the Lie superalgebra $\mathcal{K}$ of classical operators in Kähler geometry. The isomorphism φ is as follows:

$$\varphi(\triangle) = L_0, \qquad \varphi(H) = H_0, \qquad \varphi(E) = E_0, \qquad \varphi(F) = F_0,$$
$$\varphi(d) = X^1_0, \qquad \varphi(d^*) = Y^1_0, \qquad \varphi(d_c) = X^2_0, \qquad \varphi(d^*_c) = Y^2_0. \tag{2.6}$$

2.2 Hyper-Kähler manifolds

Let $N = 4$. The following 10 bosonic fields span a subalgebra of $W(4)$ isomorphic to the loop algebra of $\mathfrak{sp}(4)$:

$$H_n = t^n(\theta_1\partial_1 + \theta_2\partial_2 - \theta_3\partial_3 - \theta_4\partial_4), \qquad E^i_n = t^n(\theta_1\partial_4 + \theta_2\partial_3),$$
$$F^i_n = t^n(\theta_3\partial_2 + \theta_4\partial_1), \qquad E^j_n = it^n(\theta_1\partial_3 + \theta_2\partial_4), \qquad F^j_n = -it^n(\theta_3\partial_1 + \theta_4\partial_2),$$
$$E^k_n = t^n(\theta_1\partial_3 - \theta_2\partial_4), \qquad F^k_n = t^n(\theta_3\partial_1 - \theta_4\partial_2),$$
$$K^i_n = it^n(\theta_1\partial_1 - \theta_2\partial_2 - \theta_3\partial_3 + \theta_4\partial_4), \qquad K^j_n = t^n(\theta_1\partial_2 - \theta_2\partial_1 + \theta_3\partial_4 - \theta_4\partial_3),$$
$$K^k_n = -it^n(\theta_1\partial_2 + \theta_2\partial_1 - \theta_3\partial_4 - \theta_4\partial_3). \tag{2.7}$$

Define 8 fermionic fields X^l_n, Y^l_n, where $l \in Q$. Let

$$A^m_n = t^n\theta_m\partial_t + (n+1)t^{n-1}\theta_m \sum_{i=1}^{4} \theta_i\partial_i, \qquad m = 1, \ldots, 4. \tag{2.8}$$

Let $\mathbf{X} = (x_{lm})$ and $\mathbf{Y} = (y_{lm})$ be the following complex 4×4 matrices, where $l = 1, i, j, k$ and $m = 1, \ldots, 4$:

$$\mathbf{X} = \begin{pmatrix} 1 & -1 & i & i \\ -i & -i & 1 & -1 \\ 1 & 1 & -i & i \\ -i & i & -1 & -1 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} -1 & 1 & i & i \\ -i & -i & -1 & 1 \\ -1 & -1 & -i & i \\ -i & i & 1 & 1 \end{pmatrix}. \tag{2.9}$$

Set

$$X^l_n = \frac{1}{2}\sum_{m=1}^{2} x_{lm}A^m_n + \frac{1}{2}\sum_{m=3}^{4} x_{lm}t^{n+1}\partial_m,$$
$$Y^l_n = \frac{1}{2}\sum_{m=1}^{2} y_{lm}t^{n+1}\partial_m + \frac{1}{2}\sum_{m=3}^{4} y_{lm}A^m_n. \tag{2.10}$$

The commutation relations between L_n and the fields, defined by (2.7) and (2.10), are analogs of the relations (2.4) and (2.5), respectively.

The zero modes of the Virasoro field L_n and of the fields, defined by (2.7) and (2.10), span a Lie superalgebra, which is isomorphic to the Lie superalgebra $\mathcal{H}$ of classical operators in hyper-Kähler geometry.

The monomorphism ψ is given by the following formulas:

$$\begin{aligned}
&\psi(\triangle) = L_0, \qquad \psi(H) = H_0, \qquad \psi(E_i) = E_0^i, \qquad \psi(F_i) = F_0^i, \\
&\psi(K_i) = K_0^i, \qquad \psi(d_c^l) = X_0^l, \qquad \psi((d_c^l)^*) = Y_0^l,
\end{aligned} \tag{2.11}$$

where i runs through the set $Q \setminus \{1\}$, and $l \in Q$.

Statement. *The fields X_n^l and Y_n^l, for all $l \in Q$, generate $W(4)$.*

3 Conclusion

It is natural to expect that "affinization" of the classical operators in the case of an infinite-dimensional manifold gives a SCA, which should act on a relevant cohomology complex. Recall that the Weil complex is used for definition of the equivariant differential forms. The infinite-dimensional generalization of the classical Weil complex is the semi-infinite Weil complex of a graded Lie algebra [1].

In particular, let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra of a complex finite-dimensional Lie algebra $\mathfrak{g}$. Naturally, $\tilde{\mathfrak{g}} = \oplus_{n\in\mathbb{Z}}\tilde{\mathfrak{g}}_n$. Let $\tilde{\mathfrak{g}}' = \oplus_{n\in\mathbb{Z}}\tilde{\mathfrak{g}}'_n$ be the restricted dual of $\tilde{\mathfrak{g}}$. The linear space $U = \tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}'$ can be naturally endowed with non-degenerate skew-symmetric and symmetric bilinear forms: $(\cdot,\cdot)$ and $\{\cdot,\cdot\}$. The Weyl and Clifford algebras, $W(\tilde{\mathfrak{g}})$ and $C(\tilde{\mathfrak{g}})$, are the quotients of the tensor algebra $T^*(U)$ modulo the two-sided ideals generated by the elements of the form $xy - yx - (x, y)$ and $xy + yx - \{x, y\}$ for any $x, y \in U$, respectively; here $xy := x \otimes y$. Let u run through a fixed basis of $\mathfrak{g}$ and u' run through the dual basis. Let $\beta(u_m)$, $\gamma(u'_m)$ and $\tau(u_m)$, $\varepsilon(u'_m)$, where $m \in \mathbb{Z}$, be generators of $W(\tilde{\mathfrak{g}})$ and $C(\tilde{\mathfrak{g}})$, respectively. We can realize $S'(2, 1)$ in terms of the following quadratic expansions:

$$\begin{aligned}
&L_n = \sum_{u,m} m : \tau(u_{m-n})\varepsilon(u'_m) : + m : \beta(u_{m-n})\gamma(u'_m) : - \frac{n}{2} : \beta(u_m)\gamma(u'_{m+n}) : , \\
&H_n = -\sum_{u,m} : \beta(u_m)\gamma(u'_{m+n}) : , \qquad E_n = -\frac{i}{2}\sum_{u,m}\gamma(u'_m)\gamma(u'_{n-m}), \\
&F_n = -\frac{i}{2}\sum_{u,m}\beta(u_m)\beta(u_{-m-n}), \qquad X_n^1 = \sum_{u,m}\gamma(u'_{m+n})\tau(u_m), \\
&X_n^2 = i\sum_{u,m} m\gamma(u'_{m-n})\varepsilon(u'_m), \qquad Y_n^1 = \sum_{u,m} m\beta(u_{m-n})\varepsilon(u'_m), \\
&Y_n^2 = i\sum_{u,m}\beta(u_m)\tau(u_{-m-n}),
\end{aligned} \tag{3.1}$$

where the double colons : : denote a normal ordering operation:

$$: \tau(u_j)\varepsilon(v'_i) := \begin{cases} \tau(u_j)\varepsilon(v'_i) & \text{if } \ i \le 0, \\ -\varepsilon(v'_i)\tau(u_j) & \text{if } \ i > 0 \end{cases}$$

with the similar formula for β and γ, but without the minus sign.

The semi-infinite Weil complex of $\tilde{\mathfrak{g}}$ is

$$\left\{S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}) \otimes \Lambda^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}}), \quad \mathbf{d}+\mathbf{h}\right\}, \tag{3.2}$$

where $S^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ and $\Lambda^{\frac{\infty}{2}+*}(\tilde{\mathfrak{g}})$ are semi-infinite analogs of the modules of symmetric and exterior powers (see [1]), $\mathbf{d}$ is the analog of the differential for Lie algebra (co)homology and $\mathbf{h}$ is the analog of the Koszul differential:

$$\begin{aligned} \mathbf{d} &= \sum_{u,v,i,j} \frac{1}{2} : \tau([u_i, v_j])\varepsilon(v'_j)\varepsilon(u'_i) : + : \beta([u_i, v_j])\gamma(v'_j)\varepsilon(u'_i) : , \\ \mathbf{h} &= \sum_{u,i} \gamma(u'_i)\tau(u_i), \end{aligned} \tag{3.3}$$

The quadratic operators (3.1) define a projective action of $S'(2,1)$ on the semi-infinite Weil complex of $\tilde{\mathfrak{g}}$. The cocycle is (see [8])

$$\begin{aligned} &c(L_n, L_k) = \frac{n^3}{12}\delta_{n,-k}, \qquad c(E_n, F_k) = \frac{n-1}{6}\delta_{n,-k}, \qquad c(L_n, H_k) = -\frac{n}{6}\delta_{n,-k}, \\ &c(H_n, H_k) = \frac{n}{3}\delta_{n,-k}, \qquad c(X^i_n, Y^i_k) = \frac{n(n-1)}{6}\delta_{n,-k}, \qquad i = 1, 2. \end{aligned} \tag{3.4}$$

If $\mathfrak{g}$ has a non-degenerate invariant symmetric bilinear form, then this action commutes with $\mathbf{d}$, and the action on the (relative) semi-infinite cohomology is well-defined (see [12]). The restriction of this action to the zero modes defines a representation of $\mathcal{K}$. Note that in this way d and d^* act as the semi-infinite Koszul differential $\mathbf{h}$ and the semi-infinite homotopy operator, respectively; this is a generalization of Howe's construction [7]. Observe that our superalgebras $\mathcal{K}$ and $\mathcal{H}$ differ from the ones usually considered in examples of Howe duality on (hyper)Kähler manifolds but are contractions of $\mathfrak{osp}(2|2)$ and $\mathfrak{osp}(2|4)$, cf. [11].

It was shown in [3] that the relative semi-infinite complex of a $\mathbb{Z}$-graded complex Lie algebra with coefficients in a graded Hermitian module has a structure similar to that of the de Rham complex in Kähler geometry. In [12] we described operators on the relative semi-infinite Weil complex of the loop algebra of a complex Lie algebra, which are analogs of the classical ones in Kähler geometry and span a Lie superalgebra isomorphic to $\mathcal{K}$. Note that in this realization (for which no "affinization" seems to be possible) d acts as the differential $\mathbf{d}$.

An interesting problem is to define operators acting on a (relative) semi-infinite Weil complex, which are analogous to the classical ones in hyper-Kähler geometry, and obtain the corresponding field expansions.

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