Towards a Theory of Differential Constraints of a Hydrodynamic Hierarchy

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Abstract

We present a theory of compatible differential constraints of a hydrodynamic hierarchy of infinite-dimensional systems. It provides a convenient point of view for studying and formulating integrability properties and it reveals some hidden structures of the theory of integrable systems. Illustrative examples and new integrable models are exhibited.

1 Introduction

In this paper we are concerned with the hierarchy of evolution equations

$$\frac{\partial G}{\partial t_i} = \langle A_i, G \rangle, \qquad G = G(\lambda, x, \boldsymbol{t}), \quad \boldsymbol{t} := (t_1, t_2, \dots), \quad i \geq 1, \tag{1.1}$$

where $\langle U, V \rangle := U V_x - U_x V$, and the function G is assumed to posses an expansion

$$G = g_0(x, \boldsymbol{t}) + \frac{g_1(x, \boldsymbol{t})}{\lambda} + \frac{g_2(x, \boldsymbol{t})}{\lambda^2} + \cdots, \qquad \lambda \to \infty, \tag{1.2}$$

We consider two different forms of (1.1) corresponding to the cases $g_0 \equiv 1$ and $g_0 \not\equiv \mathrm{const}$, which will be henceforth referred to as the *normalized* $G^{(n)}$ and the *Schwarzian* $G^{(s)}$ hierarchies, respectively. They are determined by two different definitions of the functions $A_i = A_i(\lambda, x, \boldsymbol{t})$; namely,

$$A_i^{(n)} = \lambda^i + \cdots + g_{i-1}\lambda + g_i, \qquad A_i^{(s)} = g_0\lambda^i + \cdots + g_{i-1}\lambda,$$

respectively. In both cases we may write

$$A_i = (\lambda^i G)_+, \qquad i \geq 1,$$

where the projectors $P : A \to A_+$ acting on power series $A = \sum_n a_n \lambda^n$ is defined as

$$A_+^{(n)} := P^{(n)}(A) = \sum_{n \geq 0} a_n \lambda^n, \qquad A_+^{(s)} := P^{(s)}(A) = \sum_{n \geq 1} a_n \lambda^n.$$

In terms of the coefficients of the expansion of G, the hierarchy (1.1) becomes a set infinite-dimensional systems of hydrodynamic type

$$\partial_i g_n = \sum_{k=1}^{n} \langle g_{n-k}, g_{i+k} \rangle, \qquad n \geq 0, \quad \text{(normalized case)},$$

$$\partial_i g_n = \sum_{k=0}^{n} \langle g_{n-k}, g_{i+k} \rangle, \qquad n \geq 0, \quad \text{(Schwarzian case)}.$$

We will show below that the case $g_0 \not\equiv \text{const}$ can be considered as the *Schwarzian form* of the case $g_0 \equiv 1$ and that a transformation connecting both cases exists.

Although equations similar to (1.1) were considered in the literature [1, 2, 3, 4] (starting with [1]), the normalized case of the hierarchy (1.1) was recently reconsidered in [5] from the point view of the reduction theory of integrable systems. It was in this context where the notion of *differential constraint* was introduced. One of the main aims of the present work is to show that this notion is a useful one in the theory of integrable systems.

For several purposes it is convenient to use instead of G the function

$$H := \frac{1}{G} = h_0 + \frac{h_1}{\lambda} + \frac{h_2}{\lambda^2} + \cdots ,$$

$$h_0 = \frac{1}{g_0}, \qquad h_1 = -\frac{g_1}{g_0^2}, \qquad h_2 = \frac{g_1^2}{g_0^3} - \frac{g_2}{g_0^2}, \qquad \ldots . \tag{1.3}$$

It allows us to write (1.1) in the alternative form

$$\frac{\partial H}{\partial t_i} = \frac{\partial}{\partial x} \left(A_i H \right) , \tag{1.4}$$

which implies that the coefficients of the expansion of H in powers of λ supply an infinite set of conservation laws for (1.1).

The function H is particularly useful to deal with the reductions of (1.1) determined by differential constraints. Let us illustrate this fact by considering the first flow $t := t_1$ of the Schwarzian hierarchy, which according to (1.4) reads

$$h_{n,t} = \hat{h}_{n+1,x}, \qquad \hat{h}_n := \frac{h_n}{h_0}, \qquad n \geq 0. \tag{1.5}$$

Let us look for a differential constraint of the form

$$\hat{h}_1 = B(h_0, h_{0,x}), \tag{1.6}$$

allowing to reduce the first equation in (1.5) to

$$h_{0,t} = D_x B(h_0, h_{0,x}), \tag{1.7}$$

where D_x stands for the total differentiation with respect to x. The differential constraint (1.6) should be compatible with the whole hierarchy. In particular, the second equation in (1.5) implies

$$D_t(h_1) = D_t(h_0 B(h_0, h_{0,x})) \in \operatorname{Im} D_x. \tag{1.8}$$

Here D_t is the total differentiation operator with respect to t and $\operatorname{Im} D_x$ denotes the range of D_x acting on the set of polynomials in the variables h_0, $h_{0,x}$, $h_{0,xx}$, ... Now one easily finds that

$$D_t(h_1) \equiv -(h_0 B_{11} B_1) h_{0,xx}^2 + \cdots,$$

where B_{11} is the second derivative of B with respect to the argument $h_{0,x}$. Thus (1.8) yields $B_{11} = 0$ and therefore the constraint (1.6) should be of the form

$$\hat{h}_1 = b(h_0) + a(h_0) h_{0,x}.$$

One can directly prove in this way that the only constraint (1.6) compatible with (1.5)) is

$$(\log h_0)_x = h_1. \tag{1.9}$$

The corresponding reduced evolution equation (1.7) is given by

$$h_{0,t} + \left(\frac{1}{h_0} \right)_{xx} = 0.$$

Several interesting generalizations of this type of constraints will be discussed in this paper. For example it will be proved that the constraints

$$(\log h_0)_x = h_m, \qquad m \geq 1, \tag{1.10}$$

and

$$\{h_0, x\} = \left[H^2 \right]_n, \qquad n \geq 1, \tag{1.11}$$

are compatible with (1.5). Here

$$\{a, x\} := \frac{3}{4} \left(\frac{a_x}{a} \right)^2 - \frac{1}{2} \frac{a_{xx}}{a}.$$

and $\left[H^2 \right]_n$ stands for the coefficient at λ^{-n} of the expansion of H^2 in powers of λ.

The paper is organized as follows. In Section 2 we deal with the structure of symmetries of the hierarchies (1.1) and reformulate some basic results. A particular attention is devoted to applying the classical Liouville method of solution for solving the polynomial reductions of (1.1). Some apparently new results concerning τ-functions like objects (Theorem 2) and a map $G^{(s)} \to G^{(n)}$ (Theorem 3) are also presented. Section 3 is concerned with the use of the method of differential constraints to characterize nonlinear integrable models from (1.1). It is shown how the structure of wide classes of these models as well as simple strategies for finding solutions can be conveniently described by this method. Some new integrable models, including a $2+1$-dimensional one (3.22) are exhibited.

2   Symmetries

We first reformulate some already known basic properties about the symmetries of the hierarchy (1.1). Our starting point is the following theorem [5]

Theorem 1. *The evolutionary flows defined by* (1.1) *form a commuting family.*

Proof. By using the Jacobi identity for the wronskian operation $\langle U, V \rangle$ we have

$$\frac{\partial^2 G}{\partial t_i \partial t_j} - \frac{\partial^2 G}{\partial t_j \partial t_i} = \left\langle \frac{\partial A_j}{\partial t_i} - \frac{\partial A_i}{\partial t_j} + \langle A_j, A_i \rangle, G \right\rangle.$$

¿From (1.1) and taking into account that $A_i = (\lambda^i G)_+$ it follows

$$\frac{\partial A_j}{\partial t_i} - \frac{\partial A_i}{\partial t_j} = \left(\langle A_i, \lambda^j G \rangle - \langle A_j, \lambda^i G \rangle \right)_+. \tag{2.1}$$

On the other hand

$$\langle A_j, \lambda^i G \rangle = -\langle \lambda^i G, \lambda^j G - (\lambda^j G)_- \rangle = \langle \lambda^i G, (\lambda^j G)_- \rangle,$$

where $(\lambda^j G)_- := \lambda^j G - (\lambda^j G)_+$. Hence

$$\frac{\partial A_j}{\partial t_i} - \frac{\partial A_i}{\partial t_j} = \left(\langle A_i, \lambda^j G \rangle - \langle \lambda^i G, (\lambda^j G)_- \rangle \right)_+ = \left(\langle A_i, \lambda^j G \rangle - \langle A_i, (\lambda^j G)_- \rangle \right)_+$$

$$= \langle A_i, A_j \rangle_+ = \langle A_i, A_j \rangle,$$

which proves the statement.   ∎

The hydrodynamic character of the equations of the hierarchy (1.1) has important consequences. Thus, it follows that if $\lambda = \lambda_0$ is either a zero or a pole of G then

$$\frac{\partial \lambda_0}{\partial t_i} = A_i(\lambda_0, x, \boldsymbol{t}) \frac{\partial \lambda_0}{\partial x}.$$

It means that zeros and poles of G are Riemann invariants of (1.1). This is a useful property because an obvious reduction of (1.1) arises when only a finite number of coefficients in the expansion (1.2) of G are different from zero

$$G^{(n)} = 1 + \frac{g_1}{\lambda} + \cdots + \frac{g_N}{\lambda^N}, \qquad G^{(s)} = g_0 + \frac{g_1}{\lambda} + \cdots + \frac{g_{N-1}}{\lambda^{N-1}}. \tag{2.2}$$

In this case it is possible to perform the integration of (1.1) in terms of Riemann invariants. Let us illustrate this feature for the normalized case. We can rewrite G in the form

$$G = \frac{1}{\lambda^N} \prod_{i=1}^{N} (\lambda + \gamma_i(x, \boldsymbol{t})), \tag{2.3}$$

where $\gamma_i(x, \boldsymbol{t})$ are the zeros of G. Thus, under the change of dependent variables

$$\vec{g} := (g_1, \ldots, g_N) \mapsto \vec{\gamma} := (\gamma_1, \ldots, \gamma_N),$$

the system (1.1) reduces to

$$\partial_n \gamma_i = \Omega_{ni}(\vec{\gamma}) \partial_x \gamma_i, \qquad n \geq 1. \tag{2.4}$$

Here $\Omega_{ni} := A_n|_{\lambda = -\gamma_i}$ are given by

$$\Omega_{ni} = \begin{cases} (-1)^n \sum\limits_{j_l \neq i} \gamma_{j_1} \gamma_{j_2} \cdots \gamma_{j_n} & \text{if } n = 1, 2, \ldots, N-1, \\ 0 & \text{if } n \geq N. \end{cases} \tag{2.5}$$

The equations (2.4)–(2.5) form a set of $N-1$ weakly nonlinear hydrodynamic systems of Dubrovin type [2] which can be integrated by means of a version of the classical Liouville method [6] proposed by Tsarev [7] (see also [8]). It basically consists in finding a vector field D_x acting on the new dependent variables $\vec{\gamma}$

$$D_x \gamma_i = X_i(\vec{\gamma}), \qquad i = 1, \ldots, N, \tag{2.6}$$

in such a way that it commutes with the fields ∂_j, $j = 1, \ldots, N-1$. This is verified provided

$$\frac{\partial \ln X_i}{\partial \gamma_j} = \frac{1}{\gamma_i - \gamma_j}, \qquad i \neq j. \tag{2.7}$$

By applying Tsarev's method (see [7, 8]) one finds

Lemma 1. *A general solution of* (2.7) *is*

$$D_x \gamma_i = \frac{a_i(\gamma_i)}{\prod\limits_{j \neq i}(\gamma_i - \gamma_j)}, \qquad i = 1, \ldots, N, \tag{2.8}$$

where $a_i(\xi)$ *are* N *arbitrary functions.*

As a consequence we can reformulate (2.4) as a set of N dynamical systems for γ_i

$$\partial_n \gamma_i = \Omega_{ni} X_i, \qquad n = 1, \ldots, N-1,$$
$$\partial_x \gamma_i = X_i. \tag{2.9}$$

Equivalently, we have

$$\frac{d\gamma_i}{a_i(\gamma_i)} = \frac{2}{\prod\limits_{j \neq i}(\gamma_i - \gamma_j)} \left(dx + \sum_{n=1}^{N-1} \Omega_{ni}\, dt_n \right), \qquad i = 1, \ldots, N. \tag{2.10}$$

One can explicitly invert (2.10) (see [8]) and find

$$(-1)^n \, dt_n = \sum_{i=1}^N \frac{\gamma_i^{N-n-1}}{a_i(\gamma_i)} \, d\gamma_i, \qquad n = 0, \ldots, N-1, \tag{2.11}$$

where we are denoting $t_0 := x$. Therefore, we conclude that the general solution of (2.4) is determined by the following system of N implicit relations

$$(-1)^n t_n = \sum_{i=1}^N \int^{\gamma_i} \frac{\gamma_i^{N-n-1}}{a_i(\gamma_i)} \, d\gamma_i, \qquad n = 0, \ldots, N-1. \tag{2.12}$$

The theorems which follow state some apparently new results on the structure of the hierarchy (1.1)

Theorem 2. *The differential forms*

$$\omega_1 = g_1 \, \mathrm{d}\, x + g_2 \, \mathrm{d}\, t_1 + \cdots + g_n \, \mathrm{d}\, t_{n+1} + \cdots , \qquad (\textit{normalized case}),$$

$$\omega_2 = \frac{1}{g_0} \left(\mathrm{d}\, x - g_1 \, \mathrm{d}\, t_1 + \cdots - g_n \, \mathrm{d}\, t_n + \cdots \right), \qquad (\textit{Schwarzian case}),$$

are closed.

Proof. Let us consider the normalized case. Notice that

$$\omega_1 = \sum_{i \geq 0} \mathrm{Res}(\lambda^i G) \, \mathrm{d}\, t_i,$$

where $\mathrm{Res}(A(\lambda))$ stands for the coefficient of λ^{-1} of Laurent series $A(\lambda)$ in λ. Moreover, from (1.1)

$$\frac{\partial}{\partial t_j} \mathrm{Res}(\lambda^i G) = \mathrm{Res}\left(\lambda^i \frac{\partial G}{\partial t_j} \right) = \mathrm{Res}(\langle A_j, \lambda_i G \rangle). \tag{2.13}$$

On the other hand

$$\mathrm{Res}(\langle A_j, \lambda_i G \rangle) = \mathrm{Res}(\langle \lambda^j G - (\lambda G)_-, \lambda_i G \rangle) = -\mathrm{Res}(\langle (\lambda^j G)_-, (\lambda_i G)_+ \rangle)$$
$$= \mathrm{Res}(\langle A_i, (\lambda_j G)_- \rangle) = \mathrm{Res}(\langle A_i, \lambda_j G \rangle),$$

so that (2.13) implies

$$\frac{\partial}{\partial t_j} \mathrm{Res}(\lambda^i G) = \frac{\partial}{\partial t_i} \mathrm{Res}(\lambda^j G),$$

which proves that ω_1 is closed.

The proof for the Schwarzian case can be done similarly by writing

$$\omega_2 = \frac{1}{g_0} - \sum_{i \geq 1} \mathrm{Res}\left(\lambda^{i-1} \frac{G}{g_0} \right) \mathrm{d}\, t_i,$$

and by taking into account that

$$\frac{\partial}{\partial t_i} \frac{1}{g_0} = -\frac{\partial}{\partial x} \frac{g_i}{g_0}. \tag{2.14}$$

∎

As a consequence of this theorem there exist two functions $q_i = q_i(x, \boldsymbol{t})$ verifying

$$\omega_i = \mathrm{d}\, q_i, \qquad i = 1, 2, \tag{2.15}$$

so that we can rewrite the hierarchy (1.1) as a system of partial differential equations for q_i.

Theorem 3. *Let $G^{(s)} = G^{(s)}(\lambda, x, \boldsymbol{t})$ be a solution of the Schwarzian hierarchy* (1.1). *If we define $\hat{x} = q_2(x, \boldsymbol{t})$, where q_2 satisfies* (2.15), *then*

$$G^{(n)}(\lambda, \hat{x}, \boldsymbol{t}) := \frac{G^{(s)}(\lambda, x, \boldsymbol{t})}{g_0(x, \boldsymbol{t})}, \tag{2.16}$$

solves the normalized hierarchy (1.1).

Proof. Observe that

$$\frac{\partial \hat{x}}{\partial x} = \frac{1}{g_0}, \qquad \frac{\partial \hat{x}}{\partial t_i} = \frac{1}{g_i}, \quad i \geq 1.$$

Hence

$$\frac{\partial x}{\partial \hat{x}} = g_0, \qquad \frac{\partial x}{\partial t_i} = g_i, \quad i \geq 1.$$

Therefore, by taking (2.14) into account, we have

$$\frac{\partial G^{(n)}}{\partial t_i} = \left(\frac{\partial}{\partial t_i} + g_i \frac{\partial}{\partial x} \right) \frac{G^{(s)}}{g_0} = \frac{1}{g_0} \langle A_i^{(n)} + g_i, G^{(n)} \rangle = \langle A_i^{(s)}, G^{(s)} \rangle_{\hat{x}},$$

where

$$A_i^{(n)} = \lambda^i + \cdots + \hat{g}_{i-1} \lambda + \hat{g}_i, \quad \hat{g}_i := \frac{g_i}{g_0},$$

and $\langle U, V \rangle_{\hat{x}} := U \partial_{\hat{x}} V - V \partial_{\hat{x}} U.$ ∎

Remark. This theorem shows that a correspondence $G^{(s)} \to G^{(n)}$ between the Schwarzian and normalized hierarchies exists. Reciprocally, a transformation $G^{(n)} \to G^{(s)}$ can also be defined provided that the solution $G^{(n)} = G^{(n)}(\lambda, x, \boldsymbol{t})$ does not vanish at $\lambda = 0$. To prove this statement let us define

$$u := \frac{1}{G^{(n)}|_{\lambda=0}},$$

and the differential form

$$\omega := u \left(\mathrm{d}\,x + \sum_{i \geq 1} g_i \,\mathrm{d}\,t_i \right),$$

where g_n are the coefficients of the expansion of $G^{(n)}$ in powers of λ. From (1.1) it follows at once that

$$\partial_i u = (u g_i)_x.$$

Moreover, from the identities

$$\partial_j A_i - \partial_i A_j = \langle A_j, A_i \rangle,$$

and taking into account that $g_i = A_i|_{\lambda=0}$, we have

$$\partial_j g_i - \partial_i g_j = \langle g_j, g_i \rangle,$$

and therefore

$$\partial_j(u g_i) = \partial_i(u g_j),$$

which proves that ω is a closed form. It is now straightforward to demonstrate that the function

$$G^{(s)}(\lambda, \hat{x}, \boldsymbol{t}) := u(x, \boldsymbol{t}) G^{(n)}(\lambda, x, \boldsymbol{t}),$$

where

$$\hat{x} := q(x, \boldsymbol{t}), \qquad \omega = \mathrm{d}\, q,$$

verifies the Schwarzian hierarchy (1.1).

3 Reductions and differential constraints

In [5] wide classes of reductions of the normalized hierarchy (1.1) were introduced by using the notion of compatible differential constraints. They can be easily generalized to the Schwarzian hierarchy as well. We will consider here reductions defined by second-order differential constraints

$$2G_{xx}G - G_x^2 + 4a(\lambda) - 4U(\lambda, x, \boldsymbol{t})G^2 = 0, \tag{3.1}$$

where $a(\lambda)$ is an arbitrary function and

$$U(\lambda, x, \boldsymbol{t}) := \left(\frac{a(\lambda)}{G^2}\right)_+. \tag{3.2}$$

Requiring compatibility between (1.1) and (3.1) yields

$$\partial_n U = -\frac{1}{2}A_{n,xxx} + 2U A_{n,x} + U_x A_n, \qquad n \geq 1. \tag{3.3}$$

We notice that in terms of the generating function H for the conservation laws, equations (3.1) and (3.2) can be written as

$$\{H, x\} + a(\lambda)H^2 = U(\lambda, x, \boldsymbol{t}), \qquad U = \left(a(\lambda)H^2\right)_+, \tag{3.4}$$

where we are using the Schwarzian derivation operation

$$\{H, x\} := \frac{3}{4}\frac{H_x^2}{H^2} - \frac{1}{2}\frac{H_{xx}}{H}.$$

The equations (3.1) determine integrable hierarchies associated with Schrödinger spectral problems with energy-dependent potentials. Thus, for $a(\lambda) = \lambda^n$ one finds the generalized KdV hierarchies for energy-dependent potentials of the form

$$U(\lambda, x, \boldsymbol{t}) := \lambda^n + \sum_{i=0}^{n-1} \lambda^i u_i(x, \boldsymbol{t}), \qquad \text{(normalized case)}, \tag{3.5}$$

and

$$U(\lambda, x, \boldsymbol{t}) := \sum_{i=1}^{n} \lambda^i u_i(x, \boldsymbol{t}), \qquad \text{(Schwarzian case)}. \tag{3.6}$$

For the normalized case $G = G^{(n)}$ the simplest choices $n = 1$ and $n = 2$ lead to the KdV and the Zakharov–Shabat hierarchies, respectively. We observe that for the Schwarzian case the choice $a(\lambda) = \lambda^n$ in (3.4) leads to the constraint (1.11).

Theorem 4. *The differential constraint* (3.4) *is invariant under the transformation* $H^{(s)} \to H^{(n)}$ *determined by equation* (2.16). *That is to say, if* $H^{(s)}$ *satisfies*

$$\left\{ H^{(s)}, x \right\} + a(\lambda) H^{(s)2} = U^{(s)}(\lambda, x, \boldsymbol{t}),$$

then

$$\left\{ H^{(n)}, x \right\} + a(\lambda) H^{(n)2} = U^{(n)}(\lambda, x, \boldsymbol{t}).$$

The corresponding transformation law for the potential function is

$$U^{(n)} = g_0^2 U^{(s)} + \{g_0(\hat{x}), \hat{x}\}. \tag{3.7}$$

Proof. If we define $H^{(n)} := H^{(s)}/h_0$, then it follows at once that

$$\left\{ H^{(s)}, x \right\} = h_0^2 \left(\left\{ H^{(n)}, \hat{x} \right\} + \frac{1}{4} \frac{h_{0,\hat{x}}^2}{h_0^2} - \frac{1}{2} \frac{h_{0,\hat{x}\hat{x}}}{h_0} \right).$$

On the other hand

$$\left\{ \frac{1}{h_0}, \hat{x} \right\} = \frac{1}{2} \frac{h_{0,\hat{x}\hat{x}}}{h_0} - \frac{1}{4} \frac{h_{0,\hat{x}}^2}{h_0^2}.$$

Thus, it is straightforward to get

$$\left\{ H^{(n)}, \hat{x} \right\} + a(\lambda) H^{(n)2} = U^{(n)}, \quad U^{(n)} := \frac{1}{h_0^2} U^{(s)} + \left\{ \frac{1}{h_0}, \hat{x} \right\}. \tag{3.8}$$

∎

A complete and convenient description of the general class of the *energy-dependent* hierarchies can be formulated by taking advantage of the identities (3.4). The following examples illustrate the fact that, in addition to the standard hierarchies of Zakharov–Shabat and KdV types, many other integrable models may be analyzed within the theory of the second order differential constraints of the Schwarzian hierarchy (1.1).

Example 1. For $a(\lambda) = \lambda$ and $G = G^{(s)}$, the constraint (3.4) reads

$$\{H, x\} + \lambda H^2 = \lambda u_1,$$

and by identifying the coefficient of $1/\lambda$ we get

$$\{h_0, x\} + 2h_0 h_1 = 0.$$

Hence, by taking into account that in terms of the function $q := q_2(x, \boldsymbol{t})$ of (2.15)

$$h_0 = \frac{1}{g_0} = q_x, \qquad h_1 = -\frac{g_1}{g_0^2} = q_x q_t, \qquad (t := t_1), \tag{3.9}$$

we deduce at once the Schwarzian form of the KdV equation

$$2 q_t q_x = \{q_x, x\} \Leftrightarrow -8 q_t + 2 \frac{q_{xxx}}{q_x^3} - 3 \frac{q_{xx}^2}{q_x^4} = 0, \tag{3.10}$$

which, written in terms of $g_0 = 1/h_0$, reduces to the Harry–Dym equation

$$g_{0,t} = \frac{1}{4} g_0^3 g_{0,xxx}. \tag{3.11}$$

Example 2. For $a(\lambda) = \lambda^2$ and $G = G^{(s)}$, the identity (3.4) reads

$$\{H, x\} + \lambda^2 H^2 = \lambda^2 u_2 + \lambda u_1,$$

and by identifying the constant term in the expansion in powers of λ we get

$$\{h_0, x\} + h_1^2 + 2 h_0 h_2 = 0.$$

Now, from (1.4) we have

$$h_{0,t} = \left(\frac{h_1}{h_0} \right), \qquad h_{1,t} = \left(\frac{h_2}{h_0} \right), \qquad (t := t_1).$$

Hence, by taking into account (3.9) we find

$$(q_x q_t)_t = \left(\frac{1}{4} \frac{q_{xxx}}{q_x^3} - \frac{3}{8} \frac{q_{xx}^2}{q_x^4} - \frac{1}{2} q_t^2 \right)_x. \tag{3.12}$$

This nonlinear equation admits a Lagrangian given by

$$\mathfrak{L} = \iint \left(q_t^2 q_x + \frac{1}{4} \frac{q_{xx}^2}{q_x^3} \right) \mathrm{d}\,x\,\mathrm{d}\,t, \tag{3.13}$$

or, equivalently, under the point transformation $q(u(y, t), t) = y$

$$\mathfrak{L} = \iint \frac{u_t^2 + u_{yy}^2}{u_y^2} \,\mathrm{d}\,y\,\mathrm{d}\,t. \tag{3.14}$$

If we denote now $y \to x$, $u \to q$, the following equivalent Lagrangian formulation arises

$$\mathfrak{L} = \iint \left(\frac{q_t^2 + q_{xx}^2}{q_x^2} \right) \mathrm{d}\,x\,\mathrm{d}\,t = \iint \left(\frac{(q_t + q_{xx})^2}{q_x^2} - 2 \frac{q_t q_{xx}}{q_x^2} \right) \mathrm{d}\,x\,\mathrm{d}\,t$$

$$= \iint \frac{(q_t + q_{xx})^2}{q_x^2} \,\mathrm{d}\,x\,\mathrm{d}\,t.$$

Thus, the model admits a Hamiltonian density

$$H = q_t L_{q_t} - L = p_x q_x + \frac{1}{4} p^2 q_x^2, \qquad p := L_{q_t}, \qquad L := \frac{(q_t + q_{xx})^2}{q_x^2},$$

which leads to the Hamiltonian equations

$$q_t + q_{xx} + 2pq_x^2 = 0, \qquad -p_t + p_{xx} = (2p^2 q_x)_x.$$

It should be noticed that the last system is one of "canonical" forms of the isotropic Landau–Lifshitz model (see [9]).

In [5] first-order differential constraints defining reductions of (1.1) were also introduced. They adopt the form

$$G_x - a(\lambda) - U(\lambda, x, \boldsymbol{t})G = 0, \tag{3.15}$$

where $a(\lambda)$ is an arbitrary function and

$$U(\lambda, x, \boldsymbol{t}) := - \left(\frac{a(\lambda)}{G} \right)_+ . \tag{3.16}$$

Requiring compatibility between (1.1) and (3.15) gives

$$\partial_n U = -A_{n,xx} + (UA_n)_x, \qquad n \geq 1, \tag{3.17}$$

or, equivalently,

$$\partial_n U = -\partial_x (U(\lambda^n G)_-)_+, \qquad n \geq 1,. \tag{3.18}$$

In terms of the function H equations (3.15) and (3.16) read

$$H_x + a(\lambda)H^2 + U(\lambda, x, \boldsymbol{t})H = 0, \qquad U = -\left(a(\lambda)H \right)_+ . \tag{3.19}$$

We notice that for the Schwarzian case if we set $a(\lambda) = -\lambda^m$ in (3.15) the constraint (1.10) follows.

First-order differential constraints determine in turn further reductions of the second-order differential constraints. Indeed, it follows easily that

Theorem 5. *Given a solution G of* (3.15) *then it satisfies*

$$2G_{xx}G - G_x^2 + a(\lambda)^2 - 4\widehat{U}(\lambda, x, \boldsymbol{t})G^2 = 0,$$

where $U \to \widehat{U}$ is implemented by the Miura transformation

$$\widehat{U} := \frac{1}{2}U_x + \frac{1}{4}U^2.$$

Among the reductions defined by first-order differential constraints one finds the Burgers hierarchy as well as a class of hierarchies of *energy-dependent* type.

Example 3. If we impose on $G = G^{(n)}$ the constraint

$$G_x = (\lambda + u)G - \lambda, \qquad u = h_1 = -g_1,$$

we get the Burgers hierarchy

$$\partial_n u = \partial_x \left((\partial_x + u)^n u \right).$$

In particular the flow corresponding to $t := t_1$ is the Burgers equation

$$u_t = u_{xx} + 2uu_x.$$

Example 4. For a constraint on $G = G^{(n)}$ of the form

$$G_x = (\lambda^2 + \lambda u_1 + u_0)G - \lambda^2, \qquad u_1 = h_1 = -g_1, \qquad u_0 = h_2 = g_1^2 - g_2, \qquad (3.20)$$

we get the following hierarchy for the *energy-dependent* potential $U := \lambda^2 + \lambda u_1 + u_0$

$$\partial_n \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \begin{pmatrix} -\partial_x & 0 \\ 0 & -\partial_x(\partial_x - u_0) \end{pmatrix} R^{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (3.21)$$

where

$$R := \begin{pmatrix} -u_1 & \partial_x + u_0 \\ 1 & 0 \end{pmatrix}.$$

The flow corresponding to $t := t_1$ reads

$$u_{0,t} = u_{1,xx} - (u_0 u_1)_x, \qquad u_{1,t} = -\left(u_1^2 + u_0\right)_x, \qquad (3.22)$$

which in terms of the function $q := q_1(x, \boldsymbol{t})$ of (2.15) reduces to

$$q_{tt} - q_{xxx} - 3q_x q_{xt} - q_t q_{xx} + (q_x)_x^3 = 0.$$

It is possible to characterize multidimensional integrable models by imposing compatible differential constraints to (1.1). To illustrate this feature let us consider the constraint (3.20) for $G = G^{(n)}$ and denote $y := t_1$, $t := t_2$. Thus we have the system of equations

$$G_x = \left(\lambda^2 - \lambda g_1 + g_1^2 - g_2\right) G - \lambda^2, \qquad (3.23a)$$
$$G_y = \langle \lambda + g_1, G \rangle, \qquad (3.23b)$$
$$G_t = \langle \lambda^2 + \lambda g_1 + g_2, G \rangle. \qquad (3.23c)$$

Let us introduce the function

$$F := u\, G = f_0 + \frac{f_1}{\lambda} + \frac{f_2}{\lambda^2} + \cdots, \qquad u := \frac{1}{G|_{\lambda=0}}.$$

From (3.23) we get

$$u_x = \left(g_2 - g_1^2\right) u,$$
$$u_y = \left(g_{1,x} + g_1 g_2 - g_1^3\right) u,$$
$$u_t = \left(g_{2,x} + g_2^2 - g_2 g_1^2\right) u,$$

so that the equations (3.23) written in terms of F read

$$F_x = \left(\lambda^2 - \lambda g_1\right) F - \lambda^2 f_0, \qquad (3.24a)$$
$$F_y = \left(\lambda^3 - \lambda g_2\right) F - \lambda^3 f_0 - \lambda f_1, \qquad (3.24b)$$
$$F_t = \left(\lambda^4 - \lambda g_3\right) F - \lambda^4 f_0 - \lambda^3 f_1 - \lambda^2 f_2. \qquad (3.24c)$$

Now it is easy to deduce from (3.24) that

$$F_t = F_{xx} + 2g_1 F_y - g_1^2 F_x, \qquad (3.25)$$

and, as a consequence, one finds that $v := g_1$ and $w = \ln u$ satisfy the nonlinear system of equations

$$v_t = v_{xx} + 2vv_y - v^2 v_x + 2v_x w_x,$$
$$w_t = w_{xx} + 2vw_y - v^2 w_x + w_x^2. \tag{3.26}$$

We observe that equations (3.24) mean that equation (3.25) and, consequently, equation (3.26) admit separation of variables. If we expand F in powers of λ then equations (3.24) become

$$f_{n,x} = f_{n+2} - g_1 f_{n+1}, \qquad f_{-1} = 0, \qquad f_0 = u, \tag{3.27a}$$
$$f_{n,y} = f_{n+3} - g_2 f_{n+1}, \qquad f_2 = ug_2, \tag{3.27b}$$
$$f_{n,t} = f_{n+4} - g_3 f_{n+1}, \qquad f_3 = ug_3. \tag{3.27c}$$

By using (3.27a) all the coefficients f_n can be expressed as differential polynomials in u and $v = g_1$:

$$f_0 = u, \quad f_1 = uv, \qquad f_2 = u_x + uv, \qquad f_3 = (uv)_x + vu_x + uv^3, \qquad \dots$$

Furthermore, the system (3.27) implies that polynomial reductions of the form

$$f_n = 0, \qquad \forall\, n \geq N,$$

are Liouville integrable. Indeed, by noticing that $f_{N-1} = \text{const}$, we require $N-1$ commuting vector fields to integrate such a reduction. For example, for $N = 4$ the dynamical variables are f_0, f_1, f_2 and (3.27) can be written as

$$\mathrm{d}\,f_0 = \left(f_2 - \frac{f_1^2}{f_0}\right)\mathrm{d}\,x + \left(\alpha - \frac{f_1 f_2}{f_0}\right)\mathrm{d}\,y - \alpha\frac{f_1}{f_0}\,\mathrm{d}\,t,$$

$$\mathrm{d}\,f_1 = \left(\alpha - \frac{f_1 f_2}{f_0}\right)\mathrm{d}\,x - \frac{f_2^2}{f_0}\,\mathrm{d}\,y - \alpha\frac{f_2}{f_0}\,\mathrm{d}\,t,$$

$$\mathrm{d}\,f_2 = -\alpha\frac{f_1}{f_0}\,\mathrm{d}\,x - \alpha\frac{f_2}{f_0}\,\mathrm{d}\,y - \frac{\alpha^2}{f_0}\,\mathrm{d}\,t,$$

where $\alpha_3 := f_3 = \text{const}$. It is now straightforward to eliminate $\mathrm{d}\,x$, $\mathrm{d}\,y$, $\mathrm{d}\,t$ and find

$$\mathrm{d}\,x = \mathrm{d}\left(\frac{f_1}{\alpha} - \frac{f_2}{2\alpha^2}\right),$$

$$\mathrm{d}\,y = \mathrm{d}\left(\frac{f_0}{\alpha} - \frac{f_1 f_2}{\alpha^2} + \frac{f_2^3}{3\alpha^3}\right),$$

$$\mathrm{d}\,t = \mathrm{d}\left(-\frac{f_0 f_2}{\alpha^2} - \frac{f_1^2}{2\alpha^2} + \frac{f_1 f_2^2}{\alpha^3} - \frac{f_2^4}{4\alpha^4}\right),$$

In this way the solution is determined by the following system

$$f_0 = \alpha y + x f_2 - \frac{f_2^3}{2\alpha}, \qquad f_1 = \alpha x + \frac{f_2^2}{2\alpha},$$

$$-\frac{7}{8}f_2^4 + \frac{1}{2}\alpha^2 x f_2^2 + \alpha^3 f_2 + \alpha^4\left(\frac{1}{2}x^2 + t\right) = 0.$$

This example shows the complicated algebraic singularities exhibited by the solutions of (3.26).

Acknowledgements

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