The stability analysis of asset price system with delay by impulsive control

Xunxia Xu[1], Liuxiao Guo[2]
[1,2]School of Science, Jiangnan University, Jiangsu Province, China
770566754@qq.com guo-liuxiao@126.com

Abstract -The stability analysis of asset price system with delay is discussed in this paper. By adding impulsive control, dynamical properties of the system are changed, i.e., eliminating the influence of some factors such as time delay on the instability of asset price. Based on the impulsive theory and Lyapunov function stability theory, a new criteria on global exponential stability of nonlinear differential equation is established, and the conservative conditions of the global exponential stability for asset price system are obtained. Numerical simulations results verify the feasibility and validity of impulsive control method and the theoretical results. It is concluded that reasonable impulsive control can control the stability of system with delay effectively.

Index Terms - asset price system; delay; impulsive control; global exponential stability

1. Introduction

It is well recognised that one of the research focus in recent years is mathematical model of financial system. Financial model with heterogeneous beliefs and behavior, namely the heterogeneous agent model(HAM) has been put forward by many scholars. And a variety of heterogeneous agent model has been used to explain a series of the market behavior over the past twenty years. According to the research of heterogeneous agent model[1-3], Xue-Zhong He, Min Zheng analysised the dynamic of financial market model in continuous time[4]. For the stability of the impulsive control system with time delay, some scholars have also made corresponding research, for example, in 2003, Z.Luo.J analysised the stability on Impulsive Functional Differential Equations with infinite delay[5], and in 2007, Qing Wang, Xinzhi Liu realized that the situation on the stability of differential system with delay under impulsive control could be studied by Lyapunov-Razumikhin method[6].

Based on the literature[4], and combinated by the study on the stability of the system with time delay in impulsive control, this paper changes the financial model to economical model.

Price stability has been a hot issue of people's livelihood, therefore it is very necessary to strengthen the implementation for price stability under price regulation. In the economic system, when prices rise to inflation, the state can immediately raised interest, so that it can fastly reduce the amount of money in circulation in the market, which is called the impulsive phenomena. For example, in 2010, Y.-Q. Chen and H.-L.Xu made best use of impulsive phenomena for some researches[7]. What's more, J .-t. Sun, F. Qiao and Q. –D. Wu presented several new theorems on the stability of impulsive control system in 2005[8]. This paper focuses on the application of impulsive control in asset price system.

This paper mainly discusses the stability control problem of asset price system with delay. By adding impulsive control, that is the national instant interval macroeconomic regulation changes the system development trend, and eliminates or reduces the influence of some factors on the commodity price instability. According to the impulsive theory and the stability theory of Lyapunov function, the conservative conditions for global exponential stability of commodity price system are obtained. Numerical simulation example shows the effection of impulsive control method and the theory.

2. Mathematical model on asset price system with delay

Consider the dynamics of asset prices under the conditions of financial market[4], let $p(t)$ be the market price at continuous time t, $F(t)$ be the fundamental price, and assume that fundamental price is a constant $F(t) \equiv \overline{F}$, $u(t)$ be the future price in the minds of consumers.

Suppose that the financial market consists of producers and consumers (producers and consumers here are the producers and consumers of the ideal state, producers are fixed producers instead of consumers, and consumers are also fixed consumers rather than producers). Assume that market population N is a fixed constant. Here,let N_s be the numbers of producers and N_c be the numbers of consumers, then $N_s + N_c = N$. Denote by $n_s = N_s / N$ and $n_c = N_c / N$ the financial market population of producers and consumers, respectively.

The future price in the minds of consumers is related to the historical price of the same asset, which is calculated by an exponentially decaying weighted average of historical prices over a time interval $[t-\tau, t]$, namely

$$u(t) = \frac{k}{1-e^{-k\tau}} \int_{t-\tau}^{t} e^{-k(t-s)} p(s) ds, \qquad (1)$$

where delay $\tau \in (0, \infty)$ represents a price history used to calculate the price trend, and $k > 0$ is a decay rate. In general, for $0 < k < \infty$, (1) can be expressed as a delay differential equation with delay τ , that is

$$du(t)=\frac{k}{1-e^{-k\tau}}[p(t)-e^{-k\tau}p(t-\tau)-(1-e^{-k\tau})u(t)]dt, \quad (2)$$

The profits demand of producers $Z_s(t)$ and the purchase demand of consumers $Z_c(t)$ have certain relation with the market price. what's more, the market fractions of producers and consumers play an important role. Then $p(t)$ is given by

$$dp(t)=\mu(n_s Z_s+n_c Z_c)dt, \quad (3)$$

Where

$$Z_s(t)=\beta_s[p(t)-F(t)], Z_c(t)=g(p(t)-u(t)), \quad (4)$$

$\mu>0$ represents the speed of the price adjustment, $\beta_s>0$ is a constant parameter, measuring the speed of mean- reversion of the market price to the fundamental price. The demand function g satisfies:

$g^{'}(x)>0, g^{'}(0)=\beta_c>0,$ and

$xg^{''}(x)<0$ for $x\neq 0$. And β_c represents the extrapolation rate of consumers on the future price trend when the price deviation from the trend is small. For the following discussion, we let $g(x)=\tanh(\beta_c x)$.

From the above analysis,asset price in the financial market is determined by the following two-dimensional system of deterministic delay differential equations [4]

$$\begin{cases}\frac{dp}{dt}=\mu[n_s\beta_s(p-\overline{F})+n_c\tanh(\beta_c(p-u))] \\ \frac{du}{dt}=\frac{k}{1-e^{-k\tau}}[p(t)-e^{-k\tau}p(t-\tau)-(1-e^{-k\tau})u(t)]\end{cases}, \quad (5)$$

it is easy to find that $(\overline{P},\overline{u})=(\overline{F},\overline{F})$ is an only equilibrium point of system (5), where the equilibrium steady state price is given by the fundamental price. Now let $x_1=P-\overline{F}$, $x_2=u-\overline{F}$, then the system(5) becomes

$$\begin{cases}\frac{dx_1}{dt}=\mu[n_s\beta_s x_1+n_c\tanh(\beta_c(x_1-x_2))] \\ \frac{dx_2}{dt}=\frac{k}{1-e^{-k\tau}}x_1-\frac{ke^{-k\tau}}{1-e^{-k\tau}}x_1(t-\tau)-kx_2\end{cases}, \quad (6)$$

then (0,0) is the equilibrium point of system (6). It is the stability of equilibrium point in system (5) that be transformed to the stability of zero solution in system (6).

3. Impulsive control for asset price system

As to the financial system, asset price adjustment mechanism is an objective existence, which is feasible under certain conditions, who also can guarantee the stability of asset price. But there is some limitation that once beyond those conditions, dynamical properties of the financial system will be changed. Therefore, in order to ensure the healthy development of the financial market, we need a national macro-control as an impulsive effct.

In recent years, the issues of stability in impulsive differential equations with time delay have attracted increasing interest in both theoretical and practical applications. In paricular, special attention has been focused on exponential stability of delay differential equations because it has played an important role in many areas[7-10].

In the following, we shall research the global exponential stability for the system (6) under impulsive control. The system can be rewritten by

$$\frac{dx}{dt}=Ax+Dx(t-\tau)+f(t,x), \quad (7)$$

here, $x^T=(x_1,x_2), A=\begin{pmatrix}\mu n_s\beta_s & 0 \\ \frac{k}{1-e^{-k\tau}} & -k\end{pmatrix}$,

$$D=\begin{pmatrix}0 & 0 \\ \frac{-ke^{-k\tau}}{1-e^{-k\tau}} & 0\end{pmatrix}, f(t,x)=\begin{pmatrix}n_c\tanh(\beta_c(x_1-x_2)) \\ 0\end{pmatrix}.$$

Assume impulsive control $U(l,x)=B_l x, B_l=\begin{pmatrix}B_1 & 0 \\ 0 & B_2\end{pmatrix}$.

Then we have impulsive system

$$\begin{cases}\frac{dx}{dt}=Ax+Dx(t-\tau)+f(t,x), t\neq t_l \\ \Delta x\big|_{t=t_l}=U(l,x)=B_l x, t=t_l \\ x_{t_0}=\varphi(0)\end{cases}, \quad (8)$$

here, we assume that the solution for the initial problem in system (8) does exist.

4. Stability analysis

As in [7-10], we introduce some definitions as follows:

Definition 1. The function $V\in v_0$ if

(1) V is continuous in $[t_{l-1},t_l)\times R^n$, and for each $x\in R^n, t\in[t_{l-1},t_l), l\in N, \lim_{(t,y)\to(t_l^-,x)}V(t,y)=V(t_l^-,x)$ exists;

(2) $V(t,x)$ is locally Lipschitzian in all $x\in R^n$, and for all $t\geq t_0, V(t,0)\equiv 0$.

Definition 2. Given a function $V\in V_0$, let

$$\frac{dx}{dt}=Ax+Dx(t-\tau)+f(t,x)=F(t,x),$$

then the upper right-hand derivative of $V(t,x)$ is defined by

$D^+V(t,x)=\lim_{\xi\to 0^+}\sup\frac{1}{\xi}[V(t+\xi,x+\xi F(t,x))-V(t,x)]$ for $(t,x)\in[t_{l-1},t_l)\times R^n$

Definition 3. The trivial solution of system (8) is said to be globally exponentially stable if there exist some constants $\alpha>0$ and $L\ge 1$ such that for any initial data $x_{t_0}=\varphi(0)$

$\|x(t,t_0,\varphi(0)\|\le L\|\varphi(0)\|_\tau e^{-\alpha(t-t_0)},t\ge t_0.$

Our result shows that impulses do contribute to global exponential stability of dynamical systems with time delay even if they are unstable, which can be usually used as an effective control strategy to stabilize the underlying delay dynamical systems in some practical applications, the following lemma is very helpful to understand this issue[9].

Lemma 1. Assume that there exists a function $V\in v_0$ and constants $p,c,c_1,c_2>0,\alpha>\tau,\lambda>c$ such that

(1) $c_1\|x\|^p\le V(t,x)\le c_2\|x\|^p$, for any $t\in R_+$ and $x\in R^n$;

(2) $qV(t,\varphi(0))\ge V(t+h,\varphi(h))$, for $h\in[-\tau,0)$, and $D^+V(t,\varphi(0))\le cV(t,\varphi(0))$, here $q\ge e^{2\lambda\alpha}$ is a constant;

(3) $V(t_l,\varphi(0)+B_l\varphi)\le d_lV(t_l,\varphi(0))$, where $d_l>0(l\in N)$ are constants;

(4) $\tau\le t_l-t_{l-1}\le\alpha$ and $\ln(d_l)+\lambda\alpha<-\lambda(t_{l+1}-t_l)$.

Then the trivial solution of the impulsive system is globally exponentially stable.

Motivated by [9-10], and we will present some theorems on global expomemtial stability for system (8) based on Lyapunov function[11-13]. By the following theorems, we know that impulses play an important role in stabilizing asset price system in the financial market.

Theorem 1. Assume that λ_1 is the maximum value of A^T+A+2I, and λ_2 is the maximum value of D^TD, impulsive interval $\Delta t=t_{l+1}-t_l$, where $\tau\le t_l-t_{l-1}\le\alpha$, and such that

(1) $f(t,x)$ is a continuous function, and there exists a constant $M>0$ such that $\|f(t,x)\|\le M\|x\|$;

(2) $q\ge e^{2\lambda\alpha}$ is a constant, which satisfies

$\lambda_1+q\lambda_2+M^2<\lambda,$

and $d_l=\max\{(1+B_1)^2,(1+B_2)^2\}$;

(3) $\ln\|I+B_l\|+\frac{\alpha}{2}\lambda<-\frac{\lambda}{2}(t_{l+1}-t_l).$

Then the trivial solution of the impulsive system is globally exponentially stable.

Proof. Let us construct the Lyapunov function $V(t,x)=x^Tx$. There exists $c_1=\frac{1}{2},c_2=\frac{3}{2},p=2$ such that $\frac{1}{2}\|x\|^2\le V(t,x)\le\frac{3}{2}\|x\|^2$, which satisfy the condition (1) of Lemma 1.

$$\begin{aligned}
D^+V(t,x)&=(Ax+Dx(t-\tau)+f(t,x))^Tx\\
&\quad+x^T(Ax+Dx(t-\tau)+f(t,x))\\
&=x^T(A^T+A)x+x^T(t-\tau)D^Tx\\
&\quad+x^TDx(t-\tau)+f^T(t,x)x+x^Tf(t,x)\\
&=x^T(A^T+A)x+2x^TDx(t-\tau)+2x^Tf(t,x)\\
&\le x^T(A^T+A)x\\
&\quad+(x^Tx+[Dx(t-\tau)]^T[Dx(t-\tau)]\\
&\quad+(x^Tx+[f(t,x)]^T[f(t,x)])\\
&\le x^T(A^T+A+2I)x\\
&\quad+[x(t-\tau)]^TD^TDx(t-\tau)\\
&\quad+[f(t,x)]^T[f(t,x)]\\
&\le\lambda_1\|x\|^2+\lambda_2[x(t-\tau)]^Tx(t-\tau)+\|f(t,x)\|^2\\
&\le\lambda_1\|x\|^2+\lambda_2[x(t-\tau)]^Tx(t-\tau)+M^2\|x\|^2
\end{aligned}$$

Based on the condition (2) of Lemma 1, there exists $q\ge e^{2\lambda\alpha}$ such that $qV(t,\varphi(0))\ge V(t+h,\varphi(h)),h\in[-\tau,0)$. Therefore we have $qV(t,x(t))\ge V(t-\tau,x(t-\tau))$, then,

$$\begin{aligned}
D^+V(t,x)&\le\lambda_1\|x\|^2+q\lambda_2\|x\|^2+M^2\|x\|^2\\
&=(\lambda_1+q\lambda_2+M^2)\|x\|^2
\end{aligned}$$

For $I+B_l$ is diagonal matrix, then we obtain

$\rho(I+B_l)=\|I+B_l\|$, By condition (3) of Lemma 1, we know that $d_l=\max\{(1+B_1)^2,(1+B_2)^2\}$, and

$V(t_l,x+B_lx)=x^T(I+B_l)^T(I+B_l)x\le d_lV(t_l,x).$

We shall show that $\|I+B_l\|=\sqrt{\lambda_{\max}((I+B_l)^T(I+B_l)}$ $\ge\sqrt{(1+B_1)(1+B_2)}\ge\sqrt{d_l}$,therefore, $\ln(\sqrt{d_l})+\frac{\alpha}{2}\lambda$ $\le\ln\|I+B_l\|+\frac{\alpha}{2}\lambda<-\frac{\lambda}{2}(t_{l+1}-t_l)$,

that is, $\ln d_l+\alpha\lambda<-\lambda(t_{l+1}-t_l)$, here, $\tau\le t_l-t_{l-1}\le\alpha.$

All this implies that the trivial solution of system is globally exponentially stable. The proof of Theorem 1 is therefore complete.

Remark 1. Theorem 1 gives some conditions for the global exponential stability, which has the particularity with respect to the lemma 1. It is equally effective for the application of the general conditions of the lemma to Theorem 1. The proof process indicates that the conditions for global exponential stability are associated with impulsive interval Δt and some another factors.

Theorem 2. Assume that λ_1 is the maximum value of $A^T + A + 2I$, and λ_2 is the maximum value of $D^T D$, impulsive interval $\Delta t = t_{l+1} - t_l$, where $\tau \le t_l - t_{l-1} \le \alpha$, and such that

(1) $f(t,x)$ is a continuous function, and there exists a constant $M > 0$ such that $\|f(t,x)\| \le M\|x\|$;

(2) $q \ge e^{2\lambda\alpha}$ is a constant, which satisfies

$$\lambda_1 + q\lambda_2 + M^2 < \lambda,$$

and $d_l = \max\{(1+B_1)^2, (1+B_2)^2\}$;

(3) There exists a constant $\gamma < -1$, such that

$$\frac{2\ln\|I + B_l\|}{\lambda(t_{l+1} - t_l)} + \frac{\alpha}{t_{l+1} - t_l} < \gamma.$$

Then the trivial solution of the impulsive system is globally exponentially stable.

5. Numerical simulation

In this section, we give an example and its simulation to illustrate our results.

Consider an example of the system (8). When $\mu = 1$, $n_s = 0.5, \beta_s = 1, n_c = 0.5, \beta_c = 1, k = 0.06, \tau = 0.1$, then

$$A = \begin{pmatrix} 0.5 & 0 \\ 0.12 & -0.06 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ -0.06 & 0 \end{pmatrix},$$

$$f(t,x) = \begin{pmatrix} 0.5\tanh(x_1 - x_2) \\ 0 \end{pmatrix}.$$

Consider the system (6) without impulsive control, the trivial solution of system (6) is instable. When initial value $x(0) = (1, 0.5)^T$, as Fig. 1 shows the instability.

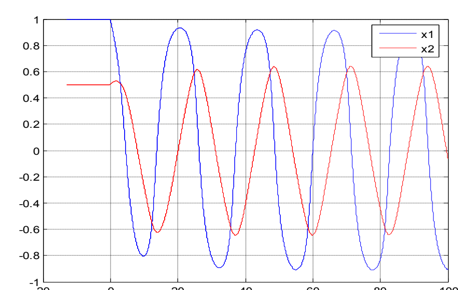

Fig.1 The system diagram without impulsive control

When adding impulsive control, we choose impulsive interval $\Delta t = 0.1$, and if we have impulsive parameter

$$B_l = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix},$$

then $d_l = \max\{(1+B_1)^2, (1+B_2)^2\} = 0.25$, $\lambda_1 = 3.0127, \lambda_2 = 0.0036$, let $\alpha = 0.25, q = 6$, $\lambda = 3.5, M = 0.5$ satisfy that

(1) $q = 6 \ge e^{2\lambda\alpha} = 5.7546$,

$\lambda_1 + q\lambda_2 + M^2 = 3.2843 < \lambda = 3.5$;

(2) $0.1 = \tau \le t_l - t_{l-1} \le \alpha = 0.25$,

$\ln d_l + \lambda\alpha = -0.511294 < -\lambda(t_{l+1} - t_l) = -0.35$.

As Fig. 2 shows that trival solution of the system (8) is globally exponentially stable.

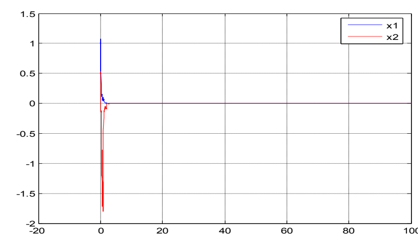

Fig.2 The system diagram when impulsive parameter is

$$B_l = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}.$$

If we have impulsive interval $\Delta t = 0.4$, and $\tau = 0.5 > \Delta t = 0.4$, and take impulsive parameter

$$B_l = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.2 \end{pmatrix},$$

then $d_l = \max\{(1+B_1)^2, (1+B_2)^2\} = 0.64$, the other parameters are the same. All above don't satisfy conditions of the theorem 1 and theorem 2. Fig 3 shows the condition.

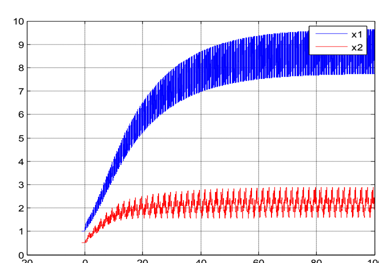

Fig.3 The system diagram when impulsive parameter is

$$B_l = \begin{pmatrix} -0.2 & 0 \\ 0 & -0.2 \end{pmatrix}$$

Comparing Fig. 2 and Fig. 3, on the premise of satisfying the conditions of theorem, the system is able to realize the global exponential stability. If don't meet the conditions of the theorem and the state regulation are not reasonable, namely the impulsive interval is less than the delay, then the system can not realize the global exponential stability. As shown in Fig. 3, asset price has deviated from the fundamental price and appears volatility. Both of Fig. 2 and Fig. 3 show the value of impulsive interval will affect the stability transition. At the same time, it shows that in the condition of the financial market, the state macro-control time and whether regulation is reasonable or not play an important role in the price stability.

If we choose impulsive parameter

$$B_l = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}$$

then $d_l = \max\{(1+B_1)^2, (1+B_2)^2\} = 0.81$, and the other parameters are the same. We know the situation still doesn't satisfies conditions of the theorem. But as shown in Fig. 4, trival solution of the system ia stable.

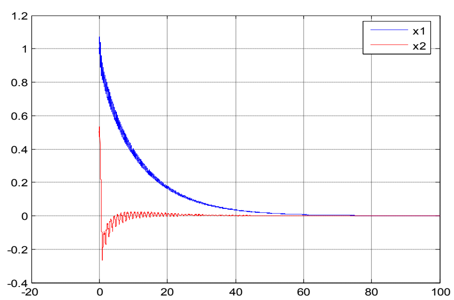

Fig.4 The system diagram when impulse parameter is

$$B_l = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \end{pmatrix}$$

Fig .4 doesn't meet the condition

$\ln\|I + B_l\| + \frac{\alpha}{2}\lambda < -\frac{\lambda}{2}(t_{l+1} - t_l)$ of the theorem 1, which shows that the theorem conditions to realize the global exponential stability are conservative. Compared with Fig.2, Fig.4 implies the exponential stability with the slow speed, which indicates that impulsive parameters affects the rate on the change of stability. Comparing Fig.3 and Fig.4, differenceis that the size of impulsive interval value, further indicating that the impulsive interval value plays an important role in the change of stability.

The system (6) is not stable in the absence of impulsive control, as shown in Fig.1 that appears periodic oscillation. When the system adds impulsive control, namely through the national macro-control, such as tax, letter loans, financial subsidies, adjusting the price stable. When adding the country macroeconomic regulation and control, if we act in accordance with market rules(here the financial market rules referred is theorem 1 and theorem 2), then it is shown in Fig.2 that the price is globally exponential stable. If it isn't in line with the conditions with theorem 1 and theorem 2, that is the national macro-control isn't reasonable, then the commodity price will deviate from the fundamental price farther and farther, as shown in Fig.3. If the national macro-control is reasonable and in order, although not quickly realize global exponential stability, but still can realize the stability at last, as shown in Fig.4.

6. Conclusion and Discussion

By the impulsive control, the stability in asset price system has been analyzed. Numerical simulations verify the validity of theoretical method. Of course, this is only a special case in financial system. At present, although many scholars have made a lot of research on Mathematical economics and achieved rich practical significant theoretical results. But a lot of problems will be further studied because of few literatures on the asset price system with time delay under impulsive control.

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