

The Poincaré–Nekhoroshev Map

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Received March 04, 2002; Accepted June 28, 2002

Abstract

We study a generalization of the familiar Poincaré map, first implicitly introduced by N N Nekhoroshev in his study of persistence of invariant tori in hamiltonian systems, and discuss some of its properties and applications. In particular, we apply it to study persistence and bifurcation of invariant tori.

Introduction

The *Poincaré map* is a classical tool in the study of a dynamical system around a known periodic solution (see e.g. [3, 7, 10, 11, 15, 16]).

Here we want to study a dynamical system around a multi-periodic solution (i.e. an invariant torus \mathbb{T}^k , $k > 1$): in this case the Poincaré map has several drawbacks, and it would be more convenient to somehow quotient out the degrees of freedom corresponding to motion along the invariant torus (and transversal to the dynamical flow). However, such a quotient is in general terms ill-defined out of the torus itself.

In a paper [12] devoted to persistence of invariant tori in partially integrable hamiltonian systems with n degrees of freedom and k integrals in involution ($1 < k < n$), N N Nekhoroshev devised a way to overcome this obstacle, and generalized the classical Poincaré–Lyapounov theorem. The main nondegeneracy condition for this theorem was expressed in terms of monodromy operators. Unfortunately, his discussion was very short and somehow not easy reading, and – as far as I know – he never published a proof of this result.

Here we note that the whole matter is better understood in terms of a generalization of the Poincaré map, which in my opinion is implicitly introduced in [12] and which I will call the Poincaré–Nekhoroshev map. This map is of interest *per se*, i.e. not just for the Poincaré–Lyapounov–Nekhoroshev theorem.

The aim of the present note is to discuss in detail the Poincaré–Nekhoroshev map (which can be defined also for non-hamiltonian systems), its geometry and its spectrum. In particular I will discuss how this bypasses the obstruction to considering a symmetry quotient (note that when such obstruction is not present, we can pass to the quotient system and apply standard Poincaré theory there), i.e. how this can be applied without

assuming regularity of global invariant manifolds near the invariant torus, and its relation with monodromy.

The **plan of the paper** is as follows. In Section 1 we fix some basic notation and recall background results concerning closed trajectories on tori. In Section 2 we recall the definition of the standard Poincaré map, and provide a geometrical interpretation of it in terms of a local fibration. Section 3 is devoted to defining the Poincaré–Nekhoroshev map, which can now be seen geometrically as a direct generalization of the standard Poincaré map; the subsequent sections deal with applications of the Poincaré–Nekhoroshev map. In Section 4 we discuss the relation between fixed points of the Poincaré–Nekhoroshev map and invariant manifolds, in particular tori. Section 5 is devoted to study persistence of invariant tori in non-hamiltonian systems from a geometrical point of view; the same question is discussed in Section 6 with the use of a coordinate system and thus providing explicit formulas. The brief Section 7 discusses invariant tori in hamiltonian systems (i.e. the subject of [12]) from the present standpoint. Finally, in Section 8 we discuss how standard results for bifurcation of fixed points of maps are to be interpreted in this frame as describing bifurcations from an invariant torus.

1 Notation and background

We consider a smooth n -dimensional manifold M (by smooth we will always mean C^r with some fixed r , $1 \leq r \leq \infty$, constant throughout the paper), and in this k independent smooth vector fields X_i , $i = 1, \dots, k$, spanning a k -dimensional Lie algebra \mathcal{G} . We are specially interested in the case – and thus we assume – that \mathcal{G} is abelian, i.e. $[X_i, X_j] = 0$.

We denote by G the connected Lie group generated by \mathcal{G} , and by $G_0 = \{\exp[\varepsilon X], X \in \mathcal{G}, -\varepsilon_0 < \varepsilon < \varepsilon_0\} \subset G$ the *local* Lie group generated by \mathcal{G} ; local Lie groups are discussed e.g. in [8, 13].

We stress that in general G is not compact, and we are *not* assuming it acts regularly or with regularly embedded orbits in M .

Suppose now that there is a smooth compact and connected submanifold $\Lambda \subset M$, which is \mathcal{G} -invariant (this means $X_i : \Lambda \rightarrow T\Lambda$ for all $i = 1, \dots, k$), and such that the X_i are linearly independent at all points $m \in \Lambda$. As \mathcal{G} is abelian and Λ is compact and connected, necessarily $\Lambda \simeq \mathbb{T}^k$ (the equivalence being a smooth isomorphism), see e.g. [2]. Note also that the linear independence of the X_i at all points of Λ implies that they are linearly independent in a tubular neighbourhood $U \subset M$ of Λ .

As the X_i are independent on Λ , we can choose coordinates $(\varphi_1, \dots, \varphi_k)$ on Λ (with $\varphi_i \in S^1$) such that $Y_i := (\partial/\partial\varphi_i)$, and the loops Γ_i corresponding to the φ_i coordinate running from 0 to 2π while the others remain constant can be chosen as basis cycles in Λ .

The homotopy class of a loop γ , which we will denote as $h(\gamma) = \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k$ (we also write $h_i(\gamma) = \alpha_i$), counts the winding of γ around the basis cycles of Λ ; with the choices mentioned above and this notation, $\alpha_i = h_i(\gamma)$ is just the increase of $\varphi_i/(2\pi)$ along the path γ .

The following lemma is well known, but we will however give a proof of it, also in order to fix some notation.

Lemma 1. *Take any loop $\tilde{\gamma}$ and any point $m \in \Lambda$. Then there is a loop γ with $h(\tilde{\gamma}) = h(\gamma)$, which is the orbit through m of a vector field of the form*

$$X_\alpha = \sum_{i=1}^k c_i(\alpha) X_i \quad (1)$$

with suitable $c_i(\alpha) \in \mathbb{R}$.

Proof. The flow on Λ under the vector field

$$X_\alpha = 2\pi \sum_{i=1}^k \alpha_i X_i \quad (2)$$

obeys the equations $(d\varphi_i/ds) = 2\pi\alpha_i$; these obviously have the solution

$$\varphi_i(s) = \varphi_i(0) + 2\pi\alpha_i s. \quad (3)$$

For $s \in [0, 1]$ this describes a loop γ in Λ with homotopy class $h(\gamma) = \alpha$. Thus for a given path $\tilde{\gamma}$ with homotopy class $h(\tilde{\gamma}) = \alpha$, equation (2) gives the required vector field, and equation (3) yields the homotopically equivalent path γ , mentioned in the statement. Obviously there is such a path through any point $m_0 = (\varphi_1(0), \dots, \varphi_k(0)) \in \Lambda$. Note we have also determined the coefficients $c_i(\alpha)$ appearing in (1): they are just $c_i(\alpha) = 2\pi\alpha_i$. ■

The notion of *monodromy operators* is also well known, but we will quickly recall it, again in order to fix some notation.

Let X be a vector field in M , and assume there is a nontrivial closed orbit γ for X passing through $m \in M$ and having finite period τ ; obviously we can always take $\tau = 1$ by rescaling t or X , or both.

We consider then the *total monodromy map* $T_X = \exp[\tau X]$, which maps the point $x = x(0) \in M$ to the point $x(\tau)$ on the flow $x(t)$ of X with $x(0) = x$ at time τ . With our assumption, $T_X(m) = m$ for all points $m \in \gamma \subset M$.

Let us denote by $A_{(X,m)}$ the linearization of T_X at $m \in \gamma$, i.e. $A_{(X,m)} := [DT_X]_m$. The $A_{(X,m)}$ is also called the *total monodromy operator*.

Note that, as the X flow through m is periodic, $A_{(X,m)}$ will always have an eigenspace tangent to $X(m) \in T_m M$ and corresponding to an eigenvalue one. We can thus – with no loss of information – consider the projection of T and A (denoted by P and L respectively) to a subspace transversal to $X(m)$ in $T_m M$; we will call them *transversal monodromy map* and *transversal monodromy operator*. (Note this terminology is not standard: in part of the literature these are defined to be the monodromy map and monodromy operator; the reader will easily avoid confusion by looking at the dimension of spaces involved.)

The eigenvalues λ_i of $L_{(X,m)}$ are called *characteristic (or Floquet) multipliers*; they carry most of the information needed to study the dynamics defined by X around the periodic orbit γ .

It is also customary to write $L_{(X,m)} = \exp[\tau Q_{(X,m)}]$; the eigenvalues χ_i of $Q_{(X,m)}$, having the obvious relation $\lambda_i = \exp[\tau\chi_i]$ with those of $L_{(X,m)}$, are called, *characteristic (or Floquet) exponents*.

Remark 1. It is well known that monodromy operators at different points in γ are conjugated, so that the spectrum of $A_{(X,m)}$ depends on γ but not on the point $m \in \gamma$. Similarly, the monodromy operator based at $m \in \gamma$ around a path γ depends only on the homotopy class of γ , and not on the actual path. See [3, 7, 15] for details.

It will follow from this remark that in our subsequent discussion – where we will need only the spectrum of monodromy operators – we can limit to consider vector fields of the form (1) and correspondingly paths of the form (3), which also sets $\tau = 1$; moreover it is easy to see that different X_α orbits in Λ are conjugated by a G -action. Note also that the monodromy map and operator are invariant under a rescaling of X and/or t (recall τ is the period and hence changes accordingly). Thus when dealing with X_α we will simply write T_α , A_α and Q_α , and call T_α the “time-one map” under X_α .

2 The Poincaré map

Let M be a n -dimensional smooth manifold. As well known, the Poincaré map is defined in the neighbourhood of (an arbitrary point on) a periodic orbit of the vector field. Let γ be a nontrivial closed orbit through the point $m \in M$ for a smooth vector field X . Consider a local manifold Σ through m , transversal to γ in m (it is well known that the Poincaré map does not depend on our choice of Σ , see e.g. [15]).

For Σ_0 a suitably small neighbourhood of m in Σ , orbits through points $x \in \Sigma_0 \subseteq \Sigma$, $x \neq m$, first intersect Σ at a point x' , in general with $x' \neq x$; the Poincaré map P is then defined as $P(x) = x'$ [3, 7, 10, 11, 15, 16]. (We need the restriction to $\Sigma_0 \subseteq \Sigma$ as the orbit through x could fail to meet again Σ if x is too far from m .)

For the present discussion it will be convenient to define the Poincaré map in a more geometric way (see also [7]). We put again the period of the periodic orbit γ for the vector field X equal to one (just rescale X or t if needed).

Consider a G -invariant neighbourhood U of m in M (note that a G -invariant neighbourhood could fail to exist). As $X(m) \neq 0$, by the flow box theorem [3] we can choose in U coordinates (ξ^1, \dots, ξ^n) , say with $m = (0, \dots, 0)$, such that $X = (\partial/\partial\xi^n)$.

We can, for the sake of simplicity, take Σ to be described by $\xi^n = 0$. Note that in this way – or however identifying locally Σ with its tangent space at m , $S \subset T_m M$ – the Poincaré map can also be thought as an application between (open sets in) linear spaces.

Consider then the time-one flow $x \mapsto T(x)$ of points in Σ_0 under X . The point m is obviously mapped again, by construction, to itself: $T(m) = m$. Nearby points $x = (\xi^1, \dots, \xi^{n-1}, 0)$ are in general not mapped to themselves, and not even mapped back to Σ . Let $\pi : U \rightarrow \Sigma$ be the projection operator to Σ , given in the ξ coordinates by $\pi(\xi^1, \xi^2, \dots, \xi^n) = (\xi^1, \dots, \xi^{n-1}, 0)$; it is clear that the Poincaré map P is described by

$$P(x) = \pi[T(x)].$$

Remark 2. In abstract terms, this description can be reformulated as follows. The quotient by the X action is well defined in U ; the Poincaré map is nothing else than the time-one flow map under X , modulo this quotient. This can also be seen as introducing in U the structure of a fiber bundle $(\pi : U \rightarrow \Sigma)$ over Σ ; the vector field X is vertical, and the Poincaré map is the projection to the base space of the time-one flow under X .

Let L be the linearization of the Poincaré map at the fixed point m , $L = (DP)(m)$. This is actually the transversal monodromy operator and, as well known and mentioned above, the spectrum of L and that of the complete monodromy operator $A = (DT)(m)$ are closely related. Indeed, let $\{\lambda_1, \dots, \lambda_{n-1}\}$ be the eigenvalues of L ; then the eigenvalues of A are $\{\lambda_1, \dots, \lambda_{n-1}; \lambda_n = 1\}$. The last eigenvalue $\lambda_n = 1$ corresponds to the eigenspace spanned by $X(m)$ in $T_m M$ (i.e. to the line $\xi^1 = \dots = \xi^{n-1} = 0$) and is always present, by construction, in the spectrum of A .

3 The Poincaré–Nekhoroshev map

Let us now come back to consider the case of a \mathcal{G} -invariant \mathbb{T}^k submanifold $\Lambda \subset M$ (as this is closed and compact, \mathcal{G} -invariance implies G -invariance). Choose a reference point $m \in \Lambda$, and a smooth local submanifold $\Sigma \subset M$, transversal to Λ in m .

Consider again a suitably small \mathcal{G} -invariant neighbourhood $U \subseteq M$ of m in M ; as the commuting vector fields X_i ($i = 1, \dots, k$) are nonzero and linearly independent in m , by the flow box theorem we can choose local coordinates (ξ^1, \dots, ξ^n) in U such that the vector fields X_i are written, in these coordinates, as $X_i = (\partial/\partial \xi^{r+i})$, for $i = 1, \dots, k$; here and below, $r := n - k$. Again for ease of discussion, choose the Σ to be identified by $\xi^{r+1} = \dots = \xi^n = 0$.

We denote now by $\pi : U \rightarrow \Sigma$ the operator of projection to Σ , given in the (ξ^1, \dots, ξ^r) coordinates by

$$\pi(\xi^1, \dots, \xi^r, \xi^{r+1}, \dots, \xi^n) = (\xi^1, \dots, \xi^r; 0, \dots, 0).$$

The time-one flow under X_α will again define a local map $T_\alpha : \Sigma_0 \rightarrow U$, where $\Sigma_0 \subseteq \Sigma$ is a suitable small neighbourhood of m in Σ .

Definition. The *Poincaré–Nekhoroshev* map $\mathcal{P}_{\alpha,m} : \Sigma_0 \rightarrow \Sigma$ associated to the vector field X_α and based at $m \in \Lambda$ is defined in this notation as $\mathcal{P}_{\alpha,m}(x) = \pi[T_\alpha(x)]$.

Analogously to the standard Poincaré case, this description can be reformulated in abstract terms.

The quotient by the \mathcal{G} action is well defined in U ; the Poincaré map is nothing else than the time-one flow map under X_α , modulo this quotient.

This can also be seen again as introducing in U the structure of a fiber bundle $(\pi : U \rightarrow \Sigma)$ over Σ ; the vector fields X_i are vertical, and the Poincaré map $\mathcal{P}_{\alpha,m}$ is the projection to the base space of the time-one flow under X_α .

Remark 3. The key point in this construction is that if we consider a tubular neighbourhood \mathcal{N} of Λ in M and the Lie group G generated by \mathcal{G} , the quotient \mathcal{N}/G is ill-defined except in situations where the G action is known *a priori* to be pretty simple (regular orbits); in this case a G -invariant neighbourhood is known to exist, and moreover one can simply consider the quotient system and apply on this the standard Poincaré theory. On the other hand, restricting to a neighbourhood of the *local* smooth manifold Σ_m we can always consider the quotient by the *local* Lie group G_0 .

Remark 4. Let Z be a contractible neighbourhood in Λ , and define transversal manifolds Σ_m through any point $m \in Z \subset \Lambda$; their union is an open set W . Inside this there

is a \mathcal{G} -invariant neighbourhood $W_0 \subseteq W$ of Z which can be seen as a trivial fiber bundle $(\mu : W_0 \rightarrow Z)$ over Z ; the vector fields X_i are horizontal in W_0 and thus define a field of horizontal k -planes, i.e. a connection in W_0 . By Frobenius' theorem, there are local smooth k -manifolds in W_0 which are everywhere tangent to this field of k -planes and thus \mathcal{G} -invariant.

Remark 5. It should be stressed that the Poincaré–Nekhoroshev map can also be seen as the composition of two maps in a slightly different way: time-one flow under X_α and the flow (for a time $t_b(x)$ which we do not need to determine) under a vector field $X_b = \sum_i b_i X_i$: indeed, any two points on the same fiber $\pi^{-1}(x)$ can be joined in this way. It is immediate from this that the Poincaré–Nekhoroshev map is the composition of two smooth maps, and is thus itself a smooth map.

We will now consider the linearization $L_{\alpha,m}$ of the Poincaré–Nekhoroshev map $\mathcal{P}_{\alpha,m}$ around the fixed point m ; we are specially interested in its spectrum.

It turns out that this spectrum is independent of the base point $\varphi_0 \in \Lambda$, i.e. depends only on the homotopy class α ; moreover, it is simply related to the spectrum of the total monodromy operator $A_{\alpha,m} := (DT_\alpha)(m)$ for X_α .

It is clear that $A_{\alpha,m}$ always has k eigenvalues equal to one; these correspond to eigenspaces spanned by the X_i (that is, tangent to Λ) at m . In the ξ coordinates, these span the subspace $\xi^{k+1} = \dots = \xi^n = 0$.

This observation shows immediately the relation between the spectra of $A_{\alpha,m}$ and of $L_{\alpha,m}$: if the spectrum of $A_{\alpha,m}$ is given by $\{\lambda_1, \dots, \lambda_r; 1, \dots, 1\}$ ($r = n - k$), then the spectrum of $L_{\alpha,m}$ is given by $\{\lambda_1, \dots, \lambda_r\}$, and viceversa. (This of course also establishes a relation between the spectra of $L_{\alpha,m}$ and that of the transversal monodromy operator.)

Lemma 2. *Given any two points φ_0 and φ_1 in Λ , and any homotopy class $\alpha \in \pi_1(\Lambda)$, the matrices $L_{\alpha;\varphi_0}$ and $L_{\alpha;\varphi_1}$ are conjugated; hence their spectra coincide.*

Proof. As recalled above, the spectra of monodromy operators $A_{\alpha,m}$ only depend on α , not on m ; hence the same holds for the spectra of the linearized Poincaré–Nekhoroshev maps $L_{\alpha,m}$, see above. ■

4 Invariant tori

In this section we discuss – to the extent needed for our goals – the relation between invariant tori and fixed points of the Poincaré–Nekhoroshev map.

We assume that there is a G -invariant submanifold $\mathcal{M}_0 \subseteq M$, with $\Lambda \subseteq \mathcal{M}_0$; we denote by $L_{\alpha,m}^{(0)}$ and $\mathcal{P}_{\alpha,m}^{(0)}$ the restrictions of $L_{\alpha,m}$ and $\mathcal{P}_{\alpha,m}$ to \mathcal{M}_0 .

Lemma 3. *If $L_{\alpha,m}^{(0)}$ has no eigenvalue of unit norm, then Λ is an isolated G -invariant torus in \mathcal{M}_0 .*

Proof. Assume there is a G -invariant torus near Λ in \mathcal{M}_0 ; it will intersect σ_m in some point $x \neq m$ near m , and necessarily $\mathcal{P}_{\alpha,m}^{(0)}(x) = x$. However, the condition on the spectrum of $L_{\alpha,m}^{(0)}$ implies it is an hyperbolic map, and thus the fixed point m is isolated in σ_m . This in turn implies the lemma. ■

Needless to say, if the spectrum of $L_{\alpha,m}$ (no restriction to \mathcal{M}_0) does not contain eigenvalues of unit norm, then Λ is isolated in M and not just in \mathcal{M}_0 .

The union of the stable and unstable manifolds for the torus Λ is obviously \mathcal{G} -invariant. Then we have immediately from Lemma 3 the

Corollary. *Let \mathcal{H} be the union of the stable and unstable manifolds for Λ ; then there is no G -invariant torus near Λ in \mathcal{H} .*

It is also obvious (and it has been used in the proof of Lemma 3 above) that a \mathcal{G} -invariant torus near Λ corresponds to a fixed point of the Poincaré–Nekhoroshev map; let us discuss if fixed point of the map correspond to invariant manifolds, and sufficient conditions for these to be tori.

Lemma 4. *Let $x \in \Sigma_m$ be a fixed point for the Poincaré–Nekhoroshev map $\mathcal{P}_{\alpha,m}$; then there is a \mathcal{G} -invariant smooth manifold through x .*

Proof. If $x = m$ the assertion is trivial, so assume $x \neq m$. By Remark 4, for Z a neighbourhood of m in Λ there is a well defined local \mathcal{G} -invariant manifold Y_0 through x ; with the construction introduced there, call $y(p)$ the point $Y_0 \cap \Sigma_p$, $p \in Z$ (so $y(m) = x$). Note that $\mathcal{P}_{\alpha,m}[y(m)] = y(m)$ implies $\mathcal{P}_{\alpha,m}[y(p)] = y(p)$ for all $p \in Z$. Consider now an atlas $\{Z_i\}$ of Λ : there is a local \mathcal{G} -invariant manifold Y_i over each chart Z_i , and by considering $Z_i \cap Z_j$ it is immediate to check that the transition functions are also smooth. Hence the Y_i blend together to give a smooth manifold Y , \mathcal{G} -invariant by construction. ■

Let us now consider the case where there is a G -invariant submanifold $\mathcal{M}_\beta \subset M$; define $\sigma_m^{(\beta)} := \Sigma_m \cap \mathcal{M}_\beta$, and note that $\mathcal{P}[\sigma_m^{(\beta)}] \cap \Sigma_m \subseteq \sigma_m^{(\beta)}$. We can thus define the restriction of the Poincaré–Nekhoroshev map to $\sigma_m^{(\beta)}$, denoted as $\mathcal{P}^{(\beta)}$.

Lemma 5. *Let \mathcal{M}_β , $\sigma_m^{(\beta)}$ and $\mathcal{P}_m^{(\beta)}$ be as above, and let $x \in \sigma_m$ be the unique fixed point for $\mathcal{P}_m^{(\beta)}$. Then there is a G -invariant k -torus through x , smoothly equivalent to Λ .*

Proof. This follows immediately from the construction used in previous lemma and the unicity of x : in this case there is a smooth one-to-one correspondence between points of Λ and points on the \mathcal{G} -invariant manifold $Y(x)$. As this is closed and compact, it is also G -invariant. ■

5 Persistence of invariant tori

We want to consider the case where the vector fields X_i depend smoothly on parameters; we aim at local results in the parameter space, so we will denote these parameters as $\varepsilon \in \mathcal{E} \subseteq E = \mathbb{R}^p$, and write $X_i^{(\varepsilon)}$. In this case we deal with a smooth manifold $\mathcal{M} = \mathcal{E} \times M$, which is foliated into G -invariant smooth submanifolds $\mathcal{M}_\varepsilon = \{\varepsilon\} \times M \simeq M$.

We assume $\Lambda \equiv \Lambda_0$ is an invariant torus for all the $X_i^{(0)}$ and wonder if – and under which conditions – this persists under perturbation, i.e. if there is some torus Λ_ε , near to Λ_0 and invariant under all the $X_i^{(\varepsilon)}$, for $\varepsilon \neq 0$ small enough.

Let us recall what is the situation for $k = 1$, i.e. for a single vector field $X^{(\varepsilon)}$ and a periodic orbit γ_0 of the vector field $X^{(0)}$. It is well known that, with an obvious extension

of the notation considered in Section 2, the *Poincaré–Lyapounov theorem* states that if the eigenvalues λ_i of the transverse monodromy operator $L^{(0)}$ associated to the path γ satisfy $|\lambda_i| \neq 1$, then the periodic orbit γ is actually part of a continuous p -parameters family of periodic orbits for X . This amounts essentially to using the implicit function theorem (see [3, 15] or e.g. [1], or any text in nonlinear analysis) for the Poincaré map, in order to ensure there is a p -parameters branch of fixed points for it, and recognizing that fixed points of the Poincaré map corresponds to periodic orbits.

A similar result, the **Poincaré–Lyapounov–Nekhoroshev theorem**, was obtained by Nekhoroshev [12] in the case of invariant tori, in terms of the spectra of the Poincaré–Nekhoroshev maps associated to a generating set of homology cycles for the torus Λ . Although his formulation was for hamiltonian dynamical systems, the theorem holds – with simple modifications – for general ones, and we will discuss it in this general setting (see Section 7 below for the hamiltonian case).

Theorem 1 (Nekhoroshev). *Let M be a n -dimensional smooth manifold, and $\mathcal{E} = E_0$ a neighbourhood of the origin in $E = \mathbb{R}^p$. Let $X_1^{(\varepsilon)}, \dots, X_k^{(\varepsilon)}$ be k smooth vector fields on M ($1 \leq k \leq n$), smoothly dependent on the p -dimensional parameter $\varepsilon \in \mathcal{E}$, independent for all $\varepsilon \in \mathcal{E}$, and such that $[X_i^{(\varepsilon)}, X_j^{(\varepsilon)}] = 0$ for all $\varepsilon \in \mathcal{E}$. We write $\mathcal{M} = \mathcal{E} \times M$, $\mathcal{M}_\varepsilon = \{\varepsilon\} \times M$, and denote by $\mathcal{G}^{(\varepsilon)}$ the Lie algebra spanned by the $X_i^{(\varepsilon)}$.*

Assume that:

- (i) *there exists a smooth k -dimensional torus $\Lambda_0 \subset \mathcal{M}_0$ invariant under all the $X_i^{(0)}$, and that these are linearly independent at all points of Λ_0 ;*
- (ii) *there is a $c \in \mathbb{R}^k$ such that the vector field $X_c^{(0)} = \sum_i c_i X_i^{(0)}$ has nontrivial closed trajectories with finite period τ in Λ_0 ;*
- (iii) *the spectrum of the linear part $L_c^{(0)}$ of the Poincaré–Nekhoroshev map associated to $X_c^{(0)}$ lies at a distance $\delta > 0$ from the unity.*

Then, in a neighbourhood \mathcal{V} of Λ_0 in \mathcal{M} , there is a smooth submanifold $\mathcal{N} \subset \mathcal{V} \subset \mathcal{M}$ which is fibered over the domain \mathcal{E} with as fibers smooth tori $\Lambda_\varepsilon \simeq \mathbb{T}^k$, smoothly equivalent to Λ_0 and $\mathcal{G}^{(\varepsilon)}$ -invariant.

Proof. We will focus on a point $m \in \Lambda_0$; choose a smooth submanifold $\Sigma \subset \mathcal{M}$ transversal to Λ_0 in m . By choosing suitable coordinates – basically, those of the tangent space $\mathbb{T}_m \Sigma \subset \mathbb{T}_m \mathcal{M}$ – we can identify a neighbourhood Σ_0 of m in Σ to a neighbourhood S_0 of the origin in a linear space S .

We define the submanifolds $\sigma^{(\varepsilon)} := \Sigma \cap \mathcal{M}_\varepsilon$, and let $\sigma_0^{(\varepsilon)} := \Sigma_0 \cap \mathcal{M}_\varepsilon$. In the same way as Σ_0 can be identified with a neighbourhood S_0 of the origin in the linear space S , the manifold $\sigma_0^{(\varepsilon)}$ can be identified with a neighbourhood $U_0^{(\varepsilon)}$ of the origin in a linear space $U_\varepsilon = \{\varepsilon\} \times U \simeq U \subset S$.

As the $X_i^{(\varepsilon)}$ do not act on the value of ε , the submanifolds \mathcal{M}_ε are trivially G -invariant, and by construction $\mathcal{P}_{\alpha, m} : \sigma_0^{(\varepsilon)} \rightarrow \sigma^{(\varepsilon)}$. We denote by $\mathcal{P}_{\alpha, m}^{(\varepsilon)}$ the restriction of $\mathcal{P}_{\alpha, m}$ to $\sigma_0^{(\varepsilon)}$.

It will also be convenient to separate the coordinates in \mathcal{E} and those in $U \simeq U_\varepsilon$: a point $x \in S_0$ will be denoted by coordinates $(\varepsilon, u) \in \mathcal{E}_0 \times U_0 \subset E \times U = S$. Thus we have coordinates $(\varphi, u; \varepsilon)$ with $\varphi \in \mathbb{T}^k$, $u \in U_0 \subset \mathbb{R}^{(n-k)}$, and $\varepsilon \in E_0 \subset \mathbb{R}^p$. We will write $N = k + r + p = n + p$.

By Lemma 1, we can consider X_α rather than X_c , where $\pi_1(c) = \alpha$. The discussion of Section 4 shows that the theorem can be restated in terms of – and proved by studying –

fixed points of the Poincaré–Nekhoroshev map $\mathcal{P}_{\alpha,m}$ associated to α and based at an arbitrary point m . We will think of α and m as fixed and omit indices referring to these, for ease of notation.

As remarked above, we can actually consider the restrictions of the Poincaré–Nekhoroshev map to the submanifolds $\sigma^{(\varepsilon)}$; we will thus look for fixed points of $\mathcal{P}^{(\varepsilon)}$.

Actually, it is convenient to slightly modify this formulation: considering the map $\Phi : S_0 \rightarrow S$ defined by

$$\Phi(x) := x - \mathcal{P}(x),$$

fixed points of the Poincaré–Nekhoroshev map correspond to zeroes of Φ , and we know that $\Phi(m) = 0$. Passing to the (ε, u) coordinates, we deal with a smooth map $\Psi : E_0 \times U_0 \rightarrow E \times U$, defined by $\Psi(\varepsilon, u) := (\varepsilon, u - \mathcal{P}^{(\varepsilon)}(u))$, and its restriction to U_ε is therefore

$$\Psi^{(\varepsilon)}(u) := (u - \mathcal{P}^{(\varepsilon)}(u)).$$

Consider the r -dimensional linear operator $\mathcal{B} : U_0 \rightarrow U$ defined as

$$\mathcal{B} := (D_u \Psi^{(0)})_m = I - L^{(0)}.$$

By the implicit function theorem (see e.g. [1]), if $\Psi^{(0)}(u_0) = 0$ and \mathcal{B} is invertible, then there are neighbourhoods $\tilde{E} \subset E_0$ of $\varepsilon_0 = 0$ and $\tilde{U} \subset U_0$ of $u_0 = 0$, and a smooth map $g : \tilde{E} \rightarrow \tilde{U}$, such that $\Psi(\varepsilon, g(\varepsilon)) = 0$ for all $\varepsilon \in \tilde{E}$; and moreover $\Psi(\varepsilon, u) = 0$ with $(\varepsilon, u) \in \tilde{E} \times \tilde{U}$ implies $u = g(\varepsilon)$.

In other words, if $\mathcal{B} : U_0 \rightarrow U$ is invertible, then there is a unique fixed point of \mathcal{P} , i.e. $(\varepsilon, g(\varepsilon))$, on each $\sigma^{(\varepsilon)}$, for $\varepsilon \in \tilde{E} = E_0$.

Due to the results of Section 4, this means that there is an invariant torus $\Lambda_\varepsilon \simeq \mathbb{T}^k$, $\Lambda_\varepsilon \subset \mathcal{M}_\varepsilon$, for each $\varepsilon \in \tilde{E}$, provided \mathcal{B} is invertible. However, the eigenvalues β_i of \mathcal{B} and λ_i of $L^{(0)}$ are simply related by $\beta_i = 1 - \lambda_i$, so \mathcal{B} is invertible provided $\lambda_i \neq 1$ for all $i = 1, \dots, n$. This concludes the proof. \blacksquare

Remark 6. Note that it is sufficient that there is one closed path with associated monodromy operator satisfying the condition (iii) of the theorem to ensure persistence of invariant tori.

Remark 7. The theorem and its proof are immediately generalized to the case of infinite dimensions; see [5] for the hamiltonian case and an application to breathers.

Remark 8. If we consider the general case of \mathcal{M} a N -dimensional smooth manifold, $\mathcal{E} \subset \mathcal{M}$ a p -dimensional manifold, and assume \mathcal{M} is foliated by regular G -invariant submanifolds \mathcal{M}_ε ($\varepsilon \in \mathcal{E}$), with $\Lambda_0 \in \mathcal{M}_0$, the theorem remains true. Indeed, our construction is purely local and is still valid (with E the tangent space to \mathcal{E} in $\varepsilon_0 = 0$, $M \simeq \mathcal{M}_0$).

6 The coordinate approach

So far our discussion has been mainly geometrical; in this section we will translate it into explicit formulas, making use of the $(\varphi, u; \varepsilon)$ coordinates defined above (recall that $\varphi \in \mathbb{T}^k$,

$u \in U_0 \subset U = \mathbb{R}^r$, $\varepsilon \in E_0 \subset E = \mathbb{R}^p$) in a neighbourhood $\mathcal{V} \subset \mathcal{M}$ of Λ . We stress that we do *not* assume \mathcal{V} is G -invariant, nor we use \mathcal{G} -adapted coordinates.

As the vector fields $X_i \equiv X_i^{(\varepsilon)}$ do not act on the parameters ε , their expression in these coordinates will be

$$X_i = \sum_{j=1}^k f_i^j(\varphi, u; \varepsilon) \frac{\partial}{\partial \varphi_j} + \sum_{\mu=1}^r F_i^\mu(\varphi, u; \varepsilon) \frac{\partial}{\partial u_\mu}.$$

For ease of notation, from now on summation over repeated indices will be tacitly understood; latin indices other than ℓ will run from 1 to k , while $\ell = 1, \dots, p$, and greek indices will run from 1 to r . We also write $\partial_i := (\partial/\partial\varphi^i)$ and $\partial_\mu := (\partial/\partial u^\mu)$.

The torus $\Lambda \equiv \Lambda_0$ corresponds to $u = 0, \varepsilon = 0$, and its invariance guarantees the vanishing of $F_i^m(\varphi, 0; 0)$. Similarly, in Section 1 the coordinates φ were chosen so that on Λ_0 we had $X_i = (\partial/\partial\varphi^i)$; hence $f_i^j(\varphi, 0; 0) = \delta_i^j$.

Expanding X_i at first order in ε and u around Λ , and considering then $X_\alpha = c^i(\alpha)X_i$, we get

$$X_\alpha = c^i \partial_i + [\tilde{P}_\nu^j u^\nu + \tilde{Q}_\ell^j \varepsilon^\ell] \partial_j + [\tilde{A}_\nu^\mu u^\nu + \tilde{B}_\ell^\mu \varepsilon^\ell] \partial_\mu + \text{h.o.t.}, \quad (4)$$

where *h.o.t.* denotes higher order terms in (u, ε) , $c^i \equiv c^i(\alpha)$, and

$$\begin{aligned} \tilde{P}_\mu^j &:= c^i (\partial f_i^j / \partial u^\mu); & \tilde{Q}_\ell^j &:= c^i (\partial f_i^j / \partial \varepsilon^\ell); \\ \tilde{A}_\nu^\mu &:= c^i (\partial \psi_i^\mu / \partial u^\nu); & \tilde{B}_\ell^\mu &:= c^i (\partial \psi_i^\mu / \partial \varepsilon^\ell). \end{aligned} \quad (5)$$

All partial derivatives are computed on Λ , so that the matrices \tilde{A} , \tilde{B} , \tilde{P} , \tilde{Q} are function of $\varphi \in \mathbb{T}^k$.

We write $\varphi^i(t) = \varphi_0^i(t) + \vartheta(t)$, where $\varphi_0^i(t) = \varphi^i(0) + c^i t$ and $\vartheta \simeq O(\varepsilon, u)$. Therefore, as we keep only first order terms in (ε, u) in the expression for X_α , see (4), we can consider $\tilde{A}(\varphi) \simeq \tilde{A}(\varphi_0^i(t)) := \tilde{A}(t)$, and similarly for the other matrices. Note that \tilde{A} and the like are explicit periodic functions of time.

The linearized flow around Λ under X_α is hence described by

$$\dot{u} = \hat{A}u + \hat{B}\varepsilon; \quad \dot{\vartheta} = \hat{P}u + \hat{Q}\varepsilon; \quad \dot{\varepsilon} = 0. \quad (6)$$

However, for the sake of discussing the Poincaré–Nekhoroshev map only the first equation is relevant.

Our discussion in the previous section shows that we can actually consider just the restriction of this dynamics to the space $\varepsilon = 0$, in which case we just deal with

$$\dot{u} = \hat{A}(t)u, \quad (7)$$

i.e. a linear ODE in \mathbb{R}^r with periodic coefficients. The method of analysis of such equations is well known (see e.g. [10, 16]), and we briefly recall it.

One considers a fundamental matrix for $\hat{P}(t)$ (this is built with a set of r independent solutions); this matrix $\Theta(t)$ satisfies $\dot{\Theta} = \hat{A}\Theta$. By Floquet's theorem [16], it is always possible to write $\Theta(t) = M(t) \exp[Bt]$, with M a periodic and B a constant matrix. Then one performs the change of variables $u = M(t)v$; using $\dot{\Theta} = \hat{A}\Theta$, and thus $\dot{M} = (\hat{A}M - MB)$, and the existence of M^{-1} , one gets $\dot{v} = Bv$.

With a matrix R of eigenvectors for B , we can further write $v = Rw$, and get $\dot{w} = Dw$, where $D = R^{-1}BR = \text{diag}(\lambda_1, \dots, \lambda_r)$. The solution of this is obviously $w^i(t) = \exp[\lambda_i t]w^i(0)$ (no sum on i), which yields $u(t) = M(t)R(\exp[Dt])(R^{-1})(M^{-1})(0)u(0)$.

At time $t = T$ we get $u(T) = Qu(0)$, where $Q = M_0R(\exp[DT])R^{-1}M_0^{-1}$, and $M_0 = M(0) = M(T)$. By definition, Q is the monodromy matrix for (7), and obviously the spectrum of Q – which is the same as that of B – is just given by $\mu_i := \exp[\lambda_i T]$. The μ_i are the characteristic multipliers, and the λ_i are the characteristic exponents, for (7).

Remark 9. Note that the situation is rather different if we want to compute the Floquet exponent for $\varepsilon \neq 0$: indeed in this case we deal with an equation of the form $\dot{u} = Au + b$, with $A = A(t)$ a periodic matrix and $b = b(t)$ a periodic vector function, $b^\mu = \widehat{B}_\ell^\mu \varepsilon^\ell$, see (6). Proceeding as above we arrive at

$$\dot{w} = Dw + f(t), \quad D = \text{diag}(\lambda_1, \dots, \lambda_r);$$

here $f(t)$, obtained by the action of periodic matrices on periodic vectors, is still periodic with the same period T .

If the periods of small u oscillations for $\varepsilon = 0$ are different from T , i.e. if the characteristic multipliers μ_i computed above satisfy $\mu_i \neq 1$, the solution will be of the form $w^i(t) = \exp[\lambda_i t]w^i(0) + F^i(t)$ (no sum on i) with F^i a periodic function; this does not affect the period maps and the discussion remain valid with the same monodromy matrix Q .

On the other hand, if there is some characteristic multiplier $\mu_i = 1$, solutions will not be of the same form, and terms proportional e.g. to $t \exp[\lambda_i t]$ will appear.

Remark 10. It should be stressed that, as clear from the discussion in this section, all we need to know in order to ensure the conditions of the theorem are satisfied are the matrices of partial derivatives $(\partial\psi_i^\mu/\partial u^\nu)$ computed at $u = 0$, $\varepsilon = 0$; they concur to form \widehat{A} , see (5).

This could be understood in a slightly different way: write $z = (u, \varepsilon) \in \mathbb{R}^{r+p}$; then the linearized evolution equations for z read $\dot{z} = Wz$, with

$$W = \begin{pmatrix} \widehat{A} & \widehat{B} \\ 0 & 0 \end{pmatrix};$$

the spectrum of W is given by $\lambda = 0$ (with multiplicity p) and by the eigenvalues $\lambda_1, \dots, \lambda_r$ of \widehat{A} . Thus we just have to check these satisfy $\lambda_i \neq 1$.

7 The hamiltonian case

In the hamiltonian case, we consider a symplectic manifold $(\mathcal{M}^{2n}, \omega)$, and k independent and mutually commuting hamiltonians H_1, \dots, H_k (commutation is meant, of course, with respect to the Poisson bracket $\{\cdot, \cdot\}$ defined by the symplectic form ω). Each of these defines a (hamiltonian) vector field X_i by $i_X(\omega) = dH_i$, and $\{H_i, H_j\} = 0$ implies $[X_i, X_j] = 0$. We denote by \mathcal{G} the abelian Lie algebra spanned by the X_i , and by G its Lie group.

Note that the (H_1, \dots, H_k) are common constants of motion for any dynamics defined by a linear combination of the H_i (equivalently, of the vector fields X_i), so that their values

(h^1, \dots, h^k) can be seen as parameters. We denote the common level manifold $H_i = h^i$ by $\mathbf{H}^{-1}(h)$.

If there exists a \mathcal{G} -invariant torus $\Lambda_0 \subset \mathcal{M}$, this is necessarily contained in $\mathbf{H}^{-1}(h_0)$ for some $h_0 \in \mathbb{R}^k$; we write then $h^i = h_0^i + \varepsilon^i$. We also assume the X_i are independent on Λ_0 .

We are thus exactly in the scheme discussed in previous sections, with $p = k$.

Remark 11. Note that, using freely the notation introduced above, the variables canonically conjugated to the $(\varphi^1, \dots, \varphi^k)$ via the symplectic structure are proportional to the $(\varepsilon^1, \dots, \varepsilon^k)$; this also implies that the characteristic multipliers relative to eigenvectors in the space $T_\Lambda E \subset T_\Lambda \mathcal{M}$ are the same as those relative to eigenvectors in $T_\Lambda \Lambda \subset T_\Lambda \mathcal{M}$, i.e. are all equal to one.

However, this is no problem as far as Nekhoroshev theorem is concerned: the eigenvalues relative to the parameter space do not affect the spectrum of the operator \mathcal{B} , see Section 5.

Actually Nekhoroshev's result [12] also include a second part, also referred to as the *Liouville–Arnold–Nekhoroshev theorem*, concerning the possibility of defining action-angle coordinates in the symplectic submanifold $N \subset M$ fibered by invariant isotropic tori; needless to say this second part is purely hamiltonian. Note that here we need that all monodromy operators associated to basis cycles are to be nondegenerate in the sense of (iii) in order to be able to extend action-angle coordinates (compare with Remark 6).

For a detailed discussion – and proof – of the Poincaré–Lyapounov–Nekhoroshev theorem in the hamiltonian case the reader is referred to [4, 9].

8 Bifurcation from an invariant torus

The Poincaré–Nekhoroshev map can be discussed in the same way as the standard Poincaré map (or any map between open sets in real spaces); this includes in particular its bifurcations when external parameters are varied. In this section we illustrate the picture emerging from such a discussion when we deal with a single parameter $\varepsilon \in \mathcal{E} \subseteq \mathbb{R}$ (thus $p = 1$). Essentially we are just interpreting the discussion of [3] (section 34) on bifurcation of fixed points of the Poincaré map in the present frame, so we will be rather sketchy.

We assume that there is a fixed point $u_0(\varepsilon) = 0$ for all values of $\varepsilon \in E_0 \subseteq \mathcal{E}$, stable for $\varepsilon < 0$ and losing stability at $\varepsilon > 0$. This corresponds, for the full dynamics, to a (parameter-dependent) invariant torus $\Lambda_0(\varepsilon) = \mathbb{T}^k$, which is transversally hyperbolically stable for $\varepsilon < 0$ and loses stability for $\varepsilon > 0$. This implies that some eigenvalues $\mu_i(\varepsilon)$ of the map $\mathcal{P}^{(\varepsilon)}$ satisfy $|\mu_i(0)| = 1$.

Let us make standard bifurcation hypotheses, i.e.: (i) the existence of a dynamically invariant neighbourhood of u_0 for all values of $\varepsilon \in E_0$; (ii) transversality for the critical eigenvalues $\mu_i(\varepsilon)$, i.e. $d|\mu_i|/d\varepsilon \neq 0$ at $\varepsilon = 0$; (iii) non-degeneracy of the spectrum of the map at the critical point (for generic dynamics, this means that there is only a pair of complex conjugate complex critical eigenvalues, or a single real one); (iv) split property of the spectrum: non-critical eigenvalues lie at a finite distance $\delta > 0$ from the unit circle at $\varepsilon = 0$.

With these, it is known that there are three elementary types of bifurcation, characterized by the value of the critical eigenvalues $\mu_i(\varepsilon)$, i.e. of the eigenvalues μ_i :

(a) $\mu(0) = -1$;

- (b) $\mu(0) = 1$;
(c) $\mu(0) = \cos(\alpha) \pm i \sin(\alpha)$ ($\alpha \neq k\pi$).

Case (a) corresponds to the appearance of two new (branches of) stable fixed points $u_{\pm}(\varepsilon)$ for the map; these corresponds to a bifurcation of the invariant torus $\Lambda_0(\varepsilon) \simeq \mathbb{T}^k$ into two new (branches of) stable tori $\Lambda_{\pm}(\varepsilon) \simeq \mathbb{T}^k$. For $\varepsilon \rightarrow 0^+$, $u_{\pm}(\varepsilon) \rightarrow u_0(\varepsilon)$, and similarly $\Lambda_{\pm}(\varepsilon) \rightarrow \Lambda_0(\varepsilon)$.

Case (b) corresponds to the appearance of two period-two points $u_{\pm}(\varepsilon)$, such that $\mathcal{P}^{(\varepsilon)} : u_{\pm}(\varepsilon) \rightarrow u_{\mp}(\varepsilon)$. This is a period-doubling bifurcation, and corresponds to the appearance of a single invariant torus $\Lambda_d(\varepsilon)$. For $\varepsilon \rightarrow 0^+$, $u_{\pm}(\varepsilon) \rightarrow u_0(\varepsilon)$, and $\Lambda_d(\varepsilon) \rightarrow \Lambda_0(\varepsilon)$. For $\varepsilon > 0$ sufficiently small, $\Lambda_d(\varepsilon)$ lies near enough to Λ_0 to make sense to consider its intersection with the transversal local manifolds to Λ_0 , and it has two such intersections on each $\sigma^{(\varepsilon)}$, given indeed by $u_{\pm}(\varepsilon)$.

Case (c) is the most interesting; we can consider $2\pi/\alpha$ irrational, as the rational case is structurally unstable (see e.g. the discussion in [3]). In this case we get a full circle of fixed points $u_{\vartheta}(\varepsilon)$ ($\vartheta \in S^1$) for the Poincaré–Nekhoroshev map. This corresponds to the appearance of a new stable torus $\Lambda_1(\varepsilon) \simeq \mathbb{T}^{k+1}$, of dimension greater than that of the original invariant torus. This is the analogue of the bifurcation of an invariant torus off a periodic solution.

Remark 12. If we discuss the (nongeneric) symmetric case, i.e. if we assume that there is an algebra – no matter if abelian or otherwise – of vector fields \mathcal{H} commuting with \mathcal{G} , then the nondegeneracy assumption (iii) should be meant in the sense that only the degeneracy imposed by the symmetry constraint is present in the critical spectrum, see [14]. In this case there will be a multiplicity if critical eigenvalues which in case (c) can lead to a bifurcation in which the new stable torus is $\Lambda_1(\varepsilon) \simeq \mathbb{T}^{k+s}$ with $s > 1$.

Acknowledgements

Common work with Dario Bambusi on the Poincaré–Lyapounov–Nekhoroshev theorem in the hamiltonian case is at the basis of this work; my interest in the general case and the geometrical aspects discussed here was also triggered by questions raised by Franco Cardin; I would like to warmly thank both of them.

This work was supported by “*Fondazione CARIPLO per la ricerca scientifica*” under the project “*Teoria delle perturbazioni per sistemi con simmetria*”.

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