On Einstein Equations on Manifolds and Supermanifolds

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Abstract

The Einstein equations (EE) are certain conditions on the Riemann tensor on the real Minkowski space M. In the twistor picture, after complexification and compactification M becomes the Grassmannian Gr_2^4 of 2-dimensional subspaces in the 4-dimensional complex one. Here we answer for which of the classical domains considered as manifolds with G-structure it is possible to impose conditions similar in some sense to EE. The above investigation has its counterpart on superdomains: an analog of the Riemann tensor is defined for any *super*manifold with G-structure with any Lie supergroup G. We also derive similar analogues of EE on supermanifolds. Our analogs of EE are not what physicists consider as SUGRA (supergravity), for SUGRA see [16, 34].

1   Introduction

This is an expanded version of a part of Leites' lectures at ICTP, Trieste, in March 1991 on our results. The description of "the left hand side of N-extended SUGRA equations", though computed several years later, appeared earlier [13] and refers to some results from this paper and [34].

Roughly speaking, in this paper, as well as in [13, 39, 40, 41, 17], for a $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g}_* = \underset{i \geq -d}{\oplus} \mathfrak{g}_i$ and its subalgebra $\mathfrak{g}_- = \underset{i < 0}{\oplus} \mathfrak{g}_i$ we calculate $H^k(\mathfrak{g}_-; \mathfrak{g}_*)$ for $k \leq 2$. In addition to a new result (analogs of EE) this paper contains a summary of [39, 40, 41]. The Nijenhuis tensor deserves a separate publication [17].

For $\mathfrak{g}_*$ simple, $k = 2$ and $d = 1$, this cohomology can be interpreted as analogs of the conformal part of the Riemann tensor, called the *Weyl tensor*, (more exactly, values thereof at a point). This cohomology coincides with the Weyl tensor on the n-dimensional manifold

when $\mathfrak{g}_* = \mathfrak{o}(n+2)$. For other Lie algebras $\mathfrak{g}_*$, not only simple ones, we obtain the so-called *structure functions* (obstructions to non-flatness in some sense) of the manifold with the G-structure, where $\mathfrak{g}_0 = \mathrm{Lie}(G)$, the Lie algebra of G. For $\mathfrak{g}_*$ simple, the cohomology $H^k(\mathfrak{g}_{-1}; \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0)$, where $\hat{\mathfrak{g}}_0$ is the semisimple part of $\mathfrak{g}_0$, corresponds to (the analogs of) the *Riemann tensor*; they consist of $H^k(\mathfrak{g}_-; \mathfrak{g}_*)$ — the "conformal part" — plus something else, and it is this "extra" part that plays the main role in the left hand side of the Einstein equations.

For $d > 1$ one obtains new invariants which we interpret as obstructions to "non-flatness" of a manifold (or supermanifolds) with a nonholonomic structure, see [32, 34, 13, 14]. These invariants eluded researchers for almost a century, see Vershik's review [48], where doubted if they existed. Similar structures appear in Manin's book [37], and our approach shows a method to describe their "non-flatness". We have only started to study such structures; the detailed exposition is in preparation.

In this paper we only consider $d = 1$ and mostly finite dimensional cases. Goncharov considered Lie algebras and cases when $\mathfrak{g}_*$ is simple; we consider also superalgebras for $\mathfrak{g}_*$ simple or close to simple and also consider $H^k(\mathfrak{g}_{-1}; \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0)$. Other cases are either open problems or will be considered elsewhere.

We are thankful to Grozman who verified our calculations of structure functions for the exceptional superdomains and in several other cases by means of his SuperLie package: even these finite dimensional calculations are almost impossible to perform without computer whereas to Grozman's package this is a matter of minutes in components; to glue the components into a module takes several hours in each case.

The main object in the study of Riemannian geometry is the *Riemann tensor*. Under the action of $O(n)$ the space of values of the Riemann tensor at the point splits into irreducible components called the *Weyl tensor*, the *traceless Ricci tensor* and the *scalar curvature*. (On 4-dimensional manifolds the Weyl tensor additionally splits into 2 subcomponents.) All these tensors are obstructions to the possibility of "flattening" the canonical (Levi–Civita) connection on the manifold they are considered.

More generally, let G be any Lie group, not necessarily $O(n)$. In what follows we will recall the definition of a G-structure on a manifold M and structure functions of this G-structure. Structure functions are obstructions to integrability or, in other words, to the possibility of "flattening" the G-structure or a connection associated with it, sometimes, canonically, see [12]. The Riemann tensor is the only nontrivial structure function for $G = O(n)$. Several most known (or popular recently) examples of G-structures and respective tensors are:

Name of the structure	G	Name of the tensor
almost conformal	$G = CO(n) = O(n) \times \mathbb{R}^+$	Weyl tensor
Riemannian structure	$G = O(n)$	Riemann tensor
Penrose' twistors	$G = S(GL(2; \mathbb{C}) \times GL(2; \mathbb{C}))$	the "α-forms" and "β-forms"
almost complex structure	$G = GL(n; \mathbb{C}) \subset GL(2n; \mathbb{R})$	Nijenhuis tensor
almost symplectic structure	$G = Sp(2n)$	no accepted name

Remark 1. The adverb "almost" should be always added until the G-structure under study is proved flat, i.e., integrable; by abuse of language people often omit it.

Infinitesimal automorphisms (with polynomial coefficients) of the flat G-structure on $\mathbb{R}^n$ ($n = \dim M$) constitute the Cartan prolong (see Section 2.2) — the Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$,

where $\mathfrak{g}_{-1}$ can be identified with the tangent space $T_m M$ at a point and $\mathfrak{g}_0 = \mathrm{Lie}(G)$. We interpret structure functions as certain Lie algebra cohomology associated with $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

The Riemannian case is the *reduction of the structure group* of the conformal case. More generally, if $\mathfrak{g}_0$ is a central extension of the (semi-)simple Lie algebra $\hat{\mathfrak{g}}_0$, the corresponding structure functions will be called, after Goncharov, generalized conformal ones, whereas the structure functions for $\hat{\mathfrak{g}}_0$ will correspond to a generalized Riemannian case — a possible candidate in search for analogs of Einstein equations.

In [12] Goncharov calculated all structure functions for the analogues of the almost conformal structure corresponding to irreducible compact Hermitian symmetric spaces (shortly CHSS in what follows); in Goncharov's examples G is the reductive part of the stabilizer of any point of the compact Hermitian symmetric space.

Here we will consider the reductions of the cases considered by Goncharov — analogues of Riemannian structures and their various generalizations to manifolds and supermanifolds, in particular, infinite dimensional, associated with Kac–Moody and stringy (super)algebras. We also review some cases considered in [39] and give an overview of [40] and [33, 34].

For prerequisites on symmetric spaces see [21]. Appendix contains preliminaries on Lie superalgebras and supermanifolds; the super analogs of classical symmetric spaces listed in [44, 36] are recalled in Tables. Observe several interesting points.

(1) Some of the spaces and superspaces we distinguish are infinite dimensional. Some of these infinite dimensional analogues of EE can only be realized on the total spaces of Fock bundles over supermanifolds with at least 3 odd coordinates; the invariance group of such an EE contains a contact Lie superalgebra. Other infinite dimensional examples are associated with Kac–Moody or loop algebras of which the examples associated with twisted versions are most intriguing.

(2) On supermanifolds, our analogues of EE are *not* what physicists consider as *super-gravity equations* (SUGRA); each N-extended SUGRA requires a nonholonomic distribution and they are considered in [34, 13]. Recall that having struggled for a decade with a conventional model of Minkowski superspace for deriving $N = 2$ SUGRA the Ogievet-sky's group GIKOS had found a solution [10]: one has to enlarge the Minkowski space underlying Minkowski superspace for $N = 2$ with an additional "harmonic" space P^1. How to advance as N grows was unclear, cf. pessimistic remarks in [10] and [51].

What was the problem?! Take the usual recipe for calculation of the Riemann tensor or even structure function of any G-structure, insert some signs to account for super flavour and that will be it! This is more or less what is suggested in [4] and [42]. The snag is that in doing so we tacitly assume that $\mathfrak{g}_{-1}$ is a commutative Lie (super)algebra, whereas on the Minkowski superspace for any N and any model (except [16]), be it a "conventional", or Manin's "exotic" one, the tangent space at any point possesses a natural structure of a nilpotent Lie superalgebra. In other words, every Minkowski superspace is a nonholonomic one, i.e., with a nonintegrable distribution.

So we need (a) a definition of structure functions for nonholonomic (super)spaces (this definition that solves the old Hertz-Vershik's problem was first published in [34]) and (b) test which of the coset spaces, or rather superspaces, satisfy a natural requirement: if we throw away all odd parameters we get the conventional Einstein equation (plus, perhaps, something else).

In [13] we executed this approach for every $N \leq 8$ and several most symmetric parabolic subgroups; in our models of $N = 8$ extended SUGRA, it is Gr_4^8 (dark matter?) together with two more copies of our Minkowski space (hell and paradise?) that constitute the space of extra parameters of the even, usual Minkowski space compulsory if we wish to satisfy the above natural requirement. These additional spaces together are analogs of "harmonic" space of [10]. Observe that the manifolds like Gr_4^8 appear in our examples of "distinguished" classical spaces, the ones on which one can write an analog of EE.

(3) The idea to apply cohomology to describe SUGRA appeared first, perhaps, in Schwarz's paper [43] and [4] but their execution of the idea is different from ours and leads astray, we think, as far as SUGRA is concerned.

(4) Among compact Hermitian symmetric spaces, some are *distinguished* by the fact that the corresponding Jordan algebra is simple; e.g., such is the Grassmannian Gr_{2n}^{4n}. In [31] there is given a number of examples of simple Jordan superalgebras corresponding to simple $\mathbb{Z}$-graded Lie superalgebras of polynomial growth. It turns out that on manifolds locally equivalent (in the sense of G-structures) to the distinguished Hermitian spaces, one can write equations resembling the conventional EE. To investigate how far can one stretch the analogy on supermanifolds is an open problem.

2 Recapitulations

In this section we recall basic definitions [45] and retell some of Goncharov's results [12] in a form convenient for us.

2.1 Principal fibre bundles

Let M be a manifold of dimension n over a field $\mathbb{K}$ (here: $\mathbb{R}$ or $\mathbb{C}$) and G a Lie group. A *principal fibre bundle $P = P(M, G)$ over M with group G* consists of a manifold P and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right;
(2) $M = P/G$ and the canonical projection $\pi : P \longrightarrow M$ is differentiable;
(3) P is locally trivial.

Example. $P = M \times G$, the trivial bundle. The free G-action on P is given by the formula $ub = (x, ab)$ for any $u = (x, a) \in P$, $b \in G$.

Example. The bundle of linear frames over M. Let $\dim M = n$. A linear frame $f(x)$ at a point $x \in M$ is an ordered basis $X_1, \ldots, X_n$ of the tangent space $T_x M$. Let $\mathcal{F}(M)$ be the set of all linear frames at all points of M and $\pi : \mathcal{F}(M) \longrightarrow M$ the map such that $\pi(f(x)) = x$. The group $GL(n)$ acts on $\mathcal{F}(M)$ on the right as follows: if $f(x) = (X_1, \ldots, X_n)$ and $(a_j^i) \in GL(n)$, then $fa = (Y_1, \ldots, Y_n)$, where $Y_i = \sum_j a_i^j X_j$ is a linear frame at x. So $GL(n)$ acts freely on $\mathcal{F}(M)$ and $\pi(u) = \pi(v)$ if and only if $v = ua$ for some $a \in GL(n)$.

2.2 Structure functions

Let $\mathcal{F}(M)$ be the principal $GL(n; \mathbb{K})$-bundle of linear frames over M. Let $G \subset GL(n; \mathbb{K})$ be a Lie group. A *G-structure on M* is a reduction of $\mathcal{F}(M)$ to a principal G-bundle.

The simplest G-structure is the *flat* G-structure defined as follows. Let V be $\mathbb{K}^n$ with a fixed frame. The flat structure is the bundle over V whose fiber over $v \in V$ consists of all frames obtained from the fixed one under the G-action, V being identified with $T_v V$.

Examples of flat structures. The classical spaces, i.e., compact Hermitian symmetric spaces, provide us with examples of manifolds with nontrivial topology but flat G-structure. We will shortly derive a well-known fact that the only possible $GL(n)$-structure on any n-dimensional manifold is always flat.

In [20] the obstructions to identification of the k-th infinitesimal neighborhood of a point on a manifold M with G-structure with the k-th infinitesimal neighborhood of a point of the flat manifold V with the above G-structure are called *structure functions of order k*, or briefly SF. In [20] and [45] it is shown that the tensors that constitute these obstructions are well-defined provided the structure functions of all orders $< k$ vanish.

We will write $M \sim N$ for two locally equivalent G-structures on manifolds M and N.

The classical description of the structure functions uses the notion of the *Spencer cochain complex*. It is defined as follows. Let S^i denote the operator of the i-th symmetric power, prime $'$ denotes the dualization. Set $\mathfrak{g}_{-1} = T_m M$, $\mathfrak{g}_0 = \mathrm{Lie}(G)$; for $i > 0$ set:

$$\mathfrak{g}_i = \{ X \in \mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{i-1}) : X(v)(w) = X(w)(v) \ \text{ for any } \ v, w \in \mathfrak{g}_{-1} \}$$
$$= (\mathfrak{g}_0 \otimes S^i \mathfrak{g}'_{-1}) \cap (\mathfrak{g}_{-1} \otimes S^{i+1} \mathfrak{g}'_{-1}).$$

Finally, set $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \underset{i \geq -1}{\oplus} \mathfrak{g}_i$. It is easy to check that $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ possesses a natural Lie algebra structure. The Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is called the *Cartan's prolong* (the result of *Cartan's prolongation*) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$.

Suppose that

the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful.   (2.1)

Then, clearly, $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* \subset \mathfrak{vect}(n) = \mathfrak{der}\mathbb{K}[x_1, \ldots, x_n]$, where $n = \dim \mathfrak{g}_{-1}$, with

$$\mathfrak{g}_i = \{ X \in \mathfrak{vect}(n)_i : [X, D] \in \mathfrak{g}_{i-1} \ \text{ for any } \ D \in \mathfrak{g}_{-1} \} \quad \text{for} \quad i \geq 1.$$

Let Λ^i be the operator of the i-th exterior power; set $C^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} = \mathfrak{g}_{k-s} \otimes \Lambda^s(\mathfrak{g}'_{-1})$; we often drop the subscript of $C^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$ or indicate only $\mathfrak{g}_0$ since the module $\mathfrak{g}_{-1}$ is clear.

Define the differential $\partial_s : C^{k,s} \longrightarrow C^{k,s+1}$ by setting for any $v_1, \ldots, v_{s+1} \in \mathfrak{g}_{-1}$ (as usual, the slot with the hatted variable is to be ignored):

$$(\partial_s f)(v_1, \ldots, v_{s+1}) = \sum_i (-1)^i \left[f(v_1, \ldots, \hat{v}_i, \ldots, v_{s+1}), v_i \right].$$

As expected, $\partial_s \partial_{s+1} = 0$, and the homology $H^{k,s}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$ of the bicomplex $\underset{k,s}{\oplus} C^{k,s}$ is called the (k, s)-th *Spencer cohomology* of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$. (In the literature various gradings of the Spencer complex are in use; ours is the most natural one.)

Proposition 1 ([20]). *The Spencer cohomology group $H^{k,2}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$ constitutes the space of values of the structure function of order k.*

2.3 The case of simple $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ over $\mathbb{C}$

The following remarkable fact, though known to experts, is seldom formulated explicitly:

Proposition 2. *Let* $\mathbb{K} = \mathbb{C}$*; let* $\mathfrak{g}_* = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ *be simple. Then only the following cases are possible:*

1) if $\mathfrak{g}_2 \neq 0$*, then* $\mathfrak{g}_*$ *is either* $\mathfrak{vect}(n)$ *or its special subalgebra* $\mathfrak{svect}(n)$*, or the subalgebra* $\mathfrak{h}(2n) \subset \mathfrak{vect}(2n)$ *of hamiltonian fields;*

2) if $\mathfrak{g}_2 = 0$*, then* $\mathfrak{g}_1 \neq 0$ *and* $\mathfrak{g}_*$ *is the Lie algebra of the complex Lie group of automorphisms of a compact Hermitian symmetric space.*

Remark 2. This Proposition gives a reason to impose the restriction (2.1) if we wish $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ to be simple. On supermanifolds, where the analogue of Proposition 2 does not imply similar restriction, (or if we do not care whether $\mathfrak{g}_*$ is simple or not) we do consider Cartan prolongs not embeddable into $\mathfrak{vect}(\dim \mathfrak{g}_{-1})$, see [40, 41].

Let us express Spencer cohomology in terms of Lie algebra cohomology. Namely, observe that:

$$\underset{k}{\oplus} H^{k,2}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} = H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*). \tag{2.2}$$

This representation has only advantages: we loose nothing, because a finer grading of Spencer cohomology is immediately recoverable from the rhs of (2.2) where it corresponds to the $\mathbb{Z}$-grading of $\mathfrak{g}_* = (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$; moreover, there are several theorems helping to compute Lie algebra cohomology ([9]) whereas in order to compute Spencer cohomology we can only use the definition.

To compute $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ is especially easy when $\mathfrak{g}_*$ is a simple finite dimensional Lie algebra over $\mathbb{C}$. Indeed, thanks to the Borel–Weil–Bott (BWB) theorem, cf. [12], the $\mathfrak{g}_0$-module $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ has as many irreducible $\mathfrak{g}_0$-modules as $H^2(\mathfrak{g}_{-1})$ which, thanks to commutativity of $\mathfrak{g}_{-1}$, is just $\Lambda^2(\mathfrak{g}'_{-1})$. The highest weights of these irreducible modules are also deducible from the theorem, as it is explained in [12]. Since [12] does not give the explicit values of these weights, we give them. We also calculate structure functions corresponding to case 1) of the Proposition 2: we did not find these calculations in the literature.

In what follows $R\left(\sum a_i \pi_i\right)$ denotes the irreducible $\mathfrak{g}_0$-module (and the corresponding representation) with highest weight $\sum a_i \pi_i$ expressed in terms of fundamental weights as in [38]; the weights of the $\mathfrak{gl}(n)$-modules, however, are given for convenience with respect to the matrix units E_{ii}.

The classical spaces are listed in Table 1 and some of them are baptized for convenience of further references.

Our next task is to superize Proposition 2 and compute the corresponding structure functions. For the list of "classical" Lie superalgebras see [22] (finite dimensional Lie superalgebras), [18] (stringy Lie superalgebras), [8] (Kac–Moody Lie superalgebras) and [35] (or [24] and [5, 6]) (vectorial Lie superalgebras). For notations of vectorial Lie superalgebras (simple and close to simple), see [35], [18].

Theorem 1. *1) In case 1) of Proposition 2 the structure functions can only be of order* 1 (Serre, see [45]). *The actual values of structure functions are as follows* ([34]):

a) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = 0$ *for* $\mathfrak{g}_* = \mathfrak{vect}(n)$ *and* $\mathfrak{svect}(m)$, $n, m > 2$;

b) $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*) = R(\pi_3) \oplus R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(2n)$, $n > 2$; $H^2(\mathfrak{g}_{-1}, \mathfrak{g}_*) = R(\pi_1)$ *for* $\mathfrak{g}_* = \mathfrak{h}(4)$.

2) (Goncharov [12]) *The structure functions for a space of type* Q_3 *can be of order* 3 *and constitute* $R(4\pi_1)$.

The cocycles representing structure functions for a space of type Gr^n_m *(when neither m nor $n - m$ is equal to 1, i.e., when Gr is not a projective space) belong to the direct sum of two irreducible (as $\mathfrak{g}_0$-modules) components. In this case* $\mathfrak{g}_0 = \mathfrak{sl}(m) \oplus \mathfrak{sl}(n - m) \oplus \mathbb{C}$ *and* $\Lambda^2(\mathfrak{g}'_{-1})$ *is*

$$\Lambda^2((\mathbb{C}^m)' \otimes (\mathbb{C}^{n-m})') = S^2(\mathbb{C}^m)' \otimes \Lambda^2(\mathbb{C}^{n-m})' \oplus \Lambda^2(\mathbb{C}^m)' \otimes S^2(\mathbb{C}^{n-m})' = \Lambda^2_+ \oplus \Lambda^2_-.$$

The space of structure functions is the sum of two irreducible components: the self-dual part, H_+, and antiself-dual part, H_-.

The following table indicates order of components $H_\pm$; the highest weight of H_+ (resp. H_-) is the sum of the highest weights of Λ^2_+ (resp. Λ^2_-) and the highest weight of $\mathfrak{g}_{k-2}$, where k is the indicated order of the structure function:

	SF of order 1	SF of order 2
$m = 2$, $n - m \neq 2$	H_-	H_+
$n - m = 2$, $m \neq 2$	H_+	H_-
$n - m$, $m = 2$		$H_- \oplus H_+$
m, $n - m \neq 2$	$H_- \oplus H_+$	

The structure functions of G-structures of the rest of the classical compact Hermitian symmetric spaces are the following irreducible $\mathfrak{g}_0$-modules, where V is the identity $\mathfrak{g}_0$-module:

CHSS	P^n	OGr_m	LGr_m
conformal SF	none	$\Lambda^2\left(\Lambda^2(V')\right) \otimes V$	$\Lambda^2\left(S^2(V')\right) \otimes V$
CHSS	Q_n, $n > 4$	$E_6^c/SO(10) \times U(1)$	$E_7^c/E_6^c \times U(1)$
conformal SF	$\Lambda^2 V' \otimes V$	$\Lambda^2(R(\pi_5)') \otimes R(\pi_5)$	$\Lambda^2(R(\pi_1)') \otimes R(\pi_1)$

Their order is equal to 1 (recall that $Q_4 = Gr^4_2$).

2.4 Connections and structure functions

(After [37].) Let $\mathcal{M}$ be a supermanifold, $\mathcal{S}$ a locally free sheaf (of sections of a vector bundle) on $\mathcal{M}$. Locally, in a sufficiently small neighbourhood $\mathcal{U}$, we may view $\mathcal{S}$ as a free module over a supercommutative superalgebra $\mathcal{F}$, which, in the general setting, is the structure sheaf of $\mathcal{M}$.

On $\mathcal{S}$, a *connection* is an *odd* map $\nabla : \mathcal{S} \longrightarrow \mathcal{S} \otimes_\mathcal{F} \Omega^1$, where $\Omega^i = \Omega^i(\mathcal{M})$ is the sheaf of differential i-forms on $\mathcal{M}$. The map ∇ can be extended to the whole de Rham complex of differential forms:

$$\mathcal{S} \xrightarrow{\nabla} \Omega^1 \otimes_\mathcal{F} \mathcal{S} \xrightarrow{\nabla} \Omega^2 \otimes_\mathcal{F} \mathcal{S} \cdots \xrightarrow{\nabla} \Omega^i \otimes_\mathcal{F} \mathcal{S} \cdots \tag{2.3}$$

by the Leibniz rule

$$\nabla(f \otimes s) = df \otimes s + (-1)^{p(f)} f \nabla(s) \quad \text{for} \quad f \in \Omega^i, \quad s \in \mathcal{S}.$$

Dualization determines the action of ∇ on the spaces of *integrable* forms, where $\Sigma_{-i} = \text{Hom}_{\mathcal{F}}(\Omega^i, \text{Vol})$ and Vol is the sheaf of volume forms:

$$\cdots \xrightarrow{\nabla} \mathcal{S} \otimes_{\mathcal{F}} \Sigma_{p-q-1} \xrightarrow{\nabla} \mathcal{S} \otimes_{\mathcal{F}} \Sigma_{p-q} \xrightarrow{\nabla} 0 \tag{2.4}$$

compatible with the Ω^*-action on Σ_* and given by the formula

$$\nabla(s \otimes \sigma) = T(\nabla(s)) \otimes \sigma + (-1)^{p(s)} s \otimes d(\sigma) \quad \text{for} \quad \sigma \in \Sigma_i, \quad s \in \mathcal{S},$$

where $T : \Omega^1 \otimes_{\mathcal{F}} \mathcal{S} \longrightarrow \mathcal{S} \otimes_{\mathcal{F}} \Omega^1$ is the twisting isomorphism (mind Sign Rule).

One connection always exists: in the $\mathcal{S}$-valued de Rham complex, set: $\nabla = d$ (more precisely, $\nabla = d \otimes \text{id}_{\mathcal{S}}$). Since, as it is easy to verify, any two connections differ by an Ω^*-linear map, then any connection is of the form $\nabla = d + \alpha$, where $\alpha \in \Omega^1$ is called the *form of the connection* or a *gauge field*. We can consider the connection as acting in the whole spaces $\Omega^* \otimes_{\mathcal{F}} \mathcal{S}$ and $\mathcal{S} \otimes_{\mathcal{F}} \Sigma_*$. Then $\nabla^2 \left(= \frac{1}{2}[\nabla, \nabla]\right)$ is a well-defined element denoted by $F_{\nabla} \in \mathcal{E}nd\, \mathcal{S} \otimes_{\mathcal{F}} \Omega^2$ and called the *curvature form* of ∇ or the *stress tensor of the Yang–Mills field α*.

A connection in $\text{Vect}(M)$ is called an *affine* one. An affine connection is *symmetric* if

$$\nabla_X(Y) - \nabla_Y(X) - [X, Y] = 0. \tag{2.5}$$

An affine connection ∇ is called *compatible with the given metric g* if

$$g(\nabla_X Y, Z) = (-1)^{p(X)p(g)} X(g(Y, Z)) - (-1)^{p(X)p(Y)} g(Y, \nabla_X Z). \tag{2.6}$$

Compatibility with a differential 2-form ω or another tensor of valency (a, b), say, a volume form, is similarly defined, only the number of variable vector fields involved is not three anymore, but $a + b + 1$.

On every Riemannian manifold we have no structure functions of order 1; hence, there always exists a unique torsion-free connection compatible with the metric (it is called *the Levi–Civita connection*) and the 2nd order structure functions are well-defined. (This is not so for certain other G-structures, cf. Theorem 5 or the case of $d > 1$, e.g., the case of Minkowski *super*spaces.)

2.5   Structure functions for Riemann-type structures

In [12] Goncharov considered *generalized conformal structures*. The structure functions for the corresponding generalizations of the Riemannian structure, i.e., when Goncharov's $\mathfrak{g}_0$ is replaced with its semisimple part $\hat{\mathfrak{g}}$ of $\mathfrak{g} = \text{Lie}(G)$, seem to be more difficult to compute because in these cases $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0$ and the BWB-theorem does not work. Fortunately, as follows from the cohomology theory of Lie algebras, we still have an explicit description of structure functions:

Proposition 3 ([12], Theorem 4.7). *For $\mathfrak{g}_0 = \hat{\mathfrak{g}}$ structure functions of order 1 are the same as for $\mathfrak{g}_0 = \mathfrak{g}$ and structure functions of order 2 for $\mathfrak{g}_0 = \hat{\mathfrak{g}}$ are the same as for $\mathfrak{g}_0 = \mathfrak{g}$ plus, additionally, $S^2(\mathfrak{g}'_{-1})$. (Clearly, there are no structure functions of order > 2 for $\mathfrak{g}_0 = \hat{\mathfrak{g}}$.)*

Let $G = O(n)$, i.e., $M \sim Q_n$. In this case $\mathfrak{g}_1 = \mathfrak{g}_{-1}$ and in $S^2(\mathfrak{g}'_{-1})$ a 1-dimensional trivial G-module is distinguished; the section through the subbundle with this subspace as a fiber is a Riemannian metric g on M.

Let now t be a structure function (the sum of its components belongs to the distinct irreducible $O(n)$-modules that constitute $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$) corresponding to the Levi–Civita connection. The process of restoring t from g (compatibility condition (2.6)) involves differentiations thus making any relation on t into a nonlinear partial differential equation. Let us consider certain other restrictions on t.

The values of the Riemann tensor $\mathcal{R}$ at a point of M constitute an $O(n)$-module $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ which contains a trivial component. Due to complete reducibility of finite dimensional $O(n)$-modules, we can consider, separately, the component of $\mathcal{R}$ corresponding to the trivial representation, denote it Scal. As is explained in Proposition 2, this trivial component is realized as a submodule in an isomorphic copy of $S^2(\mathfrak{g}'_{-1})$, the space the metric is taken from. Thus, we have two matrix-valued functions, g and Scal, each a section of the line bundle corresponding to the trivial $\mathfrak{g}_0$-module.

What is more natural than to require their ratio to be a constant (instead of a function)? This condition

$$\text{Scal} = \lambda g, \quad \text{where} \quad \lambda \in \mathbb{K}, \tag{2.7}$$

gives us "a lesser half" of what is known as *Einstein Equations* (EE).

To obtain the remaining part of EE, recall that $S^2(\mathfrak{g}'_{-1})$ consists of the two irreducible $O(n)$-components, the trivial one and another one. A section through this other component is the traceless Ricci tensor, Ric. The analogs of *Einstein equations* (in vacuum and with cosmological term proportional to λ) are the *two* conditions: (2.7) and

$$\text{Ric} = 0. \tag{2.8}$$

The remaining components of EE are invariant under conformal transformations and do not participate in EE.

3 Structure functions for reduced structures — analogs of EE on manifolds

In [12] Goncharov did not explicitly calculate the weights of structure functions for G-structures corresponding to the reduction of the generalized conformal structure. Let us fill in this gap: let us elucidate Proposition 2 for the classical compact Hermitian symmetric spaces (CHSS).

Proposition 4. *Let $\mathfrak{g}_0$ be the semisimple part $\hat{\mathfrak{g}}$ of $\mathfrak{g} = \mathrm{Lie}(G)$ corresponding to a compact Hermitian symmetric space X other than $\mathbb{O}P^2$, $\mathbb{E}$. Then nonconformal structure functions are all of order 2 and as follows:*

CHSS	$\mathbb{C}P^n$	Gr_m^{m+n}
weight of SF	$R(\pi_2)$	$R(2\pi_1) \otimes R(2\pi_1)' \oplus R(\pi_2) \otimes R(\pi_2)'$
CHSS	OGr_m	LGr_m
weight of SF	$R(2\pi_2)' \oplus R(\pi_4)'$	$R(0, \ldots, 0, -2, -2) \oplus R(0, \ldots, 0, -4)$

Let us show how to obtain equations similar to EE on some compact Hermitian symmetric spaces other than Q_n. Let R be a section of the vector bundle with the above structure function as the fiber; if the space of structure function consists of two irreducible G-components; denote the corresponding components of the structure function by $R = R_1 + R_2$ in accordance with the decomposition of the module of structure functions as indicated in the table above. We will consider structure functions corresponding to the canonical (in the same sence as Levi–Civita) connection corresponding to the G-structure considered.

The analogues of (2.7) can be defined in the following cases:
1) Gr_{2n}^{4n} (turns into the conventional (2.7) at $n = 1$);
2) P^{2n};
3) OGr_{4n} (turns into the the conventional (2.7) at $n = 1$).

These analogues are the equations:

$$v = \lambda R_2^n \quad (\text{or} \quad v = \lambda R^n \quad \text{if} \quad R \quad \text{has just one irreducible component}), \tag{3.1}$$

where v is a fixed volume element on X.

The analogues of (2.8) are the equations

$$R_1 = 0 \quad (\text{if there is such a component}). \tag{3.2}$$

Notice that if the space of structure functions is irreducible, there is no (3.2).

If structure functions of order 1 are nonzero, denote them by $T = \oplus T_i$ (here the sum runs over irreducible components). As we have quoted from [45], the equations EE are well-defined provided all the T_i vanish. This yields conditions similar to Wess–Zumino constraints in SUGRA [51]:

$$T_i = 0 \quad \text{for every} \quad i. \tag{3.3}$$

Notice that for all the compact Hermitian symmetric spaces the 1-st order structure functions vanish.

Explicit computations of the structure functions for the exceptional CHSS (see Table 1) will be given elsewhere.

4 Analogs of EE on supermanifolds

The theory of Lie supergroups and even Lie superalgebras is yet new in Mathematics. Therefore the necessary background is gathered in a condensed form in Appendix.

We have often heard that "the Riemannian geometry has parameters whereas the symplectic one does not". It is our aim to elucidate this phrase: we have shown (Theorem 1 above, Theorem 5 below and [39]) that an *almost* symplectic geometry does have parameters, the torsion, which being of order 1 should be killed, like Wess–Zumino constraints, in order to reduce the 2-form to a canonical form. The curvature, alias a structure function of order 2, might have been an obstruction to canonical form but it vanishes.

Similar is the situation for supermanifolds. But not quite: $\mathfrak{o}$ is never isomorphic to $\mathfrak{sp}$ whereas the ortho-symplectic Lie superalgebra which preserves a nondegenerate even skew-symmetric bilinear form, $\mathfrak{osp}^{sk}(V)$, is isomorphic to the Lie superalgebra preserving a non-degenerate even symmetric bilinear form, $\mathfrak{osp}(\Pi(V))$. Still their Cartan prolongs are quite

distinct: $(V, \mathfrak{osp}^{sk}(V))_* = \mathfrak{h}(\dim(V))$ whereas $(\Pi(V), \mathfrak{osp}(\Pi(V)))_* = \Pi(V) \oplus \mathfrak{osp}(\Pi(V))$, cf. [35].

Analogously, the periplectic Lie superalgebra, $\mathfrak{pe}^{sk}(V)$ which preserves a nondegenerate odd skew-symmetric bilinear form is isomorphic to the Lie superalgebra, $\mathfrak{pe}^{sy}(\Pi(V))$, preserving a nondegenerate odd symmetric bilinear form; but $(V, \mathfrak{pe}^{sk}(V))_* = \mathfrak{le}(\dim(V))$, see [35], whereas $(\Pi(V), \mathfrak{pe}^{sy}(\Pi(V)))_* = \Pi(V) \oplus \mathfrak{pe}^{sy}(\Pi(V))$.

Possible analogues of the EE on supermanifolds with a G-structure. (Here $\mathfrak{g} = \mathrm{Lie}(G)$ is a simple Lie superalgebra ($\mathbb{Z}$-graded of finite growth and not necessarily finite-dimensional) and $\mathfrak{cg}$ denotes the 1-dimensional trivial central extension of $\mathfrak{g}$.)

(1) The first idea is to replace $\mathfrak{o}(m)$ with $\mathfrak{osp}(m|2n)$ for a $\mathbb{Z}$-grading of the form

$$\mathfrak{osp}(m|2n) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{with} \quad \mathfrak{g}_0 = \mathfrak{cosp}(m-2|2n) \quad \text{and} \quad m > 2.$$

(2) The odd counterpart of this step is to replace $\mathfrak{osp}(m|2n)$ with its odd (periplectic) analogues: $\mathfrak{pe}^{sy}(n)$ and $\mathfrak{spe}^{sy}(n)$ and the "mixture" of these, $\mathfrak{spe}^{sy}(n) \Subset \mathbb{C}(az + bd)$, where in matrix realization we can take $d = \mathrm{diag}(1_n, -1_n)$, $z = 1_{2n}$, see Appendix.

Why is $m > 2$ in (1)? If $m = 2$, then $\mathfrak{g}_0 = \mathfrak{sp}(2n)$ and, as we know [40], there are no structure functions of order 2. Might it be that an analogue of EE is connected not with $\mathfrak{sp}(2n)$, the Lie algebra of *linear* symplectic transformations, but with the infinite dimensional Lie algebra of *all* symplectic transformations, i.e., the Lie algebra $\mathfrak{h}(2n|0)$ of Hamiltonian vector fields? Theorem 1 states: NO. (The structure functions are only of order 1; the corresponding eqs. written in [39], though interesting, do not resemble EE.)

Let us not give up: the Lie algebra $\mathfrak{o}(m)$, as well as $\mathfrak{h}(2n|0)$, has one more analogue — the Lie superalgebra $\mathfrak{h}(0|m)$ of Hamiltonian vector fields on $(0|m)$-dimensional supermanifold. So other possibilities are:

(3) replace $\mathfrak{osp}(m|2n)$ with $\mathfrak{h}(2n|m)$, where $m \neq 0$.

Since we went that far, let us go further still and

(4) replace $\mathfrak{h}(2n|m)$ in (3) with $\mathfrak{k}(2n+1|m)$; and, moreover, consider "odd" analogues of (3) and (4):

(5) replace $\mathfrak{pe}(n)$ and $\mathfrak{spe}(n)$ in (2) with $\mathfrak{le}(n)$ and $\mathfrak{sle}(n)$, $\mathfrak{m}(n)$ or $\mathfrak{b}_\lambda(n)$. For the definition of these and other simple vectorial Lie superalgebras see [47].

In the next sections we will list structure functions for some of the possibilities (1)–(6). The remaining ones are **open problems**.

Remark 3. One should also investigate the cases associated with $\mathbb{Z}$-grading of Kac–Moody (twisted loop) superalgebras of the form $\underset{|i| \leq 1}{\oplus} \mathfrak{g}_i$. Nobody explored yet this infinite dimensional possibility. Clearly, there are "trivial" analogues of compact Hermitian symmetric spaces, namely, the manifolds of loops with values in any finite dimensional compact Hermitian symmetric space. Remarkably, there are also "twisted" versions of these compact Hermitian symmetric spaces associated with twisted loop algebras and superalgebras, cf. [36]. It is not known, however, how to calculate the cohomology of Kac–Moody algebras with this type of coefficients, even in the "trivial cases".

5   Spencer cohomology of $\mathfrak{osp}\,(m|n)$

$\mathbb{Z}$-gradings of depth 1. All these gradings are of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_1 \simeq \mathfrak{g}'_{-1}$ as $\mathfrak{g}_0$-modules.

Proposition 5 ([23, 36]). *For $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{osp}(m|2n)$ the following cases are possible:*

a) $\mathfrak{cosp}(m-2|2n)$, $\mathfrak{g}_{-1} = \mathrm{id}$;

b) $\mathfrak{gl}(r|n)$ if $m = 2r$, $\mathfrak{g}_{-1} = \Lambda^2(\mathrm{id})$.

Cartan prolongs of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)$.

Proposition 6. *a) $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$ except for the case Proposition 5b) for $r = 3$, $n = 0$ when $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(3|0)$.*

b) $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_* = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}}_0$.

Structure functions. Cases a) and b) below correspond to cases of $\mathbb{Z}$-gradings from Proposition 5. The cases $mn = 0$ are dealt with in [12] and Introduction.

Theorem 2. *a) As $\hat{\mathfrak{g}}_0$-module, $H^{2,2}_{(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_*} = S^2(\Lambda^2(\mathfrak{g}_{-1}))/\Lambda^4(\mathfrak{g}_{-1})$ and splits into the direct sum of three irreducible components whose weights are given in Table 3.*

As $\mathfrak{g}_0$-module, $H^{2,2}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*} = H^{2,2}_{(\mathfrak{g}_{-1}, \hat{\mathfrak{g}}_0)_*}/S^2(\mathfrak{g}_{-1})$. It is irreducible and its highest weight is given in Table 3. For $k \neq 2$ structure function vanish.

b) If $r \neq n$, $n + 2$, $n + 3$, then $H^2(\mathfrak{g}_{-1}; \mathfrak{g}_*)$ is an irreducible $\mathfrak{g}_0$-module and its highest weight is given in Table 4.

The cases $r = 4$, $n = 0$ and $r = 2$, $n = 1$ coincide, respectively, with the cases considered in a) for $\mathfrak{o}(8)$ and $\mathfrak{osp}(4|2)$.

6 Spencer cohomology of $\mathfrak{spe}(n)$

Proposition 7 (Cf. [23] with [36]). *All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g}$ are listed in Table 1 of [36]. They are:*

a)$\mathfrak{g}_0 = \mathfrak{sl}(m|n-m)$, $\mathfrak{g}_{-1} = \Pi(S^2(\mathrm{id}))$, $\mathfrak{g}_1 = \Pi(\Lambda^2(\mathrm{id}'))$;

b) $\mathfrak{g}_0 = \langle \tau + (n-1)z \rangle \oright \mathfrak{spe}(n-1)$, $\mathfrak{g}_{-1} = \mathrm{id}$, $\mathfrak{g}_1 = \mathrm{id}' = \Pi(\mathrm{id})$, $\mathfrak{g}_1 = \Pi(\langle \mathbf{1} \rangle)$.

Here $\tau = \mathrm{diag}(1_{n-1}, -1_{n-1})$, $z = 1_{2n-2}$, the sign $\mathfrak{a} \oright \mathfrak{b}$ denotes a semidirect sum of Lie superalgebras, the ideal is on the right, id is endowed with a nondegenerate supersymmetric odd bilinear form. In these cases $\mathfrak{g}_* = \mathfrak{g}$.

If $\mathfrak{g}_0 = \mathfrak{cpe}(n-1)$, $\mathfrak{g}_{-1} = \mathrm{id}$, then $\mathfrak{g}_* = \mathfrak{pe}(n)$. If $\mathfrak{g}_0 = \mathfrak{spe}(n-1)$, $\mathfrak{pe}(n-1)$ or $\mathfrak{cspe}(n-1)$ and $\mathfrak{g}_{-1} = \mathrm{id}$, then $\mathfrak{g}_* = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.

Theorem 3. *a) The nonvanishing structure function are of order 1, and in the cases when they constitute a completely reducible $\mathfrak{g}_0$-module, the corresponding highest weights are given in Table 5.*

b) For $\mathfrak{g}_0 = \mathfrak{spe}(n-1)$, $\mathfrak{pe}(n-1)$, $\mathfrak{cspe}(n-1)$, $\langle \tau + (n-1)z \rangle \oright \mathfrak{spe}(n-1)$, $\mathfrak{cpe}(n-1)$, and $\mathfrak{g}_{-1} = \mathrm{id}$ all structure functions vanish except for $H^{1,2}_{\mathfrak{spe}(n-1)} = \Pi(\mathrm{id}) = \Pi(V_{\varepsilon_1})$ and there are the following nonsplit exact sequence of $\mathfrak{spe}(n-1)$-modules: (here $\varepsilon_1, \ldots, \varepsilon_{n-1}$ is the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{pe}(n-1)$. V_λ denotes the irreducible $\mathfrak{pe}(n-1)$-module with highest weight λ and even highest vector)

$$0 \longrightarrow V_{\varepsilon_1 + \varepsilon_2} \longrightarrow H^{2,2}_{\mathfrak{spe}(n-1)} \longrightarrow \Pi(V_{2\varepsilon_1 + 2\varepsilon_2}) \longrightarrow 0 \quad for \quad n > 4,$$

$$0 \longrightarrow X \longrightarrow H^{2,2}_{\mathfrak{spe}(3)} \longrightarrow \Pi(V_{3\varepsilon_1}) \longrightarrow 0,$$

where X is determined from the following nonsplit exact sequences of $\mathfrak{spe}(3)$ -modules:

$$0 \longrightarrow V_{\varepsilon_1+\varepsilon_2} \longrightarrow X \longrightarrow \Pi(V_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow 0.$$

Also

$$0 \longrightarrow H^{2,2}_{\mathfrak{spe}(n-1)} \longrightarrow H^{2,2}_{\mathfrak{pe}(n-1)} \longrightarrow V_{2\varepsilon_1} \longrightarrow 0, \quad if \quad n > 3;$$

$$0 \longrightarrow H^{2,2}_{\mathfrak{spe}(n-1)} \longrightarrow H^{2,2}_{\mathfrak{cspe}(n-1)} \longrightarrow V_{2\varepsilon_1} \longrightarrow 0, \quad if \quad n > 3;$$

$H^{2,2}_{\langle \tau+(n-1)z\rangle \, \in \, \mathfrak{spe}(n-1)} = \Pi(V_{2\varepsilon_1+2\varepsilon_2})$ *is an irreducible* $\mathfrak{spe}(n-1)$*-module if* $n > 4$ *and*

$$0 \longrightarrow \Pi(V_{2\varepsilon_1+2\varepsilon_2}) \longrightarrow H^{2,2}_{\langle \tau+3z\rangle \, \in \, \mathfrak{spe}(3)} \longrightarrow \Pi(V_{3\varepsilon_1}) \longrightarrow 0.$$

Finally,

$$0 \longrightarrow H^{2,2}_{\langle \tau+(n-1)z\rangle \, \in \, \mathfrak{spe}(n-1)} \longrightarrow H^{2,2}_{\mathfrak{cpe}(n-1)} \longrightarrow V_{2\varepsilon_1} \longrightarrow 0 \quad if \quad n > 3.$$

Moreover, $H^{2,2}_{\mathfrak{cpe}(n-1)} = \Pi(S^2(\Lambda^2(\mathrm{id})/\Pi(\langle 1\rangle))/\Lambda^4(\mathrm{id})).$

7   An analogue of a theorem by Serre: on involutivity of $\mathbb{Z}$-graded Lie superalgebras

Theorem 1, part of which we have attributed above to Serre, is actually a corollary of Serre's initial statement [45]. Before we formulate it, recall that the notion of involutivity comes from very practical problems: how to solve differential equation with the help of a computer [25]. Let $\pi : E \longrightarrow B$ be the bundle (it sufficies to consider the trivial bundle with base $B = \mathbb{R}^n$ and the fiber $\mathbb{R}^m$); let $J_q E$ be the space of q-jets of sections of the bundle π. Let $V(E) \subset TE$ be the vertical bundle, i.e., the kernel of the map $T\pi$. With every system of differential equations $DE_q \subset J_q E$ of order q in m unknown functions of n variables we can associate a subbundle $N_q \subset V^{(q)} J_q E$, where $V^{(q)} J_q E$ is the vertical bundle with respect to the projection $\pi^q_{q-1} : J_q E \longrightarrow J_{q-1} E$ as a subbundle of $S^q(T^*B) \otimes VE$. The subbundle N_q is called the *geometric symbol* of the system DE_q. Set

$$N_q^{(s)} := \left\{ f \in N_q \Big| \frac{\partial f}{\partial x_i} = 0 \text{ for } i = 1, \ldots, s \right\},$$

where $f \in \mathbb{R}^m$ and the derivatives are taken coordinate-wise. Let P_q^m be the mth tensor (symmetric, actually) power of the space of degree $\leq q$ polynomials (in n variables). The first prolongation of N_q is defined to be

$$N_{q+1} := \left\{ f \in P_{q+1}^m \Big| \frac{\partial f}{\partial x_i} \in N_q \text{ for } i = 1, \ldots, n \right\}.$$

The symbol N_q is said to be *involutive* if

$$\dim N_{q+1} = \dim N_q + \dim N_q^{(1)} + \cdots + \dim N_q^{(n-1)}.$$

(usually, the lhs is smaller).

Similarly, let $g \subset \mathrm{Hom}(V, W)$ be a subspace and $g^{(i)}$ the ith Cartan prolongation of g (defined above for $W = V$). For any subspace $H \subset V$ set:

$$g_H := \{F \in g \mid F(h) = 0 \text{ for any } h \in H\}.$$

Let $r_k = \min\limits_{\dim H = k} g_H$. It is not difficult to show that

$$\dim g^{(1)} \le r_0 + r_1 + \cdots + r_{k-1}. \tag{$*$}$$

The space g is called *involutive* if there is an equality in $(*)$. It is not difficult to see that if g is involutive, then $g^{(1)}$ is also involutive. So, speaking about Lie algebras which are Catran prolongs it suffices to consider involutivity of their linear parts.

Let $\mathfrak{g} = \bigoplus\limits_{k \ge -1} \mathfrak{g}_k$ be a $\mathbb{Z}$-graded Lie algebra, $\{a_1, \ldots, a_n\}$ be a basis of $\mathfrak{g}_{-1}$. Clearly, the map

$$\mathrm{ad}_{a_r} : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad x \mapsto [a_r, x]$$

is a homomorphism of $\mathfrak{g}_{-1}$-modules. In accordance with the above, we say that a $\mathbb{Z}$-graded Lie algebra of the form $\mathfrak{g} = \bigoplus\limits_{k \ge -1} \mathfrak{g}_k$ is called *involutive* if all the maps ad_{a_r} are onto.

Serre observed that involutivity property considerably simplifies computation of cohomology: if $\mathfrak{g}_*$ is involutive, then for every i

$$H^i(\mathfrak{g}_-; \mathfrak{g}_*) \text{ is supported in the lowest possible degree.} \tag{$**$}$$

To superize the notion of involutivity, we have to require surjectivity of the maps ad_{a_r} for a_r even. Additionally we must demand vanishing of the homology with respect to each differential given by the odd map ad_{a_r} (the homology is well-defined thanks to the Jacobi identity). More precisely, for any Lie superalgebra $\mathfrak{g} = \bigoplus\limits_{k \ge -1} \mathfrak{g}_k$ set:

$$\mathfrak{g}^r = \ker \mathrm{ad}_{a_1} \cap \ker \mathrm{ad}_{a_2} \cap \cdots \cap \ker \mathrm{ad}_{a_r}.$$

Clearly, $\mathfrak{g}^r = \bigoplus\limits_{k \ge -1} \mathfrak{g}_k^r$, where $\mathfrak{g}_k^r = \mathfrak{g}^r \cap \mathfrak{g}_k$. Notice that $\mathrm{ad}_{a_r} \left(\mathfrak{g}_k^{r-1} \right) \subset \mathfrak{g}_{k-1}^{r-1}$. The Lie superalgebra $\mathfrak{g} = \bigoplus\limits_{k \ge -1} \mathfrak{g}_k$ will be called *involutive* if the following conditions are fulfilled:

(1) $\mathfrak{g}^n = \mathfrak{g}_{-1}$ (recall that $n = \dim \mathfrak{g}_{-1}$);
(2) $\mathrm{ad}_{a_r} \left(\mathfrak{g}^{r-1} \right) = \mathfrak{g}^{r-1}$ if a_r is even;
(3) $\mathrm{ad}_{a_r} \left(\mathfrak{g}^{r-1} \right) = \mathfrak{g}^r$ if a_r is odd.
The cohomology group $H^i(\mathfrak{g}_{-1}; \mathfrak{g})$ has a natural $\mathbb{Z}$-grading:
$H^i(\mathfrak{g}_{-1}; \mathfrak{g}) = \oplus_{k \ge -1} H^{i,k}(\mathfrak{g}_{-1}; \mathfrak{g})$ induced by the $\mathbb{Z}$-grading of $\mathfrak{g}$.

Theorem 4. ([39]) *Let $\mathfrak{g}$ be involutive. Then if $i \ge 0$ and $k \ge 0$, then $H^{i,k}(\mathfrak{g}_{-1}; \mathfrak{g}) = 0$.*

8 Spencer cohomology of vectorial Lie superalgebras in their standard grading

Theorem 5 (cf. Theorem 1). *1) For $\mathfrak{g}_* = \mathfrak{vect}(m|n)$ and $\mathfrak{svect}(m|n)$ the structure functions vanish except for $\mathfrak{svect}(0|n)$ when the structure functions are of order n and constitute the $\mathfrak{g}_0$-module $\Pi^n(\mathbf{1})$.*

2) For $\mathfrak{g}_* = \mathfrak{h}(0|m)$ for $m > 4$, and $\mathfrak{g}_* = \mathfrak{h}(2n|m)$ for $mn \neq 0$, the nonzero structure functions are $\Pi(R(3\pi_1) \oplus R(\pi_1))$ of order 1.

3) For $\mathfrak{g}_* = \mathfrak{h}^o(0|m)$, $m > 4$, the nonzero structure functions are same as for $\mathfrak{h}(0|m)$ plus an additional direct summand $\Pi^{m-1}(R(\pi_1))$ of order $m - 1$.

4) For $\mathfrak{g}_* = \mathfrak{sle}(n)$, $n > 1$, the nonzero structure functions are $H^{1,2}_{\mathfrak{sle}(n)} = S^3(\mathfrak{g}_{-1})$, $H^{2,2}_{\mathfrak{sle}(n)} = \Pi(\mathbf{1})$, $H^{n,2}_{\mathfrak{sle}(n)} = \Pi^n(\mathbf{1})$.

Thus, on almost symplectic manifolds with nondegenerate and non-closed form ω, there is an analog of torsion — structure function of order 1, namely $d\omega$. Since the space of 3-forms splits into the space of forms proportional to ω and the complementary space of "primitive" forms, there are two components of this torsion: $d\omega = \lambda\omega + P$. If the primitive component vanishes, we have a nice-looking equation:

$$d\omega = \lambda\omega \quad \text{for some} \quad \lambda \in \Omega^1. \tag{8.1}$$

The other component of "torsion" must also vanish for the supermanifold to be symplectic, not almost symplectic.

An analogue of Einstein equation on almost periplectic supermanifolds. Let ω_1 be the canonical odd 2-form and R a 2-form which is a section through $H^{2,2}_{\mathfrak{sle}(n)} = \Pi(\mathbf{1})$. This gives rise to an analogue of (2.7) for $\mathfrak{sle}(n)$:

$$\omega_1 = \lambda R, \quad \lambda \in \mathbb{C}. \tag{8.2}$$

The equation (8.2) are well-defined *provided the irreducible components of the 1st order structure functions, the elements from $H^{1,2}_{\mathfrak{sle}(n)}$ vanish.* Denote by $\mathbf{Tor}_i$ $(i = 1, 2)$ the components of the torsion tensor; then these conditions are:

$$\mathbf{Tor}_i = 0 \quad \text{for} \quad i = 1, 2.$$

When the torsion components vanish, we can reduce the nondegenerate odd 2-form to the canonical form (cf. [26] and [46]).

9 Proof of Theorem 4

The long exact sequence. Let $\mathfrak{g}$ be a Lie superalgebra and

$$0 \longrightarrow A \xrightarrow{\partial_0} C \xrightarrow{\partial_1} B \longrightarrow 0,$$
$$\text{where} \quad p(\partial_0) = \bar{0} \quad \text{and} \quad \partial_1 \quad \text{is either even or odd,} \tag{9.1}$$

be a short exact sequence of $\mathfrak{g}$-modules. Let d be the differential in the standard cochain complex of the Lie superalgebra $\mathfrak{g}$, cf. [9].

Consider the long sequence of cohomology:

$$\cdots \xrightarrow{\partial} H^i(\mathfrak{g}; A) \xrightarrow{\partial_0} H^i(\mathfrak{g}; C) \xrightarrow{\partial_1} H^i(\mathfrak{g}; B) \xrightarrow{\partial} H^{i+1}(\mathfrak{g}; A) \xrightarrow{\partial_0} \cdots, \tag{9.2}$$

where ∂_i is the differential induced by the namesake differential in (9.1), and $\partial = \partial_0^{-1} \circ d \circ \partial_1^{-1}$. Since ∂_0 and ∂_1 commute with d, the sequence (9.2) is well-defined and the same

arguments as for Lie algebras [9] demonstrate that the long sequence of cohomology (9.2) induced by (9.1) is exact.

Thus, the short exact sequences of $\mathfrak{g}_{-1}$-modules, where $\partial_1 = \mathrm{ad}_{a_r}$ and ∂_0 is the embedding $\mathfrak{g}^r \subset \mathfrak{g}^{r-1}$:

$$0 \longrightarrow \mathfrak{g}^r \xrightarrow{\partial_0} \mathfrak{g}^{r-1} \xrightarrow{\partial_1} \mathfrak{g}^{r-1} \longrightarrow 0 \quad \text{for} \quad p(a_r) = \bar{0},$$

$$0 \longrightarrow \mathfrak{g}^r \xrightarrow{\partial_0} \mathfrak{g}^{r-1} \xrightarrow{\partial_1} \mathfrak{g}^r \longrightarrow 0 \quad \text{for} \quad p(a_r) = \bar{1}$$

induce the long exact sequences of cohomology

$$\cdots \longrightarrow H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right) \xrightarrow{\partial_0} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right) \xrightarrow{\partial_1} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right)$$
$$\xrightarrow{\partial} H^{i+1}\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right) \longrightarrow \cdots, \tag{9.3}$$

$$\cdots \longrightarrow H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right) \xrightarrow{\partial_0} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right) \xrightarrow{\partial_1} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right)$$
$$\xrightarrow{\partial} H^{i+1}\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right) \longrightarrow \cdots. \tag{9.4}$$

Lemma 1. *In sequences (9.3) and (9.4) we have $\partial_1 = 0$.*

Proof. First, consider sequence (9.3). Let $f \in C^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right)$ and $df = 0$. Define $c \in C^{i-1}\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right)$ by the formula

$$c(x_1, \ldots, x_{i-1}) = f(x_1, \ldots, x_{i-1}, a_r).$$

Then (here the sign $\pm$ is determined by Sign Rule)

$$dc(x_1, \ldots, x_i) = \sum \pm \mathrm{ad}_{x_j}\, c(x_1, \ldots, \hat{x}_j, \ldots, x_i)$$
$$= \sum \pm \mathrm{ad}_{x_j}\, f(x_1, \ldots, \hat{x}_j, \ldots, x_i, a_r) = df(x_1, \ldots, x_i, a_r) - \pm \mathrm{ad}_{a_r}\, f(x_1, \ldots, x_i).$$

Since $df = 0$, we have $dc = \pm a_r f$. Thus, $a_r f$ is exact, and, therefore, $\partial_1 = 0$.

Let us prove now (9.4). Let $l = \mathbb{C}a_r$. Then $H^p(l; \mathfrak{g}^{r-1}) = 0$ for $p > 0$ and $H^0(l; \mathfrak{g}^{r-1}) = \mathfrak{g}^r$. Since $\mathfrak{g}_{-1} = l \oplus \mathfrak{g}_{-1}/l$, the spectral sequence with respect to the ideal $l \subset \mathfrak{g}_{-1}$ immediately gives

$$H^i(\mathfrak{g}_{-1}; \mathfrak{g}^r) \simeq H^i(\mathfrak{g}_{-1}/l; \mathfrak{g}^r)$$

Therefore, $H^i(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}) \subset H^i(\mathfrak{g}_{-1}; \mathfrak{g}^r)$ and ∂_0 is surjective. Thus, $\partial_1 = 0$. ∎

Corollary 1. *The long exact sequences (9.3) and (9.4) can be reduced to the following short exact sequences*

$$0 \longrightarrow H^{i-1}\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right) \xrightarrow{\partial} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right) \xrightarrow{\partial_0} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right) \xrightarrow{\partial_1} 0, \tag{9.5}$$

$$0 \longrightarrow H^{i-1}\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right) \xrightarrow{\partial} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^r\right) \xrightarrow{\partial_0} H^i\left(\mathfrak{g}_{-1}; \mathfrak{g}^{r-1}\right) \xrightarrow{\partial_1} 0. \tag{9.6}$$

Now we can prove the theorem by induction on r. First of all, $\mathfrak{g}^n = \mathfrak{g}_{-1}$ by condition (1) of involutivity. So we have $\mathfrak{g}^n_k = 0$ for $k \geq 0$ and

$$H^{i,k}\left(\mathfrak{g}_{-1}; \mathfrak{g}^n\right) = 0 \quad \text{if} \quad i \geq 0 \quad \text{and} \quad k \geq 0.$$

Then consider the term of degree k in (9.5) and (9.6). We obtain the exact sequences

$$0 \longrightarrow H^{i-1,k}\left(\mathfrak{g}_{-1};\mathfrak{g}^{r-1}\right) \xrightarrow{\partial} H^{i,k}\left(\mathfrak{g}_{-1};\mathfrak{g}^{r}\right) \xrightarrow{\partial_0} H^{i,k}\left(\mathfrak{g}_{-1};\mathfrak{g}^{r-1}\right) \xrightarrow{\partial_1} 0, \tag{9.7}$$

$$0 \longrightarrow H^{i-1,k}\left(\mathfrak{g}_{-1};\mathfrak{g}^{r}\right) \xrightarrow{\partial} H^{i,k}\left(\mathfrak{g}_{-1};\mathfrak{g}^{r}\right) \xrightarrow{\partial_0} H^{i,k}\left(\mathfrak{g}_{-1};\mathfrak{g}^{r-1}\right) \xrightarrow{\partial_1} 0. \tag{9.8}$$

It follows immediately from (9.7) for $p(a_r) = \bar{0}$ and from (9.8) for $p(a_r) = \bar{1}$ that $H^{i,k}\left(\mathfrak{g}_{-1};\mathfrak{g}^{r-1}\right) = 0$. The theorem is proved.

10   Open problems: Riemann tensors on curved supergrassmannians

Denote by $\mathfrak{g}(m|n)$ either of the Lie superalgebras $\mathfrak{h}(2m|n)$, $\mathfrak{h}^o(n)$ or $\mathfrak{k}(2m+1|n)$; let $F(m|n-2)$ be the superspace of "functions" which in our case are polynomials or power series on which $\mathfrak{g}_0$ naturally acts.

In [36], Table 5, there are listed all $\mathbb{Z}$-gradings of $\mathfrak{g} = \mathfrak{g}(m|n)$ of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. For them, $\mathfrak{g}_1 \simeq \mathfrak{g}'_{-1}$, $\mathfrak{g}_{-1} = F(m|n)$, $\mathfrak{g}_0 = \mathfrak{g}(m|n-2) \oplus F(m|n-2)$ for $n > 1$, and if $n > 2$, then $\mathfrak{g}_{-1}$ is not purely odd and is isomorphic to the tangent space to the total space of the what is called Fock bundle over a $(2m|n-2)$-dimensional symplectic supermanifold or its version for the contact supermanifold.

In 1985 Yu Kochetkov informed us that he showed (unpublished) that for $\mathfrak{g}(m|n) = \mathfrak{h}(2m|n)$ or $\mathfrak{h}(0|n)$ there is always a trivial component (perhaps, there are several) in the space of 2nd order structure functions for the Riemann-like tensors, so there are analogues of (2.7). Observe, that for Weyl-like tensors (for conformal structures) there is no trivial modules and, this is expected since trivial modules correspond to filtered deformations, cf. e.g., [5].

One of us (EP) managed to calculate structure functions of order 1 for $\mathfrak{g} = \mathfrak{h}^o(0|6)$. The space of these structure functions is nonzero; in addition, it is not completely reducible, some of the indecomposable components look as complicated as follows, where x and y are some irreducible components (the same symbol denotes an isomorphic copy):

$$
\begin{array}{ccccc}
x & \longrightarrow & y & \longleftarrow & x \\
\downarrow & & \uparrow & & \downarrow \\
y & \longleftarrow & x & \longrightarrow & y \\
\uparrow & & \downarrow & & \uparrow \\
x & \longrightarrow & y & \longleftarrow & x
\end{array}
$$

Since structure functions of order 1 must vanish in order for the analogues of EE be well-defined, these structure functions constitute constraints similar to the Wess–Zumino constraints in supergravity. Here we encounter an amazing situation: the lack of complete reducibility implies that only part of these constraints (depicted by x) are relevant.

We have no idea how to approach analytically other, especially infinite dimensional, cases: the number of structure function grows quickly with m and n. The only way we see at the moment is to arm ourselves with computers, e.g., Grozman's package SuperLie [14]. A first result in this direction is calculation of structure functions for curved supergrassmannian $\mathcal{G}_{0|2}^{0|4}$ and its "relatives" resulting in an unconventional and unexpected version of supergravity equations [16].

11 Tables

Notations in tables. We use the notational conventions of [44] and definitions adopted there.

In Table 1: $\mathfrak{s} = (\mathrm{Lie}(S_c)) \otimes \mathbb{C}$, NCHSS is an abbreviation for *noncompact Hermitian symmetric space*, in the diagram of $\mathfrak{s}$ the maximal parabolic subalgebra $\mathfrak{p} = \mathrm{Lie}(P)$, such that X can be represented as $(S_c)^{\mathbb{C}}/P$, is determined by one vertex: the last one in cases 0, 2, 3, E_7, E_8, the first one in case 4, the p-th one in case 1. Symbol $\mathfrak{cg}$ denotes the trivial central extension of the Lie (super) algebra $\mathfrak{g}$.

In Table 2: we call a homogeneous space G/P, where G is a simple Lie supergroup P its parabolic subsupergroup corresponding to several omitted generators of a Borel subalgebra (description of these generators can be found in [15]), of *depth d* and *length l* if such are the depth and length of $\mathrm{Lie}(G)$ in the $\mathbb{Z}$-grading compatible with that of $\mathrm{Lie}(P)$. Note that all superspaces of Table 2 possess an hermitian structure (hence are of depth 1) except $PeGr$ (no hermitian structure), PeQ (no hermitian structure, length 2), $CGr_{0,k}^{0,n}$ and $SCGr_{0,k}^{0,n}$ (no hermitian structure, lengths $n-k$ and, resp. $n-k-1$). The dots stand for the notions without name or those that require too much space. The sign $\mathfrak{a} \mathbin{\oplus\!\!\!\!+} \mathfrak{b}$ denotes the semidirect sum of Lie superalgebras, the ideal on the right; Lie superalgebras $\mathfrak{osp}_\alpha(4|2)$ and $\mathfrak{ab}(3)$ are described via their Cartan matrices, see [15].

In Table 3: $m = 2r + 2$ or $m = 2r + 3$, $\varepsilon_1, \ldots, \varepsilon_r$ and $\delta_1 \ldots, \delta_n$ are the standard bases of the dual spaces to the spaces of diagonal matrices in $\mathfrak{o}(m-2)$ and $\mathfrak{sp}(n)$, respectively.

In Table 4: $\varepsilon_1, \ldots, \varepsilon_r, \delta_1 \ldots, \delta_n$ is the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{gl}(r|n)$. In Table 5: $\varepsilon_1, \ldots, \varepsilon_m, \delta_1 \ldots, \delta_{n-m}$ is the standard basis of the dual space to the space of diagonal matrices in $\mathfrak{gl}(m|n-m)$.

Table 1. Hermitian symmetric spaces

	Name of CHSS X	$X = S_c/G_c$	$\mathfrak{s}_0 = (\mathfrak{g}_c)^{\mathbb{C}}$	$\mathfrak{s}_{-1} \sim T_0 X$	$(S_c)^*$	Name of NCHSS
0	$\mathbb{C}P^n$	$SU(n+1)/U(n)$	$\mathfrak{gl}(n)$	id	$SU(1,n)$	$^*\mathbb{C}P^n$
1	Gr_p^{p+q}	$SU(p+q)/S(U(p) \times U(q))$	$\mathfrak{gl}(p) \oplus \mathfrak{gl}(q)$	$\mathrm{id} \odot \mathrm{id}^*$	$SU(p,q)$	$^*Gr_p^{p+q}$
2	OGr_n	$SO(2n)/U(n)$	$\mathfrak{gl}(n)$	$\Lambda^2\, \mathrm{id}$	$SO(n,n)$	$^*OGr_n$
3	LGr_n	$Sp(2n)/U(n)$	$\mathfrak{gl}(n)$	$S^2\, \mathrm{id}$	$Sp(2n;\mathbb{R})$	$^*LGr_n$
4	Q_n	$SO(n+2)/(SO(2) \times SO(n))$	$\mathfrak{co}(n)$	id	$SO(n,2)$	$^*Q_n$
E_7	$\mathbb{O}P^2$	$E_7/(SO(10) \times U(1))$	$\mathfrak{co}(10)$	$R(\pi_5)$	E_6^*	$^*(\mathbb{O}P^2)$
E_8	$\mathbb{E}$	$E_8/(E_7 \times U(1))$	$\mathfrak{ce}_7$	$R(\pi_1)$	E_7^*	$^*\mathbb{E}$

Occasional isomorphisms: $Gr_p^{p+q} \cong Gr_q^{p+q}$, $Q_1 \cong \mathbb{C}P^1$, $Q_2 \cong S^2 \times S^2$, $Q_3 \cong LGr_2$, $Q_4 \cong Gr_2^4$, $OGr_2 \cong LGr_1 \cong \mathbb{C}P^1$, $OGr_3 \cong Gr_3^4$.

Table 2. Classical superspaces of depth 1

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	Interpretation	Underlying domain	Name of the superdomain
$\mathfrak{sl}(m\|n)$	$\mathfrak{s}(\mathfrak{gl}(p\|q) \oplus \mathfrak{gl}(m-p\|n-q))$	$\mathrm{id} \otimes \mathrm{id}^*$	Supergrassmannian of the $(p\|q)$-dimensional subsuperspaces in $\mathbb{C}^{m\|n}$	$Gr_p^m \times Gr_q^n$	$Gr_{p,q}^{m,n}$
$\mathfrak{psl}(m\|m)$	$\mathfrak{ps}(\mathfrak{gl}(p\|p) \oplus \mathfrak{gl}(m-p\|m-p))$	$\mathrm{id} \otimes \mathrm{id}^*$	Same for $m=n$, $p=q$	$G_p^m \times Gr_p^m$	$Gr_{p,p}^{m,m}$
$\mathfrak{osp}(m\|2n)$	$\mathfrak{cosp}(m-2\|2n)$	id	Superquadric of $(1\|0)$-dimensional isotropic (with respect to a non-degenerate even form) lines in $\mathbb{C}^{m\|n}$	Q_{m-2}	$Q_{m-2,n}$
$\mathfrak{osp}(2m\|2n)$	$\mathfrak{gl}(m\|n)$	$\Lambda^2(\mathrm{id})$	Orthosymplectic Lagrangian supergrassmannian of $(m\|m)$-dimensional isotropic (with respect to a nondegenerate even form) subsuperspaces in $\mathbb{C}^{2m\|n}$	$^*OGr_m \times LGr_n$	$OLGr_{m,n}$
$\mathfrak{sq}(n)$ $\mathfrak{psq}(n)$	$\mathfrak{s}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$ $\mathfrak{ps}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$	$\mathrm{id} \odot \mathrm{id}^*$	Queergrassmannian of Π-symmetric $(p\|p)$-dimensional subsuperspaces in $\mathbb{C}^{n\|n}$	Gr_p^n	QGr_p^n
$\mathfrak{pe}(n)$ $\mathfrak{spe}(n)$	$\mathfrak{cpe}(n-1)$ $\mathfrak{cspe}(n-1)$	id	Periplectic superquadric of $(1\|0)$-dimensional isotropic (with respect to a nondegenerate odd form) lines in $\mathbb{C}^{n\|n}$	$\mathbb{C}P^{n-1}$	PeQ_{n-1}
$\mathfrak{pe}(n)$ $(\mathfrak{spe}(n))$	$\mathfrak{gl}(p\|n-p)$ $(\mathfrak{sl}(p\|n-p))$	$\Pi(S^2(\mathrm{id}))$ or $\Pi(\Lambda^2(\mathrm{id}))$	Periplectic Lagrangian supergrassmannian of $(p\|n-p)$-dimensional (and with a fixed volume for $\mathfrak{spe}$) subsuperspaces in $\mathbb{C}^{n\|n}$ isotropic with respect to an odd symmetric or skew-symmetric form	Gr_p^n	$PeGr_p^n$

Table 2 (continued). Classical superspaces of depth 1

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$	Interpretation	Underlying domain	Name of the superdomain
$\mathfrak{osp}_\alpha(4\|2)$ $= \mathfrak{osp}(4\|2)_\alpha$	$\mathfrak{cosp}(2\|2) \simeq \mathfrak{gl}(2\|1)$	id	$\cdots$	$\mathbb{C}P^1 \times \mathbb{C}P^1$	$\cdots$
$\mathfrak{ab}(3)$	$\mathfrak{cosp}(2\|4)$	$L_{3\varepsilon_1}$	$\cdots$	$\mathbb{C}P^1 \times Q_5$	$\cdots$
$\mathfrak{vect}(0\|n)$	$\mathfrak{vect}(0\|n-k) \niplus \mathfrak{gl}(k; \Lambda(n-k))$	$\Lambda(k) \otimes \Pi(\mathrm{id})$	Curved supergrassmanian of $(0\|1)$-dimensional subsupermanifolds in $\mathbb{C}^{0\|n}$	$\cdots$	$CGr_{0,k}^{0,n}$
$\mathfrak{svect}(0\|n)$	$\mathfrak{vect}(0\|n-k) \niplus \mathfrak{sl}(k; \Lambda(n-k))$	$\Pi(Vol)$ if $k=1$	Same with volume elements preserved in the sub- and ambient supermanifolds	$\cdots$	$SCGr_{0,k}^{0,n}$
$\mathfrak{h}(0\|m)$	$\mathfrak{h}(0\|m-2) \niplus \Lambda(m-2) \cdot z$		Curved superquadric of $(0\|1)$-dimensional subsupermanifolds in $\mathbb{C}^{0\|m}$	$\cdots$	$CQ_{m-2,0}$
$\mathfrak{h}^o(m)$	$\mathfrak{h}^o(m-2) \niplus \Lambda(m-2) \cdot z$	$\Pi(\mathrm{id})$	isotropic (with respect to a (partly) split symmetric form)		

Table 3.

r	n	$H^{2,2}_{(\mathfrak{g}_{-1},\widehat{\mathfrak{g}}_0)_*}$	$S^2(\mathfrak{g}_{-1})$
0	1	—	0, if $m=2$
			$0, \delta_1$ if $m=3$
0	≥ 2	$2\delta_1 + 2\delta_2$	$0, \delta_1 + \delta_2$
1	1	$\varepsilon_1 + \delta_1$	$0, 2\varepsilon_1$
1	≥ 2	$2\varepsilon_1 + \delta_1 + \delta_2$	$0, 2\varepsilon_1$
≥ 2	≥ 1	$2\varepsilon_1 + 2\varepsilon_2$	$0, 2\varepsilon_1$

Table 4.

r	n	$H^{1,2}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}$	$H^{2,2}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}$
≥ 5	0	$\varepsilon_1 + \varepsilon_2 - \varepsilon_{r-2} - \varepsilon_{r-1} - 2\varepsilon_r$	—
0	≥ 3	$2\delta_1 - \delta_{n-1} - 3\delta_n$	—
1	≥ 3	$\varepsilon_1 + \delta_1 - \delta_{n-1} - 3\delta_n$	—
≥ 5	1	$\varepsilon_1 + \varepsilon_2 - \varepsilon_r - 3\delta_n$	—
≥ 2	≥ 2	$\varepsilon_1 + \varepsilon_2 - \delta_{n-1} - 3\delta_n$	—
2	1	—	$-\varepsilon_2 - 3\delta_1$

Table 5.

m	$n-m$	$H^{1,2}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}$
0	≥ 4	$\delta_1 + \delta_2 - 2\delta_{n-1} - 2\delta_n; \quad -\delta_{n-1} - \delta_n$
≥ 1	≥ 2	$2\varepsilon_1 - 2\delta_{n-m-1} - 2\delta_{n-m}; \quad -\delta_{n-m-1} - \delta_{n-m}$
2	1	$2\varepsilon_1 - 3\varepsilon_2 - \delta_1; \quad -\varepsilon_2 - \delta_1$
≥ 3	0	$2\varepsilon_1 - 4\varepsilon_m; \quad 2\varepsilon_1 - 2\varepsilon_{m-1} - 2\varepsilon_m; \quad -2\varepsilon_m$
≥ 3	1	$2\varepsilon_1 - \varepsilon_{m-1} - \varepsilon_m - 2\delta_1; \quad 2\varepsilon_1 - 3\varepsilon_m - \delta_1$

12   Structure functions for exceptional superdomains

In this section we set $Y_i = X_i^+$, $X_i = X_i^-$.

$\mathfrak{g} = \mathfrak{osp}_\alpha(4|2)$. There are five types of Cartan matrices obtained from each other via odd reflections, see [15], but we only consider realization with the Cartan matrices

$$1) \begin{pmatrix} 0 & 1 & -1-\alpha \\ -1 & 0 & -\alpha \\ -1-\alpha & \alpha & 0 \end{pmatrix} \quad \text{and} \quad 2) \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -\alpha \\ 0 & -1 & 2 \end{pmatrix}$$

because the classical superdomains corresponding to other matrices are the same as the ones obtained from these Cartan matrices.

Theorem 6. *The structure functions are only of order* 2.

1) For the parabolic subalgebra generated by X_1^+, $X_2^\pm$, $X_3^\pm$, *set* $X_4^- = [X_1^-, X_2^-]$, $X_5^- = [X_1^-, X_3^-]$, $X_6^- = [X_2^-, X_3^-]$, $X_7^- = [X_1^-, [X_2^-, X_3^-]]$.

The $\mathfrak{g}_0 = \mathfrak{gl}(1|2)$*-module of structure functions is irreducible, the highest weight vector (a representative of the cohomology class) is odd and its highest weight (with respect to Borel subalgebra given by* X_2^+, X_3^+*) in basis* H_1, H_2, H_3 *are as follows:*

$$-\alpha(\alpha+1)H_1 dY_4 dY_7 + \alpha^2 H_2 dY_4 dY_7 + (1+\alpha)X_2 dY_4 dY_5 + X_6 dY_1 dY_4$$

$$\left(-1, -\tfrac{4}{\alpha} - 1, 1\right)$$

2) *For the parabolic subalgebra generated by* X_1^+, $X_2^\pm$, $X_3^\pm$, *set* $X_4^- = [X_1^-, X_2^-]$, $X_5^- = [X_2^-, X_3^-]$, $X_6^- = [X_3^-, [X_1^-, X_2^-]]$, $X_7^- = [[X_1^-, X_2^-], [X_2^-, X_3^-]]$.

The $\mathfrak{g}_0 = \mathfrak{gl}(1|2)$-*module of structure functions is irreducible, the highest weight vector (a representative of the cohomology class) is odd and its heighest weight (with respect to Borel subalgebra given by* X_1^+, X_2^+) *in basis* H_1, H_2, H_3 *are as follows:*

$$(2+\alpha)H_1 dY_1 dY_6 + 2H_2 dY_1 dY_6 + \alpha H_3 dY_1 dY_6 + 2(1+\alpha)Y_2 dY_1 dY_7$$

$$(-1, -1-\alpha, 1)$$

$\mathfrak{g} = \mathfrak{ab}_3$. We consider realization with the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

Theorem 7. *The structure functions are only of order* 1. *The* $\mathfrak{g}_0 = \mathfrak{osp}(2|4)$-*module of structure functions is reducible, so we describe it in terms of the highest weight vectors (representatives of the cohomology classes) with respect to* $(\mathfrak{g}_0)_{\bar{0}}$) *in the following table (where p is parity,* deg *is the degree (*$\deg x_1 = \cdots = \deg x_4 = 1$*; in general this "deg" is more convenient than weight which may depend on a complex parameter, e.g., as for* $\mathfrak{osp}_\alpha(2|4)$); $\mathfrak{sp}$ *is the weight wrt* $\mathfrak{sp}(4)$ *(it does not matter how it is embedded; the weight is given for those who wonder how the dimensions were computed) and the operator that grades* $\mathfrak{g}$):

#	weight	$\mathfrak{sp}$	deg	vector	p	dim
1	$3, -2, 1, 1$	$2, 1$	$1, -1, 1, 1$	$Y_{11}dY_1 dY_{15}$	1	16
2	$2, -2, 0, 1$	$1, 1$	$1, 0, 1, 1$	$Y_5 dY_1 dY_{12} + Y_8 dY_1 dY_{15}$	0	5
3	$2, -1, 2, 0$	$2, 0$	$1, 0, 2, 1$	$Y_5 dY_1 dY_{15}$	0	10
4	$2, -1, 0, 2$	$2, 2$	$1, 0, 2, 2$	$Y_{11}dY_1 dY_{18}$	0	14
5	$2, 0, 2, 1$	$3, 1$	$1, 0, 3, 2$	$Y_{11}dY_{15}dY_{15}$	0	35
6	$1, -1, 1, 0$	$1, 0$	$1, 1, 2, 1$	$Y_1 dY_1 dY_{15}$	1	4
7	$1, 0, 1, 1$	$2, 1$	$1, 1, 3, 2$	*a:* $Y_5 dY_1 dY_{18}$	1	16
				b: $Y_5 dY_{12}dY_{15} + Y_8 dY_{15}dY_{15}$		16
8	$1, 1, 3, 0$	$3, 0$	$1, 1, 4, 2$	$Y_5 dY_{15}dY_{15}$	1	20
9	$1, 1, 1, 2$	$3, 2$	$1, 1, 4, 3$	$Y_1 dY_{11}dY_{18}$	1	40
10	$0, 0, 0, 1$	$1, 1$	$1, 2, 3, 2$	$Y_1 dY_1 dY_{18}$	0	5
11	$0, 1, 2, 0$	$2, 0$	$1, 2, 4, 2$	*a:* $Y_1 dY_1 dY_{15}$	0	10
				b: $-2Y_5 dY_8 dY_{18} - 4Y_5 dY_{12}dY_{17}$		10
				$+Y_5 dY_{15}dY_{16}$		
12	$0, 1, 0, 2$	$2, 2$	$1, 2, 4, 3$	$Y_5 dY_{12}dY_{18} + Y_8 dY_{15}dY_{18}$	0	14
13	$0, 2, 2, 1$	$3, 1$	$1, 2, 5, 3$	$Y_5 dY_{15}dY_{18}$	0	35
14	$-1, 1, 1, 0$	$1, 0$	$1, 3, 4, 2$	$-2Y_1 dY_8 dY_{18} - 4Y_1 dY_{12}dY_{17}$	1	4
				$+Y_1 dY_{15}dY_{16}$		
15	$-1, 2, 1, 1$	$2, 1$	$1, 3, 5, 3$	$Y_1 dY_{15}dY_{18}$	1	16
16	$-1, 3, 3, 0$	$3, 0$	$1, 3, 6, 3$	$Y_5 dY_{17}dY_{18}$	1	20
17	$-2, 3, 2, 0$	$2, 0$	$1, 4, 6, 3$	$Y_1 dY_{17}dY_{18}$	0	10

Irreducible $\mathfrak{osp}(2|4)$-*submodules of* $H^{1,2}$:

$$A = [1] \oplus [2] \oplus [4] \oplus [5] \oplus [7 : 2a - 3b] \oplus [9] \oplus [12] \quad (\dim = 68|72),$$
$$B = [11 : a - b] \oplus [14] \oplus [16] \oplus [17] \quad (\dim = 20|24).$$

The quotient of $H^{1,2}$ *modulo* $A \oplus B$ *is an irreducible* $\mathfrak{osp}(2|4)$-*module.*
   The maximal parabolic subalgebra corresponds to the first Chevalley generator;

$$X_5^- = [X_1^-, X_2^-], \quad X_6^- = [X_2^-, X_3^-], \quad X_7^- = [X_3^-, X_4^-],$$
$$X_8^- = [X_3^-, [X_1^-, X_2^-]], \quad X_9^- = [X_3^-, [X_3^-, X_4^-]], \quad X_{10}^- = [X_4^-, [X_2^-, X_3^-]],$$
$$X_{11}^- = [[X_1^-, X_2^-], [X_2^-, X_3^-]], \quad X_{12}^- = [[X_1^-, X_2^-], [X_3^-, X_4^-]],$$
$$X_{13}^- = [[X_2^-, X_3^-], [X_3^-, X_4^-]], \quad X_{14}^- = [[X_1^-, X_2^-], [X_4^-, [X_2^-, X_3^-]]],$$
$$X_{15}^- = [[X_3^-, X_4^-], [X_3^-, [X_1^-, X_2^-]]],$$
$$X_{16}^- = [[X_3^-, [X_1^-, X_2^-]], [X_4^-, [X_2^-, X_3^-]]],$$
$$X_{17}^- = [[X_3^-, [X_3^-, X_4^-]], [[X_1^-, X_2^-], [X_2^-, X_3^-]]],$$
$$X_{18}^- = [[[X_1^-, X_2^-], [X_3^-, X_4^-]], [[X_2^-, X_3^-], [X_3^-, X_4^-]]].$$

13   Structure functions for the "odd Penrous" tensor

All $\mathbb{Z}$-gradings of depth 1 of $\mathfrak{g} = \mathfrak{psq}(n)$ are of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{ps}(\mathfrak{q}(p) \oplus \mathfrak{q}(n-p))$ for $p > 0$, and $\mathfrak{g}_1 \cong \mathfrak{g}_{-1}^*$, as $\mathfrak{g}_0$-modules, where $\mathfrak{g}_{-1}$ is either one of the two irreducible $\mathfrak{g}_0$-modules in $V(p|p)^* \otimes V(n-p|n-p)$. Explicitly:

$$\mathfrak{g}_{-1} = \text{Span}((x \pm \Pi(x)) \otimes (y \pm \Pi(y))), \text{ where } x \in V(p|p)^*, \quad y \in V(n-p|n-p).$$

Let $\varepsilon_1, \ldots, \varepsilon_p$ and $\delta_1, \ldots, \delta_{n-p}$ be the standard bases of the dual spaces to the spaces of diagonal matrices in $\mathfrak{q}(p)$ and $\mathfrak{q}(n-p)$, respectively.

Theorem 8. *1)* $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{g}$.
   2) all structure functions are of order 1; they split into the direct sum of two irreducible $\mathfrak{g}_0$-*submodules with highest weights* $2\varepsilon_1 - \varepsilon_p + \delta_1 - 2\delta_{n-p}$ *and* $\varepsilon_1 - \delta_{n-p}$.

Appendix

A   Background

Linear algebra in superspaces. Generalities. A *superspace* is a $\mathbb{Z}/2$-graded space; for a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $(\Pi(V))_{\bar{i}} = V_{\bar{i}+\bar{1}}$. The *superdimension* of V is $\dim V = p + q\varepsilon$, where $\varepsilon^2 = 1$ and $p = \dim V_{\bar{0}}$, $q = \dim V_{\bar{1}}$. (Usually $\dim V$ is shorthanded as a pair (p, q) or $p|q$; with the help of ε the fact that $\dim V \otimes W = \dim V \cdot \dim W$ becomes lucid.)
   A *superalgebra* is a superspace A with an even multiplication map $m : A \otimes A \longrightarrow A$.
   A superspace structure in V induces the superspace structure in the space $\text{End}(V)$. A *basis of a superspace* always consists of *homogeneous* vectors; let $\text{Par} = (p_1, \ldots, p_{\dim V})$ be an ordered collection of their parities. We call Par the *format* of the basis of V. A square

supermatrix of format (size) Par is a $\dim V \times \dim V$ matrix whose ith row and ith column are of the same parity p_i. The matrix unit E_{ij} is supposed to be of parity $p_i + p_j$ and the bracket of supermatrices (of the same format) is defined via Sign Rule: *if something of parity p moves past something of parity q the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.*

Examples of application of Sign Rule: setting $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$ we get the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra and the Lie superalgebra (that in addition to superskew-commutativity satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule). The *superderivation* of a superalgebra A is a linear map $D : A \longrightarrow A$ such that satisfies the Leibniz rule (and Sign rule)

$$D(ab) = D(a)b + (-1)^{p(D)p(a)}aD(b).$$

Usually, Par is of the form $(\bar{0}, \ldots, \bar{0}, \bar{1}, \ldots, \bar{1})$. Such a format is called *standard*. In this paper we can do without nonstandard formats. But they are vital in various questions related with the study of distinct systems of simple roots that the reader might be interested in.

The *general linear* Lie superalgebra of all supermatrices of size Par is denoted by $\mathfrak{gl}(\mathrm{Par})$; usually, $\mathfrak{gl}(\bar{0}, \ldots, \bar{0}, \bar{1}, \ldots, \bar{1})$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar{0}}| \dim V_{\bar{1}})$. Any matrix from $\mathfrak{gl}(\mathrm{Par})$ can be expressed as the sum of its even and odd parts; in the standard format this is the block expression:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p\left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\right) = \bar{0}, \quad p\left(\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}\right) = \bar{1}.$$

The *supertrace* is the map $\mathfrak{gl}(\mathrm{Par}) \longrightarrow \mathbb{C}$, $(A_{ij}) \mapsto \sum(-1)^{p_i}A_{ii}$. Since $\mathrm{str}[x, y] = 0$, the space of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra $\mathfrak{sl}(\mathrm{Par})$.

There are, however, two super versions of $\mathfrak{gl}(n)$, not one. The other version is called the *queer* Lie superalgebra and is defined as the one that preserves the complex structure given by an *odd* operator J, i.e., is the centralizer $C(J)$ of J:

$$\mathfrak{q}(n) = C(J) = \{X \in \mathfrak{gl}(n|n) : [X, J] = 0\}, \quad \text{where} \quad J^2 = -\,\mathrm{id}\,.$$

It is clear that by a change of basis we can reduce J to the form $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1 & 0 \end{pmatrix}$. In the standard format we have

$$\mathfrak{q}(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.$$

On $\mathfrak{q}(n)$, the *queer trace* is defined: $\mathfrak{q}tr : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \mathrm{tr}\, B$. Denote by $\mathfrak{sq}(n)$ the Lie superalgebra of *queertraceless* matrices.

Observe that the identity representations of $\mathfrak{q}$ and $\mathfrak{sq}$ in V, though irreducible in super sence, are not irreducible in the nongraded sence: take homogeneous linearly independent vectors $v_1, \ldots, v_n$ from V; then $\mathrm{Span}(v_1 + J(v_1), \ldots, v_n + J(v_n))$ is an invariant subspace of V, which is not a subsuperspace, singled out by a Π-symmetry.

We will stick to the following terminology, cf. [27, 29]. The representation of a super-algebra A in the superspace V is irreducible of *general type* or just of *G-type* if it does not contain homogeneous (with respect to parity) subrepresentations distinct from 0 and V itself, otherwise it is called *irreducible of Q-type*. Thus, an irreducible representation of Q-type has no invariant sub*super*space but *has* a nontrivial invariant subspace.

So, there are two types of irreducible representations: those that do not contain any nontrivial subrepresentations (called of *general type* or of type G) and those that contain *in*homogeneous invariant subspaces (called them of *type Q*). If V is of finite dimension, then in the first case its centralizer, as of A-module, is isomorphic to $\mathfrak{gl}(1)$, whereas in the second case to $\mathfrak{q}(1)$.

Let V_1 and V_2 be finite dimensional irreducible modules over A_1 and A_2, respectively. Then $V_1 \otimes V_2$ is an irreducible $A_1 \otimes A_2$-module except for the case when both V_1 and V_2 are of type Q. In the latter case, the centralizer of the $A_1 \otimes A_2$-module $V_1 \otimes V_2$ is isomorphic to Cl_2, the Clifford superalgebra with 2 generators.

If $e \in Cl_2$ is a minimal idempotent, then $e(V_1 \otimes V_2)$ is an irreducible $A_1 \otimes A_2$-module of type G that we will denote by $V_1 \odot V_2$, see Tables 1 and 2.

More generally, we can consider matrices with the elements from a supercommutative superalgebra Λ. Then the parity of the matrix with only one nonzero i,j-th element $X_{i,j} \in \Lambda$ is equal to $p_i + p_j + p(X_{i,j})$.

The berezinian and the module of volume forms. On $GL(p|q; \Lambda)$, the group of even $p|q \times p|q$ invertible matrices with elements from a supercommutative superalgebra Λ, a multiplicative function — an analog of determinant — is defined. In honor of F Berezin Leites baptized it *berezinian*, cf. [26]. Its explicit expression in the standard format is

$$\mathrm{ber}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left(A - BD^{-1}C\right)\det D^{-1}$$

or

$$\mathrm{ber}^{-1}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\left(D - CA^{-1}B\right)\det A^{-1}.$$

The berezinian is a rational function and this is a reason why the structure of the algebra of invariant polynomials on $\mathfrak{gl}(p|q)$ is much more complicated than that for the Lie algebra $\mathfrak{gl}(n)$.

Clearly, the derivative of the berezinian is supertrace and the relation between them is as expected: $\mathrm{ber}\,X = \exp\mathrm{str}\log X$.

The one-dimensional representation $\mathrm{Vol}(V)$ of $GL(V; \Lambda)$ corresponding to ber and at the same time to the representation str of $\mathfrak{gl}(V)$ is called the space of *volume forms*. It can be only realized in the space of tensors as a quotient module: recall that for $\mathfrak{gl}(V)$ there is no complete reducibility, cf. [30].

The odd analog of berezinian. On the group $GQ(n; \Lambda)$ of invertible even matrices from $Q(n; \Lambda)$, the berezinian is identically equal to 1. Instead, on $GQ(n; \Lambda)$ there is defined its own *queer determinant*

$$\mathfrak{q}et\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \sum_{i \geq 0} \frac{1}{2i+1}\,\mathrm{tr}\left(A^{-1}B\right)^{2i+1}.$$

This strange function is $GQ(n;\Lambda)$-invariant and additive, i.e., $\mathfrak{q}et\,XY = \mathfrak{q}et\,X + \mathfrak{q}et\,Y$, cf. [3].

Superalgebras that preserve bilinear forms: two types. To the linear map $F : V \longrightarrow W$ of superspaces there corresponds the dual map $F^* : W^* \longrightarrow V^*$ of the dual superspaces; if A is the supermatrix corresponding to F in a basis of the format Par, then, in the dual basis, to F^* the *supertransposed* matrix A^{st} corresponds:

$$(A^{st})_{ij} = (-1)^{(p_i+p_j)(p_i+p(A))} A_{ji}.$$

The supermatrices $X \in \mathfrak{gl}(\text{Par})$ such that

$$X^{st} B + (-1)^{p(X)p(B)} BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\text{Par})$$

constitute the Lie superalgebra $\mathfrak{aut}(B)$ that preserves the bilinear form on V with matrix B. Most popular is the nondegenerate supersymmetric form whose matrix in the standard format is the canonical form B_{ev} or B'_{ev}:

$$B_{\text{ev}}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \text{where} \quad J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B'_{\text{ev}}(m|2n) = \begin{pmatrix} \text{antidiag}(1,\ldots,1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for $\mathfrak{aut}(B_{\text{ev}}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or $\mathfrak{osp}^{sy}(m|2n)$.

Recall that the "upsetting" map $u : \text{Bil}(V,W) \longrightarrow \text{Bil}(W,V)$ becomes for $V = W$ an involution $u : B \mapsto B^u$ which on matrices acts as follows:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \mapsto B^u = \begin{pmatrix} B_{11}^t & (-1)^{p(B)}B_{21}^t \\ (-1)^{p(B)}B_{12}^t & B_{22}^t \end{pmatrix}.$$

The forms B such that $B = B^u$ are called *supersymmetric* and *superskew-symmetric* if $B = -B^u$. The passage from V to $\Pi(V)$ identifies the space of supersymmetric forms on V with that of superskew-symmetric ones on $\Pi(V)$, preserved by the "symplectico-orthogonal" Lie superalgebra $\mathfrak{osp}^{sk}(m|2n)$ which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization. We never use notation $\mathfrak{sp}'\mathfrak{o}(2n|m)$ in order not to confuse with the special Poisson superalgebra.

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}, \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$

where $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n)$, $E \in \mathfrak{o}(m)$ and ^{t} is the usual transposition.

A nondegenerate supersymmetric odd bilinear form $B_{\text{odd}}(n|n)$ can be reduced to the canonical form whose matrix in the standard format is J_{2n}. A canonical form of the super-skew odd nondegenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\mathfrak{aut}(B_{\text{odd}}(\text{Par}))$ is $\mathfrak{pe}(\text{Par})$. The passage from V to $\Pi(V)$ sends the supersymmetric

forms to superskew-symmetric ones and establishes an isomorphism $\mathfrak{pe}^{sy}(\mathrm{Par}) \cong \mathfrak{pe}^{sk}(\mathrm{Par})$. This Lie superalgebra is called, as A Weil suggested to Leites, *periplectic*. In the standard format these superalgebras are shorthanded as in the following formula, where their matrix realizations is also given:

$$\mathfrak{pe}^{sy}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t, \ C = C^t \right\};$$

$$\mathfrak{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t, \ C = -C^t \right\}.$$

The *special periplectic* superalgebra is $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) : \mathrm{str}\, X = 0\}$.

Observe that though the Lie superalgebras $\mathfrak{osp}^{sy}(m|2n)$ and $\mathfrak{pe}^{sk}(2n|m)$, as well as $\mathfrak{pe}^{sy}(n)$ and $\mathfrak{pe}^{sk}(n)$, are isomorphic, the difference between them is sometimes crucial.

Projectivization. If $\mathfrak{s}$ is a Lie algebra of scalar matrices, and $\mathfrak{g} \subset \mathfrak{gl}(n|n)$ is a Lie subsuperalgebra containing $\mathfrak{s}$, then the *projective* Lie superalgebra of type $\mathfrak{g}$ is $\mathfrak{pg} = \mathfrak{g}/\mathfrak{s}$.

Projectivization sometimes leads to new Lie superalgebras, for example: $\mathfrak{pgl}(n|n)$, $\mathfrak{psl}(n|n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$; whereas $\mathfrak{pgl}(p|q) \cong \mathfrak{sl}(p|q)$ if $p \neq q$.

B   Certain constructions with the point functor

The point functor is well-known in algebraic geometry since at least 1953 [49]. The advertising of ringed spaces with nilpotents in the structure sheaf that followed the discovery of supersymmetries caused many mathematicians and physicists to realize the usefulness of the language of points; most interesting are numerous ideas due to Witten (for some of them see [50]), for their clarifications, and further developments and references see [37, 7]. F A Berezin [2] was the first who applied the point functor to study Lie superalgebras.

All superalgebras and modules are supposed to be finite dimensional over $\mathbb{C}$.

What a Lie superalgebra is. Lie superalgebras had appeared in topology in 1930's or earlier. So when somebody offers a "better than usual" definition of a notion which seemed to have been established about 70 year ago this might look strange, to say the least. Nevertheless, the answer to the question "what is a Lie superalgebra?" is still not a common knowledge. Indeed, the naive definition ("apply the Sign Rule to the definition of the Lie algebra") is manifestly inadequate for considering the (singular) supervarieties of deformations and applying representation theory to mathematical physics, for example, in the study of the coadjoint representation of the Lie supergroup which can act on a supermanifold but never on a superspace (an object from another category). So, to deform Lie superalgebras and apply group-theoretical methods in "super" setting, we must be able to recover a supermanifold from a superspace, and vice versa.

A proper definition of Lie superalgebras is as follows, cf. [27, 7]. The *Lie superalgebra* in the category of supermanifolds corresponding to the "naive" Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ is a linear supermanifold $\mathcal{L} = (L_{\bar{0}}, \mathcal{O})$, where the sheaf of functions $\mathcal{O}$ consists of functions on $L_{\bar{0}}$ with values in the Grassmann superalgebra on $L_{\bar{1}}^*$; this supermanifold should be such that for "any" (say, finitely generated, or from some other appropriate category) supercommutative superalgebra C, the space $\mathcal{L}(C) = \mathrm{Hom}(\mathrm{Spec}\, C, \mathcal{L})$, called *the space of C-points of $\mathcal{L}$*, is a Lie algebra and the correspondence $C \longrightarrow \mathcal{L}(C)$ is a functor in C. (In super setting Weil's approach called *the language of points* or was rediscovered in [27] as

families, see also [37, 7].) This definition might look terribly complicated, but fortunately one can show that the correspondence $\mathcal{L} \longleftrightarrow L$ is one-to-one and the Lie algebra $\mathcal{L}(C)$, also denoted $L(C)$, admits a very simple description: $L(C) = (L \otimes C)_{\bar{0}}$.

A *Lie superalgebra homomorphism* $\rho : L_1 \longrightarrow L_2$ in these terms is a functor morphism, i.e., a collection of Lie algebra homomorphisms $\rho_C : L_1(C) \longrightarrow L_2(C)$ compatible with morphisms of supercommutative superalgebras $C \longrightarrow C'$. In particular, a *representation* of a Lie superalgebra L in a superspace V is a homomorphism $\rho : L \longrightarrow \mathfrak{gl}(V)$, i.e., a collection of Lie algebra homomorphisms $\rho_C : L(C) \longrightarrow (\mathfrak{gl}(V) \otimes C)_{\bar{0}}$.

Example. Consider a representation $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$. The tangent space of the moduli superspace of deformations of ρ is isomorphic to $H^1(\mathfrak{g}; V \otimes V^*)$. For example, if $\mathfrak{g}$ is the $0|n$-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the trivial one, $\mathbf{1}$, and $\Pi(\mathbf{1})$. Clearly, $\mathbf{1} \otimes \mathbf{1}^* \simeq \Pi(\mathbf{1}) \otimes \Pi(\mathbf{1})^* \simeq \mathbf{1}$, and because the superalgebra is commutative, the differential in the cochain complex is trivial. Therefore, $H^1(\mathfrak{g}; \mathbf{1}) = \Lambda^1(\mathfrak{g}^*) \simeq \mathfrak{g}^*$, so there are $\dim \mathfrak{g}$ odd parameters of deformations of the trivial representation. If we consider $\mathfrak{g}$ "naively" all of the odd parameters will be lost.

Which of these infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved *ad hoc*.

Thus, $\mathfrak{q}tr$ is not a representation of $\mathfrak{q}(n)$ according to the naive definition ("a representation is a Lie superalgebra homomorphism", hence, an even map), but is a representation, moreover, an irreducible one, if we consider odd parameters.

Acknowledgements

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