Journal of Nonlinear Mathematical Physics Volume 9, Number 4 (2002), 475–482 Article

On the Lie Symmetries of Kepler–Ermakov Systems

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Received February 12, 2002; Revised May 4, 2002; Accepted May, 2002

Abstract

In this work, we study the Lie-point symmetries of Kepler–Ermakov systems presented by C Athorne in *J. Phys.* **A24** (1991), L1385–L1389. We determine the forms of arbitrary function $H(x, y)$ in order to find the members of this class possessing the $\mathbf{sl}(2, \mathbb{R})$ symmetry and a Lagrangian. We show that these systems are usual Ermakov systems with the frequency function depending on the dynamical variables.

1 Introduction

The basic idea behind the symmetry methods is to reduce the order of differential equations under considerations as much as possible so that the integration can be done easily [11]. It is well known that a dynamical system is integrable if a sufficient number of sufficiently simple (e.g. polynomial) independent first integrals can be found. The knowledge of a first integral amounts to the reduction of the order of the integration procedure by one. For systems of N second order differential equations this order has to be counted as $2N$. Ermakov systems are time-dependent dynamical systems [9]. They contain one arbitrary function of time, the so-called frequency function, and two arbitrary homogeneous functions of dynamical variables. A trend in the latest developments on the subject is to focus attention on some special features of subclasses of generalized Ermakov systems in which the frequency function may depend on time, the dynamical variables and their derivatives [5, 6, 8]. These subclasses may be tailored to suit some particular applications or special purposes. A central feature of Ermakov systems is their property of always having a first integral [4]. This invariant plays a central role in the linearization of Ermakov systems [2, 6, 10]. A class of dynamical systems was presented by Athorne [1] which can be regarded as perturbations of the classical Kepler problem or of an autonomous Ermakov system. Kepler–Ermakov systems generalize the usual Ermakov systems while preserving the property of being amenable to linearization [1, 3, 6]. In the next section, we study the Lie point symmetries of Kepler–Ermakov systems. We find the Kepler–Ermakov–Lagrangian systems possessing the $\mathbf{sl}(2, \mathbb{R})$ symmetry.

2 Symmetry generators

Systems of second order differential equations often appear in classical mechanics. So we use the notations common in that field [11]. We consider the generalized coordinates q^a as dependent variables, t as the independent variable, and denote dq^a/dt by $\dot{q}^a$. Then,

$$\ddot{q}^a = w^a\left(q^i, \dot{q}^i, t\right), \qquad a, i = 1, \ldots, N, \tag{1}$$

is the general form for a system of second order differential equations. The corresponding linear partial differential equation of first order is

$$\mathbf{A}f = \left(\frac{\partial}{\partial t} + \dot{q}^a \frac{\partial}{\partial q^a} + w^a\left(q^i, \dot{q}^i, t\right)\frac{\partial}{\partial \dot{q}^a}\right) f = 0, \tag{2}$$

which admits $2N$ functionally independent solutions $\varphi^\alpha = \varphi^\alpha\left(q^a, \dot{q}^a, t\right)$ that are first integrals of the system (1). So every solution of (1) can locally be written as $q^a = q^a(\varphi^\alpha, t)$, where $d\varphi^\alpha/dt = 0$.

When the operator $\mathbf{A}$ is written in the form of (2), we can write the infinitesimal generator of the Lie point symmetry admitted by (1) as

$$\mathbf{X} = \xi\left(q^i, t\right)\frac{\partial}{\partial t} + \eta^a\left(q^i, t\right)\frac{\partial}{\partial q^a}. \tag{3}$$

Its extension is

$$\dot{\mathbf{X}} = \xi\frac{\partial}{\partial t} + \eta^a\frac{\partial}{\partial q^a} + \left(\frac{d\eta^a}{dt} - \dot{q}^a\frac{d\xi}{dt}\right)\frac{\partial}{\partial \dot{q}^a}, \tag{4}$$

and the symmetry condition is

$$[\dot{\mathbf{X}}, \mathbf{A}] = \lambda\mathbf{A}, \tag{5}$$

where λ is a non-constant factor, in general. By writing (5) explicitly and matching the terms on both sides, we obtain the following equations

$$-\mathbf{A}\xi = -\frac{d\xi}{dt} = \lambda, \tag{6}$$

$$\dot{\mathbf{X}}w^a = \mathbf{A}\left(\frac{d\eta^a}{dt} - \dot{q}^a\frac{d\xi}{dt}\right) - w^a\frac{d\xi}{dt}. \tag{7}$$

If (7) is written in full, it is

$$\begin{aligned}\xi w^a_t + \eta^b w^a_b + (\eta^b_t + \dot{q}^c\eta^b_c - \dot{q}^b\xi_t - \dot{q}^b\dot{q}^c\xi_c)\frac{\partial w^a}{\partial \dot{q}^b} + 2w^a(\xi_t + \dot{q}^b\xi_b) + w^b(\dot{q}^a\xi_b - \eta^a_b)\\ + \dot{q}^a\dot{q}^b\dot{q}^c\xi_{bc} + 2\dot{q}^a\dot{q}^c\xi_{tc} - \dot{q}^c\dot{q}^b\eta^a_{bc} + \dot{q}^a\xi_{tt} - 2\dot{q}^b\eta^a_{tb} - \eta^a_{tt} = 0,\end{aligned} \tag{8}$$

where the subscripts $b, c = 1, \ldots, N$, and t denote partial differentiations with respect to generalized coordinates and time.

Kepler–Ermakov systems are defined [1] as the system of equations

$$\begin{aligned}\ddot{x} + w^2(t)x &= -\frac{x}{r^3}H + \frac{1}{x^3}f(y/x),\\ \ddot{y} + w^2(t)y &= -\frac{y}{r^3}H + \frac{1}{y^3}g(y/x),\end{aligned} \tag{9}$$

where H is a function of x, y and $r = \left(x^2+y^2\right)^{1/2}$. Here f and g are arbitrary functions of the indicated arguments and w is an arbitrary function of time. If H, f and g are zero, we have a time-dependent harmonic oscillator which occurs in many practical contexts. In the case that H is taken to be zero we have generalized Ermakov systems.

In order to calculate the Lie point symmetries of system (9) we consider the generator of the group of point transformations

$$\mathbf{X} = \xi(x,y,t)\frac{\partial}{\partial t} + \eta_1(x,y,t)\frac{\partial}{\partial x} + \eta_2(x,y,t)\frac{\partial}{\partial y}. \tag{10}$$

By inserting (9) into the symmetry condition (8) we obtain a polynomial equation in $\dot{x}$ and $\dot{y}$. The coefficients of all monomials of the form $\dot{x}^m\dot{y}^n$ must be identically zero. This yields the following system of partial differential equations satisfied by ξ, η_1 and η_2,

$$\begin{gathered}\xi_{xx} = \xi_{yy} = \xi_{xy} = 0, \qquad \eta_{1xx} - 2\xi_{tx} = 0, \qquad \eta_{1xy} - \xi_{ty} = 0,\\ \eta_{2xy} - \xi_{tx} = 0, \qquad \eta_{2yy} - 2\xi_{ty} = 0, \qquad \eta_{1yy} = \eta_{2xx} = 0,\end{gathered} \tag{11}$$

where we use subscripts to denote partial derivatives. The solutions of equations (11) can be written as

$$\begin{gathered}\xi = \kappa(t)x + \delta(t)y + \sigma(t), \qquad \eta_1 = \alpha_1(x,t)y + \beta_1(x,t),\\ \eta_2 = \alpha_2(y,t)x + \beta_2(y,t),\end{gathered} \tag{12}$$

where the functions κ, δ, σ, α_1, β_1, α_2 and β_2 must satisfy the conditions

$$\begin{gathered}\alpha_{1xx} = \alpha_{2yy} = 0, \qquad \beta_{1xx} - 2\dot{\kappa} = 0, \qquad \beta_{2yy} - 2\dot{\delta} = 0,\\ \alpha_{1x} - \dot{\delta} = 0, \qquad \alpha_{2y} - \dot{\kappa} = 0.\end{gathered} \tag{13}$$

The solutions of these equations are

$$\begin{gathered}\alpha_1(x,t) = \dot{\delta}x + \phi_1(t), \qquad \alpha_2(y,t) = \dot{\kappa}y + \phi_2(t),\\ \beta_1(x,t) = \dot{\kappa}x^2 + \phi_3(t)x + \phi_5(t), \qquad \beta_2(y,t) = \dot{\delta}y^2 + \phi_4(t)y + \phi_6(t).\end{gathered} \tag{14}$$

Substituting (12) and (14) into the symmetry condition (8) and setting the coefficients of the terms linear in $\dot{x}$ and $\dot{y}$ to zero we obtain

$$\phi_3(t) = (\dot{\sigma} - c_1)/2, \qquad \phi_4(t) = (\dot{\sigma} - c_2)/2, \qquad \kappa = \delta = 0. \tag{15}$$

We note that, up to here, f, g, H and w are arbitrary functions of their arguments. Inserting (15) into the rest of (8), we find that

$$c_1 = c_2 = 0, \qquad \phi_1 = \phi_2 = \phi_5 = \phi_6 = 0, \tag{16}$$

$$(xH_x + yH_y + 2H)\frac{\dot{\sigma}}{r^3} + \dddot{\sigma} + 4\dot{w}w\sigma + 4\dot{\sigma}w^2 = 0. \tag{17}$$

The last equation implies that the function $H(x,y)$ is not completely arbitrary but must be a function on the integral surface of the partial differential equation

$$xH_x + yH_y + 2H = C\left(x^2+y^2\right)^{3/2} \tag{18}$$

where C is a constant. As a special case, for example, if C is zero, H must be in the form

$$H = -\frac{h(x/y)}{y^2}, \tag{19}$$

where h is an arbitrary function of its argument and the function $\sigma(t)$ must be the solution of

$$\dddot{\sigma} + 4\dot{w}w\sigma + 4\dot{\sigma}w^2 = 0 \tag{20}$$

which is the third order form of Ermakov–Pinney equation. Then equations (9) have the symmetry

$$\mathbf{X} = \sigma(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{\sigma}(t)x\frac{\partial}{\partial x} + \frac{1}{2}\dot{\sigma}(t)y\frac{\partial}{\partial y} \tag{21}$$

which is also a symmetry of

$$x\ddot{y} - y\ddot{x} - \frac{x}{y^3}g(y/x) + \frac{y}{x^3}f(y/x) = 0 \tag{22}$$

obtained [4, 8] by eliminating $w(t)$ and H terms in (9). If $w \neq 0$, equation (20) can be integrated once to obtain, $\sigma\ddot{\sigma} - \frac{1}{2}\dot{\sigma}^2 + 2\sigma^2 w^2 = c_1$ which can be reduced to the Pinney equation, $\ddot{\rho} + w^2\rho = \frac{c_2}{\rho^3}$, by the change of variable $\sigma = \rho^2(t)$. On the other hand, equation (20) can be solved easily if $w = 0$. Hence the system (9) with $w = 0$ has the symmetry

$$\mathbf{X} = \left(c_1 t^2 + c_2 t + c_3\right)\frac{\partial}{\partial t} + \left(c_1 tx + \frac{c_2}{2}x\right)\frac{\partial}{\partial x} + \left(c_1 ty + \frac{c_2}{2}y\right)\frac{\partial}{\partial y} \tag{23}$$

which splits into three components

$$G_1 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}, \qquad G_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}, \qquad G_3 = \frac{\partial}{\partial t} \tag{24}$$

which represent conformal and self-similar transformations and time translation. These generators satisfy the commutation relations

$$[G_3, G_2] = G_3, \qquad [G_3, G_1] = 2G_2, \qquad [G_2, G_1] = G_1. \tag{25}$$

By redefining the generators as $G_3 = \sigma_1$, $G_2 = \sigma_2/2$, $G_1 = \sigma_3$, where

$$[\sigma_1, \sigma_2] = 2\sigma_1, \qquad [\sigma_1, \sigma_3] = \sigma_2, \qquad [\sigma_2, \sigma_3] = 2\sigma_3,$$

we see that the symmetry algebra is $\mathbf{sl}(2,\mathbb{R})$. Actually, it is the characteristic algebra of generalized Ermakov systems [5]. We have to note that equation (22) can be integrated and the result is

$$\frac{1}{2}(\dot{x}y - x\dot{y})^2 + \int^{y/x}\left[uf(u) - u^{-3}g(u)\right]du = \text{const} \tag{26}$$

which is known as the Ermakov–Lewis invariant and (23) is also a symmetry of the Ermakov–Lewis invariant [4].

If C is a nonzero constant, the function $H(x, y)$ must be the solution of equation (18), that is

$$H = -\frac{h(x/y)}{y^2} + \frac{C}{5}\left(x^2 + y^2\right)^{3/2}, \tag{27}$$

and the function $\sigma(t)$ must be the solution of

$$\ddot{\sigma} + \frac{4}{5} C\sigma = \zeta, \tag{28}$$

where $w = 0$ and ζ is a constant. This equation can be integrated easily and the result is

$$\sigma(t) = c_1 e^{\beta t} + c_2 e^{-\beta t} - \frac{\zeta}{\beta^2}, \tag{29}$$

where $\beta = 2i\sqrt{\frac{C}{5}}$ and c_1, c_2 are constants of integration. Substituting (29) into (21) we obtain the symmetries

$$G_1 = e^{\beta t}\frac{\partial}{\partial t} + \frac{\beta}{2} e^{\beta t}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right),$$
$$G_2 = e^{-\beta t}\frac{\partial}{\partial t} - \frac{\beta}{2} e^{-\beta t}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right), \qquad G_3 = -\frac{1}{\beta^2}\frac{\partial}{\partial t} \tag{30}$$

with commutation relations

$$[G_1, G_3] = \frac{1}{\beta} G_1, \qquad [G_1, G_2] = \frac{5}{2}\beta^3 G_3, \qquad [G_3, G_2] = \frac{1}{\beta} G_2. \tag{31}$$

The Lie algebra is again $\mathbf{sl}(2, \mathbb{R})$ which can be shown easily by choosing $G_1 = 2\sigma_1/\sqrt{5\beta}$, $G_2 = 2\sigma_3/\sqrt{5\beta}$, $G_3 = \sigma_2/2\beta$. We conclude that Kepler–Ermakov systems have the Lie algebra $\mathbf{sl}(2, \mathbb{R})$ if the function H is in the form of (27).

By using (27) in (9) we have the equations

$$\ddot{x} = -x\left[\frac{C}{5} - \frac{h(x/y)}{y^2\left(x^2 + y^2\right)^{3/2}}\right] + \frac{f(y/x)}{x^3},$$
$$\ddot{y} = -y\left[\frac{C}{5} - \frac{h(x/y)}{y^2\left(x^2 + y^2\right)^{3/2}}\right] + \frac{g(y/x)}{y^3} \tag{32}$$

possessing the $\mathbf{sl}(2, \mathbb{R})$ symmetry. These equations can be considered as the equations of motions of a particle with unit mass and can be obtained from a Lagrangian function,

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{C}{10}\left(x^2 + y^2\right) - \frac{1}{2}\left[\frac{f(y/x)}{x^2} + \frac{g(y/x)}{y^2}\right] - \Psi(x, y), \tag{33}$$

if

$$y^2 f'(y/x) + x^2 g'(y/x) = 0, \tag{34}$$

where the prime denotes the derivative with respect to the argument and

$$\frac{\partial\Psi}{\partial x} = -\frac{xh(x/y)}{y^2\,(x^2+y^2)^{3/2}}, \qquad \frac{\partial\Psi}{\partial y} = -\frac{yh(x/y)}{y^2\,(x^2+y^2)^{3/2}}, \tag{35}$$

so that the force is derivable from the potential function. The function $\Psi(x, y)$ is integrable if

$$h(x/y) = \frac{C_0 y^2}{x^2+y^2}, \tag{36}$$

where C_0 is an arbitrary constant. As a result we have the equations of motion

$$\begin{aligned} \ddot{x} &= -x\left[\frac{C}{5} - \frac{C_0}{(x^2+y^2)^{5/2}}\right] + \frac{f(y/x)}{x^3}, \\ \ddot{y} &= -y\left[\frac{C}{5} - \frac{C_0}{(x^2+y^2)^{5/2}}\right] + \frac{g(y/x)}{y^3}, \end{aligned} \tag{37}$$

for Kepler–Ermakov systems possessing the $\mathbf{sl}(2,\mathbb{R})$ symmetry that are obtained from the Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right) - \frac{C}{10}\left(x^2+y^2\right) - \frac{C_0}{3}\left(x^2+y^2\right)^{-3/2} - \frac{1}{2}\left[\frac{f(y/x)}{x^2} + \frac{g(y/x)}{y^2}\right] \tag{38}$$

with the condition (34).

In polar coordinates the Lagrangian and equations of motion can be written as

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{C}{10}r^2 - \frac{C_0}{3r^3} - \frac{G(\theta)}{2r^2}, \tag{39}$$

where $G(\theta) = \sec^2\theta f(\tan\theta) + \csc^2\theta g(\tan\theta)$, and

$$\ddot{r} - r\dot{\theta}^2 - \frac{G(\theta)}{r^3} + \frac{C}{5}r - \frac{C_0}{r^4} = 0, \tag{40}$$

$$\frac{d(r^2\dot{\theta})}{dt} + \frac{G'(\theta)}{r^2} = 0. \tag{41}$$

The Ermakov–Lewis invariant (26) comes from the integration of (41) and is

$$I = \frac{1}{2}(\dot{\theta}r^2)^2 + \int^{\tan\theta}\left[uf(u) - u^{-3}g(u)\right]du. \tag{42}$$

Very recently [7], it was shown that the Ermakov–Lewis invariant is the result of a Noether dynamical symmetry for a class of Lagrangian–Ermakov systems. A Noether symmetry is a Lie point transformation that leaves the action functional invariant up to an additive constant [11]. It is also a Lie symmetry of the corresponding Euler–Lagrange equations. If

$$\mathbf{X} = \xi\frac{\partial}{\partial t} + \eta^a\frac{\partial}{\partial q^a} + (\dot{\eta}^a - \dot{q}^a\dot{\xi})\frac{\partial}{\partial \dot{q}^a}, \tag{43}$$

is the generator of a Noether symmetry, then

$$\varphi = \xi \left[\dot{q}^k \frac{\partial L}{\partial \dot{q}^k} - L \right] - \eta^k \frac{\partial L}{\partial \dot{q}^k} + \Lambda(q^i, t) \tag{44}$$

is a first integral that satisfies $\mathbf{X}\varphi = 0$. For the Lagrangian in (38) a Noether symmetry generator is G_3 and the first integral determined by (44) is the Hamiltonian.

Finally we calculate the generator of Ermakov invariant for the Lagrangian (38). In general, if the Lagrangian L and a first integral φ of a dynamical system are known one can determine the corresponding symmetry of dynamical character [11]. If

$$\mathbf{X} = \xi(q^k, \dot{q}^k, t)\frac{\partial}{\partial t} + \eta^a(q^k, \dot{q}^k, t)\frac{\partial}{\partial q^a} + \dot{\eta}^a(q^k, \dot{q}^k, t)\frac{\partial}{\partial \dot{q}^a}, \tag{45}$$

is the generator of a dynamical symmetry, then there exist a first integral φ such that

$$\frac{\partial^2 L}{\partial \dot{q}^a \partial \dot{q}^b}(\eta^a - \dot{q}^a \xi) = -\frac{\partial \varphi}{\partial \dot{q}^b} \tag{46}$$

holds. Conversely, if φ is a first integral, then $\eta^a - \dot{q}^a\xi$ determines a dynamical symmetry which is called a Cartan symmetry [11]. We can now use (46) to determine the dynamical symmetry corresponding to the Ermakov invariant (26) and then have

$$\begin{aligned} \mathbf{X} = (x\dot{y} - y\dot{x})\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \left[\dot{y}(x\dot{y} - y\dot{x}) + \frac{x}{y^2}g(y/x) - \frac{y^2}{x^3}f(y/x)\right]\frac{\partial}{\partial \dot{x}} \\ - \left[\dot{x}(x\dot{y} - y\dot{x}) + \frac{x^2}{y^3}g(y/x) - \frac{y}{x^2}f(y/x)\right]\frac{\partial}{\partial \dot{y}}, \end{aligned} \tag{47}$$

which is not a point symmetry [11] because it is not possible to add it a multiple of $\mathbf{A}$ and thereby to get rid of all derivatives in the coefficients of ∂_x and ∂_y.

3 Conclusion

In this work we concentrated on the Kepler–Ermakov systems, with $w(t) = 0$, possessing the $\mathbf{sl}(2, \mathbb{R})$ symmetry. We found that these systems are usual Ermakov systems with the frequency function depending on the dynamical variables. We obtained the Lagrangian for these systems and calculated the Cartan symmetry corresponding to the Ermakov invariant.

Acknowledgements

We would like to thank S Yu Sakovich for his careful reading and suggestions on the manuscript. We also gratefully acknowledge the editor, P G L Leach, and the referee for valuable comments and suggestions. This work is supported in part by the Scientific and Technical Research Council of Turkey (TUBITAK).

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