

Cohomology of Groups of Diffeomorphisms Related to the Modules of Differential Operators on a Smooth Manifold

Sofiane BOUARROUDJ

Department of Mathematics, Keio University, Faculty of Science & Technology
3-14-1, Hiyoshi, Kohoku-ku, Yokohama, 223-8522, Japan
E-mail: sofbou@math.keio.ac.jp

Received December 21, 2001; Revised April 11, 2002; Accepted April 17, 2002

Abstract

Let M be a manifold and T^*M be the cotangent bundle. We introduce a 1-cocycle on the group of diffeomorphisms of M with values in the space of linear differential operators acting on $C^\infty(T^*M)$. When M is the n -dimensional sphere, S^n , we use this 1-cocycle to compute the first-cohomology group of the group of diffeomorphisms of S^n , with coefficients in the space of linear differential operators acting on contravariant tensor fields.

1 Introduction

The origin of our investigation is the relationship between the space of linear differential operators acting on densities and the corresponding space of symbols, both viewed as modules over the group of diffeomorphisms and the Lie algebra of smooth vector fields. These two spaces are obviously isomorphic as vector spaces; however, these spaces are not isomorphic as modules (cf. [3, 6, 7, 9, 10]). More precisely, the module of linear differential operators is a non-trivial deformation of the module of symbols in the sense of Neijinhuis–Richardson’s theory of deformation (see [11]). This theory of deformation can be summarized briefly as follows. Let a Lie algebra \mathfrak{g} and a \mathfrak{g} -module V be given. Deformation of the \mathfrak{g} -module V means we extend the action of the Lie algebra \mathfrak{g} on the formal power series $V[[t]]$, where t is a parameter. The problem of deformation is related to the cohomology groups $H^1(\mathfrak{g}; \text{End}(V))$ and $H^2(\mathfrak{g}; \text{End}(V))$. The first cohomology group classifies all *infinitesimal* deformation of the module V , while the second cohomology group measures the obstruction to extend this infinitesimal deformation to a formal deformation.

In this paper, we deal with the space of symbols \mathcal{S} ; they are functions on the cotangent bundle which are polynomial on the fibers. This space is naturally a module over the Lie algebra of smooth vector fields $\text{Vect}(M)$ and the group of diffeomorphisms $\text{Diff}(M)$. Since the module of linear differential operators is a deformation of the space of symbols, it is interesting to exhibit 1-cocycles generating the infinitesimal deformation. For the Lie algebra $\text{Vect}(M)$, Lecomte and Ovsienko [10] have computed the cohomology group

$H^1(\text{Vect}(M); \mathcal{D}(\mathcal{S}))$, where $\mathcal{D}(\mathcal{S})$ is the space of linear differential operators acting on \mathcal{S} . In this paper we will study the counterpart cohomology of the group of diffeomorphisms

$$H^1(\text{Diff}(M); \mathcal{D}(\mathcal{S})). \quad (1.1)$$

First, we introduce a non-trivial 1-cocycle on the group of diffeomorphisms $\text{Diff}(M)$ with values in $\mathcal{D}(C^\infty(T^*M))$. The construction of this 1-cocycle was inspired from the Lecomte–Ovsienko’s result [10] where they show that one of the generators of the cohomology group $H^1(\text{Vect}(M); \mathcal{D}(\mathcal{S}))$ is given by the so-called Vey cocycle (see, e.g., [13])—written with a (lifted) connection on the cotangent bundle T^*M . We will explain in the Section 4 the relation between our 1-cocycle and the Vey cocycle. At last, we use this 1-cocycle to compute the cohomology group (1.1) for the case when the manifold is the n -dimensional sphere.

2 Introducing a 1-cocycle on $\text{Diff}(M)$

Let M be a manifold of dimension n . Fix a symmetric affine connection on it. Let us recall here the natural way to lift a connection to the cotangent bundle T^*M (see [17] for more details).

2.1 Lift of a connection

Denote by Γ_{ij}^k , for $i, j, k = 1, \dots, n$ the Christoffel symbols of the connection on M . There exists a symmetric affine connection on T^*M whose Christoffel symbols $\tilde{\Gamma}_{\mathbf{i}\mathbf{j}}^{\mathbf{k}}$, where $\mathbf{i}, \mathbf{j}, \mathbf{k} = 1, \dots, 2n$, are as follows: denote the superscript \mathbf{i} by i if it varies from 1 to n , and by \bar{i} if it varies from $n+1$ to $2n$. In local coordinates (x^i, ξ_i) on T^*M , the Christoffel symbols of the lifted connection are as follows:

$$\begin{aligned} \tilde{\Gamma}_{ij}^k &= \Gamma_{ij}^k, & \tilde{\Gamma}_{i\bar{j}}^k &= 0, & \tilde{\Gamma}_{\bar{i}\bar{j}}^k &= 0, & \tilde{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} &= 0, \\ \tilde{\Gamma}_{ij}^{\bar{k}} &= \xi_a (\partial_k \Gamma_{ij}^a - \partial_i \Gamma_{jk}^a - \partial_j \Gamma_{ik}^a + 2\Gamma_{kt}^a \Gamma_{ij}^t), \\ \tilde{\Gamma}_{i\bar{j}}^{\bar{k}} &= -\Gamma_{ik}^j, & \tilde{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} &= -\Gamma_{kj}^i, & \Gamma_{\bar{i}\bar{j}}^{\bar{k}} &= 0. \end{aligned} \quad (2.1)$$

Denote by $\tilde{\nabla}$ the covariant derivative associated with the connection $\tilde{\Gamma}$.

Remark 1. The connection $\tilde{\Gamma}$ is actually the Levi–Civita connection of some metric on T^*M (see [17]).

2.2 Main definition

First we shall construct canonically a tensor field on T^*M .

It is well known that the difference between two connections is a well-defined tensor field of type $(2, 1)$. It follows from this fact that the following expression

$$C(F) := F^*(\tilde{\Gamma}) - \tilde{\Gamma}, \quad (2.2)$$

where F is a lift to T^*M of a diffeomorphism $f \in \text{Diff}(M)$ and $\tilde{\Gamma}$ is the connection above, is a well-defined tensor field on T^*M of type $(2, 1)$. It is easy to see that the map

$$F \mapsto C(F^{-1})$$

defines a non-trivial 1-cocycle on the group of diffeomorphisms of T^*M with values in the space of tensor fields of type $(2, 1)$.

Denote by g^{ij} the standard bivector, dual to the standard symplectic structure on T^*M .

For all $f \in \text{Diff}(M)$, denote by \tilde{f} its symplectic lift to T^*M .

The main tool of this paper is the third-order linear differential operator $\mathcal{L}(f)$ on $C^\infty(T^*M)$ defined by¹

$$\begin{aligned} \mathcal{L}(f) := & \text{Sym}_{j,i,k} \left(C_{ml}^j(\tilde{f}) \cdot g^{im} \cdot g^{kl} \right) \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\nabla}_k \\ & - \frac{3}{2} \text{Sym}_{n,m,i} \left(C_{lk}^n(\tilde{f}) \cdot g^{ml} \cdot g^{ik} \right) \cdot C_{mn}^j(\tilde{f}) \tilde{\nabla}_i \tilde{\nabla}_j, \end{aligned} \quad (2.3)$$

where Sym denotes symmetrization.

Recall that the space $C^\infty(T^*M)$ is naturally a module over the group $\text{Diff}(M)$. The action is as follows: take $f \in \text{Diff}(M)$ and $Q \in C^\infty(T^*M)$, then

$$f^* \cdot Q := Q \circ \tilde{f}^{-1}. \quad (2.4)$$

By differentiating the action above, one obtains the action of the Lie algebra $\text{Vect}(M)$.

The action (2.4) induces an action of the group $\text{Diff}(M)$ on $\mathcal{D}(C^\infty(T^*M))$ as follows: take $f \in \text{Diff}(M)$ and $T \in \mathcal{D}(C^\infty(T^*M))$, then

$$f^*T := f^* \circ T \circ f^{-1*}.$$

In this paper we shall study the cohomology arising in this context.

We have the following Theorem

Theorem 1. *The map*

$$f \mapsto \mathcal{L}(f^{-1}),$$

*defines a non-trivial 1-cocycle on $\text{Diff}(M)$ with values in $\mathcal{D}^3(C^\infty(T^*M))$.*

Proof. To prove that the operator (2.3) is 1-cocycle one has to verify the 1-cocycle condition

$$\mathcal{L}(f \circ h) = \tilde{h}^* \mathcal{L}(f) + \mathcal{L}(h), \quad (2.5)$$

for all $f, h \in \text{Diff}(M)$.

To verify (2.5) we use the formulae

$$\begin{aligned} & \text{Sym}_{i,m,n} \left(C_{kl}^i(\tilde{f}) \cdot g^{mk} \cdot g^{nl} \right) \cdot C_{mn}^j(\tilde{f}) \tilde{\nabla}_i \tilde{\nabla}_j (\tilde{h}^{-1*} Q) \\ & = \text{Sym}_{i,m,n} \left(C_{kl}^i(\tilde{f}) \cdot g^{mk} \cdot g^{nl} \right) \cdot C_{mn}^j(\tilde{f}) \tilde{h}^{-1*} \tilde{\nabla}_i \tilde{\nabla}_j Q, \\ & \text{Sym}_{j,i,k} \left(C_{ml}^j(\tilde{f}) \cdot g^{im} \cdot g^{kl} \right) \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\nabla}_k (\tilde{h}^{-1*} Q) \\ & = \text{Sym}_{j,i,k} \left(C_{ml}^j(\tilde{f}) \cdot g^{im} \cdot g^{kl} \right) \left(3C_{ij}^t(\tilde{h}^{-1}) \tilde{h}^{-1*} \tilde{\nabla}_t \tilde{\nabla}_k Q + \tilde{h}^{-1*} \tilde{\nabla}_i \tilde{\nabla}_j \tilde{\nabla}_k Q \right), \end{aligned} \quad (2.6)$$

for all $f, h \in \text{Diff}(M)$ and for all $Q \in C^\infty(T^*M)$.

¹Here and below we use the convention of summation over repeated indices.

Let us prove that this 1-cocycle is not trivial. Suppose that there exists an operator A of degree three such that

$$\mathcal{L}(f) = \tilde{f}^* A - A.$$

The principal symbol of the operator A transforms under coordinates change as a contravariant tensor field of degree three. Thus, it depends only on the first jet of the symplectomorphism \tilde{f} , whereas the principal symbol of the operator (2.3) depends on the second jet of the symplectomorphism f , a contradiction. \blacksquare

Remark 2. The fact that the first-order coefficients in the formulae (2.6) are trivial is due to particular properties of the connection (2.1).

2.3 Expression in local coordinates

Denote by $(x^i, \xi_i), i = 1, \dots, n$, the local coordinates on T^*M . In these coordinates the components of the symplectic form g are

$$g_{ij} = g_{\bar{i}\bar{j}} = 0, \quad g_{i\bar{j}} = \delta_{ij}, \quad g_{\bar{i}j} = -g_{\bar{j}i},$$

where $i, j = 1, \dots, n$.

For any diffeomorphism $f(x) = (f^1(x), \dots, f^n(x))$ on M , its symplectic lift to T^*M is written as $\tilde{f}(x, \xi) = \left(f^1(x), \dots, f^n(x), \frac{\partial x^i}{\partial f^1} \xi_i, \dots, \frac{\partial x^i}{\partial f^n} \xi_i \right)$.

An easy computation shows that the operator (2.3) (to some factor) has the form

$$\begin{aligned} & 3C_{jk}^i(\tilde{f}) \frac{\partial^2}{\partial \xi^j \partial \xi^k} \frac{\partial}{\partial x^i} + 3 \left(2C_{ik}^m(\tilde{f}) \Gamma_{jm}^k + C_{ki}^m(\tilde{f}) C_{mj}^k(\tilde{f}) \right) \frac{\partial^2}{\partial \xi^i \partial \xi^j} \\ & + C_{jk}^{\bar{i}}(\tilde{f}) \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k}, \end{aligned} \quad (2.7)$$

where $C(\tilde{f})$ is the tensor (2.2).

In the particular case when the connection Γ is Euclidean (i.e. $\Gamma \equiv 0$), the operator above takes the form

$$\begin{aligned} & 3 \frac{\partial^2 f^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial f^l} \frac{\partial^2}{\partial \xi^j \partial \xi^k} \frac{\partial}{\partial x^i} + 3 \frac{\partial^2 f^k}{\partial x^q \partial x^i} \frac{\partial x^m}{\partial f^k} \frac{\partial^2 f^l}{\partial x^j \partial x^m} \frac{\partial x^q}{\partial f^l} \frac{\partial^2}{\partial \xi^i \partial \xi^j} \\ & + \frac{\partial x^m}{\partial f^q} \xi_m \left(3 \frac{\partial x^l}{\partial f^p} \frac{\partial^2 f^p}{\partial x^i \partial x^j} \frac{\partial^2 f^q}{\partial x^l \partial x^k} - \frac{\partial^3 f^q}{\partial x^i \partial x^j \partial x^k} \right) \frac{\partial^3}{\partial \xi^i \partial \xi^j \partial \xi^k}. \end{aligned} \quad (2.8)$$

3 Cohomology of the group of diffeomorphisms

Let M be an oriented manifold. Denote by $\text{Diff}_+(M)$ the group of diffeomorphisms of M that preserve the orientation on M .

3.1 Space of linear differential operators and space of symbols

Let $\mathcal{D}(\mathcal{F}_\lambda)$ be the space of linear differential operators acting on the space of tensor densities of degree λ on M . This space admits a structure of module over the group $\text{Diff}(M)$ as follows: take $f \in \text{Diff}(M)$ and $A \in \mathcal{D}(\mathcal{F}_\lambda)$, then

$$f^*A := f^* \circ A \circ f^{-1*}, \quad (3.1)$$

where f^* is the natural action of a diffeomorphism on λ -densities.

By differentiating this action, one obtains the action of $\text{Vect}(M)$ on $\mathcal{D}(\mathcal{F}_\lambda)$.

Consider the space \mathcal{S}^k of symmetric contravariant tensor fields on M of degree k . This space is naturally isomorphic to the space of functions on T^*M that are polynomials of degree k on the fibers. We shall identify these two spaces throughout this paper.

One can define a $\text{Diff}(M)$ -module structure on \mathcal{S}^k by the formula (2.4). We have then a graduation of $\text{Diff}(M)$ -module

$$\mathcal{S} = \bigoplus_{k \geq 0} \mathcal{S}^k.$$

The module \mathcal{S} and the modules $\mathcal{D}(\mathcal{F}_\lambda)$ are not isomorphic with respect to the action of $\text{Diff}(M)$ given above (cf. [6, 7, 9, 10]).

3.2 Cohomology of $\text{Vect}(M)$ and cohomology of $\text{Diff}(M)$

The problem of the infinitesimal deformation of the module \mathcal{S} with respect to the Lie algebra $\text{Vect}(M)$ is related to the cohomology group $H^1(\text{Vect}(M); \text{End}(\mathcal{S}))$. Consider now the space of differential operators $\mathcal{D}(\mathcal{S})$ as a submodule of $\text{End}(\mathcal{S})$. This module can be decomposed as a $\text{Vect}(M)$ -module into $\bigoplus_{k,l} \mathcal{D}(\mathcal{S}^k, \mathcal{S}^l)$. The following result is proved in [10].

$$H^1(\text{Vect}(M); \mathcal{D}(\mathcal{S}^k, \mathcal{S}^m)) = \begin{cases} \mathbb{R}, & \text{if } k - m = 2, \\ \mathbb{R}, & \text{if } k - m = 1, \ m \neq 0, \\ \mathbb{R} \oplus H_{\text{DR}}^1(M), & \text{if } k - m = 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.2)$$

Let us give explicit formulae for 1-cocycles that generate the cohomology group (3.2).

For $k - m = 0$. Denote by Div the divergence operator associated with the volume form on M . The map $X \mapsto a \text{Div}(X) + i_X \omega$, where a is a constant and ω is a closed 1-form on M , defines a 1-cocycle on $\text{Vect}(M)$ with values in $C^\infty(M)$ (cf. [8])

The 1-cocycle generating the cohomology group (3.2), for $k - m = 0$, is a multiplication by k -tensor fields with the cocycle above.

For $k - m = 1$, and $m \neq 0$. The map $X \mapsto L_X \nabla$, where $L_X \nabla$ is the Lie derivative of the connection ∇ (see [10] for more details), defines a 1-cocycle on $\text{Vect}(M)$ with values in the space of tensor fields on M of type $(2, 1)$.

The 1-cocycle generating the cohomology group (3.2), for $k - m = 1$, is obtained by contracting k -tensor fields with the above $(2, 1)$ -tensor fields.

For $k - m = 2$. It is well known that the cohomology group $H^2(C^\infty(T^*M); C^\infty(T^*M))$ is isomorphic to $\mathbb{R} \oplus H_{\text{DR}}^1(M)$ (see, e.g. [12] and the references therein). The component

\mathbb{R} is generated by the so-called Vey cocycle [16], usually denoted by S_{Γ}^3 (see [13] for an explicit construction which uses a connection). The Vey cocycle is important in the theory of deformation quantization; it appears in the third-order term in the Moyal product (see, e.g. [12]).

Consider now the natural embedding of $\text{Vect}(M)$ in $C^\infty(T^*M)$ given by $X \mapsto X^i \xi_i$. The map $X \mapsto S_{\tilde{\Gamma}}^3(X^i \xi_i, \cdot)$, where $\tilde{\Gamma}$ is the connection (2.1), turns out to be a 1-cocycle on $\text{Vect}(M)$ with values in $\mathcal{D}(\mathcal{S}^k, \mathcal{S}^{k-2})$ and generates the cohomology group (3.2) (see [10]).

Remark 3. In the one dimensional case, the cohomology group (3.2) is calculated in [5] (see [4] for the complex case).

To study the cohomology of the group of diffeomorphisms, we deal with differential cohomology “Van Est Cohomology”. This means we consider only differential cochains (see [8]).

We have the following:

Theorem 2. *Let $M := S^n$ be the n -dimensional sphere. For $n = 2, 3$, the first-cohomology group*

$$H^1(\text{Diff}_+(S^n); \mathcal{D}(\mathcal{S}^k, \mathcal{S}^m)) = \begin{cases} \mathbb{R}, & \text{if } k - m = 0, \\ \mathbb{R}, & \text{if } k - m = 2, \\ \mathbb{R}, & \text{if } k - m = 1, m \neq 0, \\ 0, & \text{otherwise.} \end{cases} \quad (3.3)$$

We start by giving explicit formulae for 1-cocycles generating the cohomology group (3.3).

For $k - m = 0$. Any diffeomorphism $f \in \text{Diff}_+(S^n)$ preserves the volume form on S^n up to some factor. The logarithm function of this factor defines a 1-cocycle on $\text{Diff}(S^n)$, say $c(f)$, with values in $C^\infty(S^n)$.

The 1-cocycle generating the component \mathbb{R} of the cohomology group (3.3) is given as a multiplication operator of any k -tensor field by the 1-cocycle $c(f)$.

For $k - m = 1, m \neq 0$. The difference $\ell(f) := f^* \Gamma - \Gamma$, where Γ is a connection on S^n , is a well-defined tensor field of type $(2, 1)$. The map $f \mapsto \ell(f^{-1})$ defines a 1-cocycle on $\text{Diff}_+(S^n)$ with values in the space of tensor fields of type $(2, 1)$.

The 1-cocycle generating the cohomology group (3.3) is obtained by contracting every k -tensor field with the above 1-cocycle.

For $k - m = 2$. First we have the following

Proposition 1. *For any $f \in \text{Diff}_+(S^n)$, the restriction $\mathcal{L}(\tilde{f})|_{\mathcal{S}^k}$, where \mathcal{L} is the operator (2.3), defines a map from \mathcal{S}^k to \mathcal{S}^{k-2} .*

The proof is immediate from the formula (2.7) and the properties of the connection (2.1).

It is easy to see that the map $f \mapsto \mathcal{L}(\tilde{f}^{-1})|_{\mathcal{S}^k}$ defines a 1-cocycle on $\text{Diff}_+(S^n)$ with values in $\mathcal{D}(\mathcal{S}^k, \mathcal{S}^{k-2})$, and generates the cohomology group (3.3).

Remark 4. It is worth noticing that the 1-cocycles above can be defined on any oriented manifold.

Now we are in position to prove Theorem 2.

It is well-known that the maximal compact group of “rotations” of S^n , $SO(n + 1)$, is a deformation retract of the group $\text{Diff}_+(S^n)$, for $n = 1, 2, 3$ (see [15]). Since the space $\text{Diff}_+(S^n)/SO(n + 1)$ is acyclic, the Van Est cohomology of the Lie group $\text{Diff}_+(S^n)$ can be computed using the isomorphism (see, e.g., [8, p. 298])

$$H^1(\text{Diff}_+(S^n); \mathcal{D}(\mathcal{S}^k, \mathcal{S}^m)) \simeq H^1(\text{Vect}(S^n), SO(n + 1); \mathcal{D}(\mathcal{S}^k, \mathcal{S}^m)).$$

Thus $H^1(\text{Diff}_+(S^n); \mathcal{D}(\mathcal{S}^k, \mathcal{S}^m)) = 0$ for $k - m \neq 0, 1, 2$, and for $(k, m) = (1, 0)$.

First, observe that the De Rham classes in the cohomology group (3.2) turn out to be trivial since $H_{\text{DR}}^1(S^n) = 0$. Suppose that there are two 1-cocycles representing cohomology classes in the cohomology group (3.3). The isomorphism above shows that these two 1-cocycles induce two non-cohomologue classes in the cohomology group (3.2), which is absurd. Theorem 2 follows from explicit constructions of the 1-cocycles above. ■

Remark 5. (i) Theorem 2 remains true for $n \geq 4$ if the subgroup $SO(n + 1)$ is a deformation retract of the group $\text{Diff}_+(S^n)$ for all n . As far as I know, this problem is not solved yet.

(ii) It is possible to integrate the De Rham class to the group $\text{Diff}_+(M)$, when M is an (oriented) simply connected manifold, if one considers non-differential cochains. To define these 1-cocycles, let α be a closed 1-form on M . The function

$$c^\alpha(f)(x) := \int_{\gamma(x, f(x))} \alpha, \tag{3.4}$$

where $\gamma(x, f(x))$ is a path joining the point x to the point $f(x)$, is well-defined on M since α is closed and M is simply connected. It is easy to show that the map $f \mapsto c^\alpha(f^{-1})(x)$ defines a 1-cocycle on $\text{Diff}_+(M)$ with values in $C^\infty(M)$. The contraction of any tensor by the 1-cocycle above defines a 1-cocycle that integrate, for $k - m = 0$, the De Rham class in the cohomology group (3.2).

4 Discussion

1. In the one dimensional case $M := S^1$, the analogue of Theorem (3.3) was given in [5]: the first cohomology group

$$H^1(\text{Diff}_+(S^1), \text{PSL}_2(\mathbb{R}); \mathcal{D}(\mathcal{F}_\lambda, \mathcal{F}_\mu)) = \begin{cases} \mathbb{R}, & \text{if } \mu - \lambda = 2, 3, 4, (\lambda \text{ generic}) \\ \mathbb{R}, & \text{if } (\lambda, \mu) = (-4, 1), (0, 5) \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

The 1-cocycles generating the cohomology group (4.1) are differential operators whose coefficients are a linear combination by the so-called Schwarzian derivative and its derivatives (see [6] for explicit formulae).

From this point of view, Ovsienko and the author [6] (see also [1]) generalize the Schwarzian derivative on S^n endowed with a projective structure (i.e. coordinates change are projective transformations): the multi-dimensional Schwarzian derivative is a 1-cocycle on the group $\text{Diff}(S^n)$ with values in $\mathcal{D}(\mathcal{S}^2, \mathcal{S}^0)$, vanishing on the subgroup $\text{PSL}_{n+1}(\mathbb{R})$. It

is interesting to ask whether there is a relation between the 1-cocycle (2.8) and the multi-dimensional Schwarzian derivative. Observe that the 1-cocycle (2.8) does not vanish on the subgroup $\mathrm{PSL}_{n+1}(\mathbb{R})$.

Another approach for the *conformal* Schwarzian derivative on S^n was introduced in [2]. This derivative is a 1-cocycle on the group $\mathrm{Diff}(S^n)$ with values in $\mathcal{D}(\mathcal{S}^2, \mathcal{S}^0)$, vanishing on the subgroup $O(p+1, q+1)$ (group of all conformal diffeomorphisms with respect to a given metric on S^n , where $p+q=n$). It is interesting to ask whether there is a relation between the 1-cocycle (2.8) and the *conformal* Schwarzian derivative introduced in [2].

2. It is interesting to generalize the 1-cocycle (2.3) to any arbitrary symplectic manifold. In this case, the 1-cocycle (2.3) can be interpreted as a cocycle integrating on one argument the Vey 2-cocycle S_{Γ}^3 (see Section 3.2). The 1-cocycle (2.3) integrates the Vey cocycle only on the cotangent bundle T^*M .

Another 2-cocycle was constructed by Tabachnikov [14] integrating the Vey 2-cocycle. It is also interesting to compare this 2-cocycle with the operator (2.3).

Acknowledgments

I am grateful to V Ovsienko for his constant support and for helpful remarks concerning this paper. I am also grateful to H Gargoubi, B Kolev, Y Maeda, P Lecomte and H Moriyoshi for numerous fruitful discussions. Research supported by the Japan Society for the Promotion of Science.

References

- [1] Bouarroudj S, Multi-Dimensional Schwarzian Derivative Revisited, *math. DG/0101056*.
- [2] Bouarroudj S, Conformal Schwarzian Derivatives and Conformally Invariant Quantization, *Inter. Math. Res. Not.*, Nr. 29 (2002), 1553–1570.
- [3] Bouarroudj S, Projectively Equivariant Quantization Map, *Lett. Math. Phys.* **51**, Nr. 4 (2000), 265–274.
- [4] Bouarroudj S and Gargoubi H, Projectively Invariant Cocycles of Holomorphic Vector Fields on an Open Riemann Surface, *Tokyo J. Math.* **25**, Nr. 1 (2002), 33–40.
- [5] Bouarroudj S and Ovsienko V, Three Cocycles on $\mathrm{Diff}(S^1)$ Generalizing the Schwarzian Derivative, *Inter. Math. Res. Not.*, Nr. 1 (1998), 25–39.
- [6] Bouarroudj S and Ovsienko V, Schwarzian Derivative Related to the Space of Linear Differential Operators on a Locally Projective Manifold, *Banach Cent. Pub.*, Vol. 51, 2000, 15–23.
- [7] Duval C and Ovsienko V, Conformally Equivariant Quantum Hamiltonians, *Selecta Math. (N.S.)* **7**, Nr. 3 (2001), 291–320.
- [8] Fuks D B, *Cohomology of Infinite-Dimensional Lie Algebras*, *Contemp. Soviet Math.*, Consultants Bureau, New York, 1986.
- [9] Lecomte P B A and Ovsienko V, Projectively Invariant Symbol Calculus, *Lett. Math. Phys.* **49**, Nr. 3 (1999), 173–196.
- [10] Lecomte P B A and Ovsienko V, Cohomology of the Vector Fields Lie Algebra and Modules of Differential Operators on a smooth manifold, *Compositio Mathematica* **119** (2000), 95–110.

- [11] Neijnhuis A and Richardson R W (jr.), Deformation of Homomorphisms of Lie Groups and Lie Algebras, *Bull. Amer. Math. Soc.* **73** (1967), 175–179.
- [12] Ovsienko V and Roger C, Deformation of Poisson Brackets and Extensions of Lie Algebras of Contact Vector Fields, *Russian Math. Survey* **47** (1992), 141–194.
- [13] Roger C, Déformation algébriques et applications à la physique, *Gaz. Math.* Nr. 49 (1991), 75–94.
- [14] Tabachnikov S, Projective Connections, Group Vey Cocycle, and Deformation Quantization, *Internat. Math. Res. Not.* Nr. 14 (1996), 705–722.
- [15] Thurston W P, Three-Dimensional Geometry and Topology, Editor: Silvio Levy, Princeton Mathematical Series, Vol. 35, Princeton University Press, Princeton, NJ, 1997.
- [16] Vey J, Déformation du crochet de Poisson sur une variété symplectique, *Comment. Math. Helv.* **50**, Nr. 4 (1975), 421–454.
- [17] Yano K and Ishihara S, Tangent and Cotangent Bundles: Differential Geometry, Pure and Applied Mathematics, Nr. 16, Marcel Dekker, Inc., New York, 1973.