The Landau–Ginzberg Theory for the Two-Dimensional Bose Gas

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Abstract

Using results from sheaf theory combined with the phenomenological theory of the two-dimensional superfluid, the precipitation of quantum vortices is shown to be the genesis of a macroscopic order parameter for a phase transition in two dimensions.

1 Introduction

The underlying mechanism for superfluidity in three-dimensional Bose systems is based upon the spectral decomposition of the Bose field operator $\Psi(x)$. The zero-momentum component of the operator becomes the macroscopic order parameter for the associated phase transition [4] in the thermodynamic limit. The existence of this order parameter, which defines the condensed phase [2], is a consequence of the breaking of a $U(1)$ gauge symmetry. Indeed, the states of the condensate and the states of the normal fluid belong to two inequivalent representations of the algebra generated by $\Psi(x)$ and $\Psi^\dagger(x)$.

In two-dimensions, however, the situation is much more complicated since the breaking of a $U(1)$ gauge symmetry is explicitly prohibited [5, 11, 12, 6]. There are instead two phenomenological theories of the phase transition, both of which presupposes the existence of vortices with integral vorticity, or “charge”, in the system. In the Kosterlitz–Thouless–Nelson [9, 15, 16, 13] theory the superfluid transition is marked by the precipitation of essentially free vortices in the normal fluid into vortices that are tightly bound in dipole pairs in the superfluid. There are effectively no free vortices in the superfluid state in this theory. In theory given in [17], on the other hand, vortices are only present in the superfluid state and the phase transition is due to the annihilation of oppositely charged vortices at the transition temperature. In this case, there are no *quantum* vortices in the normal fluid state. An essential feature of both theories, however, is that the net vorticity, or charge, of the system must be zero. Indeed, it was shown in [17] that this is a necessary condition for the phase transition to occur.

In this paper we use the existence of quantized vortices in the superfluid state to demonstrate by construction the existence of an order parameter for the superfluid phase transition in spite of the absence of a Bose-condensation-based phase transition in two-dimensions. We do so by using the results from analysing the classical equations of motion for the vortices given in [9, 15, 16, 13] and [17] as a motivation for studying sheaf theory. Results from sheaf theory are then combined with an effective-field description of the system to construct the order parameter. Global sheaf-theoretic results on the geometry of two-dimensional surfaces are then used to make statements about the possibility of having the superfluid phase transitions on various two-dimensional surfaces.

The rest of this paper is organized as follows. In Section 2 we review the properties of the vortex gas used by both phenomenological theories of the superfluid phase transition. Results from this section is meant to motive the use of sheaf theory as a tool for constructing the order parameter for the phase transition. In Section 3 we review those aspects of sheaf theory that are needed for this construction. The results stated in this section are all well known and no proofs are given. In Section 4 we combine the results of Section 3 with an effective free energy functional to construct the order parameter for the phase transition. Results from sheaf theory will also be used to comment on the existence of the superfluid phase transition for various two-dimensional surfaces. Final concluding remarks can be found in Section 5.

2 Vortices in two-dimensions

The description of the two-dimensional quantum vortices used in [9, 15, 16, 13] and [17] follows in direct analogy with the description of vortex lines used in three-dimensional superfluids. In this description vortices are characterized by $\{\kappa_i, z_i\}$, where κ_i is the circulation associated with the i^{th} vortex at the position z_i (in complex coordinates) on a Riemann surface $\mathbf{M}$. Vortices are formed due to a current flow j_z in the fluid. Because of the underlying quantum mechanical nature of the superfluid, their circulation (or vorticity) is quantized and

$$\kappa_i \equiv \int_{\gamma_i} \frac{j_{\bar{z}} dz}{\rho} = \frac{h}{m} n_i, \tag{2.1}$$

where ρ is the (constant) superfluid density, γ_i is a closed contour on $\mathbf{M}$ encompassing the i^{th} vortex and n_i is a non-zero integer.

One then traditionally appeals to classical fluid dynamical arguments [3] to describe the interaction between vortices and their dynamics. Because of the particular properties of two dimensional fluid flow, one can treat the vortices as though they were particles in and of themselves with a hamiltonian

$$H = -\frac{\hbar^2}{m^2} \sum_{k>l} n_k n_l \log |z_k - z_l|. \tag{2.2}$$

This hamiltonian is identical to the interaction hamiltonian of a gas of charges particles with charge n_i, hence the analogy between a gas of vortices and a gas of charged particles. Accordingly, n_k is often interpreted as the charge of the k^{th} vortex, and the total charge of the vortices is

$$Q = \sum_k n_k. \tag{2.3}$$

The evolution equations for vortices is quite different from that of charged particles, however, [3, 1, 14, 18] in that

$$\begin{aligned} n_k \frac{dz_k}{dt} &= -2i \frac{m}{\hbar} \frac{\partial H}{\partial \bar{z}_k}, \\ n_k \frac{d\bar{z}_k}{dt} &= 2i \frac{m}{\hbar} \frac{\partial H}{\partial z_k}. \end{aligned} \tag{2.4}$$

Notice that vortex motion is determined by its *velocity*, and not its acceleration.

It is straightforward to see that H admits the following affine transformation in the complex plane

$$H(z_1, \ldots, z_n) = H(\eta z_1 + \xi, \ldots, \eta z_n + \xi), \tag{2.5}$$

where ξ is an uniform time dependent translation, and η (with the constraint that $|\eta| = 1$) is a time dependent uniform rotation. Translational invariance gives the constant of the motion

$$M = \sum_k n_k z_k, \tag{2.6}$$

while rotational invariance gives

$$I = \sum_k n_k |z_k|^2. \tag{2.7}$$

To motive our use of sheaf theory to study this system, we associate to each M a meromorphic function

$$f(z) = \prod_k (z - z_k)^{n_k}, \tag{2.8}$$

with poles or zeros of order n_k at the points z_k on $\mathbf{M}$. Such a function is often identified with the flow potential of an ideal fluid. This association is clearly not unique, of course, for if $g(z)$ is any non-vanishing holomorphic function on an open set $U \subset \mathbf{M}$, then $f(z)g(z)$ can also be associated to the same M; g does not add any zeros or poles to f. We should instead consider equivalence classes of meromorphic functions in which two meromorphic functions $f(z)$ and $h(z)$, not identically zero, are equivalent if f/h is a non-vanishing holomorphic function. This leads naturally to the consideration of all such equivalence classes over the manifold $\mathbf{M}$. In sheaf theory this equivalence class is the sheaf of germs of divisors with M identified with $\mathfrak{d}$, the divisor of a line bundle over $\mathbf{M}$. Correspondingly, Q is identified with the bundle's chern class. Sheaf theory is therefore a natural framework for describing and understanding the behavior and dynamics of quantum vortices.

3 Overview of sheaf theory

In this section we present a brief review of sheaf theory, highlighting the topics needed in our analysis. We follow exclusively the treatment and notation found in [8] (see also [7]). No new results in sheaf theory is presented, however, and we do not present any proofs for the results stated here.

We begin with with an open covering $\mathcal{U}$ of a Riemann surface $\mathbf{M}$ consisting of open sets $\{U_\alpha\}$. As an additional structure, the notion of a sheaf is introduced:

A *sheaf* of abelian groups over a topological space $\mathbf{M}$ is a topological space $\mathcal{S}$, together with a mapping $\pi : \mathcal{S} \to \mathbf{M}$ such that:

1. π is a local homomorphism;
2. for each point $p \in \mathbf{M}$, the set $\mathcal{S}_p = \pi^{-1}(p)$, $\mathcal{S}_p \supset \mathcal{S}$ has the structure of an abelian group;
3. the group operations are continuous in the topology of $\mathcal{S}$.

$\mathcal{S}_p$ is called the *stalk* of the sheaf $\mathcal{S}$ at the point p. A *section* s of the sheaf $\mathcal{S}$ over U_α is a continuous map $s : U_\alpha \to \mathcal{S}$ such that $\pi \circ s : U_\alpha \to U_\alpha$ is the identity map on $\mathbf{M}$. $\Gamma(U_\alpha, \mathcal{S})$ is the set of all sections of $\mathcal{S}$ over U_α and $\Gamma(\mathcal{U}, \mathcal{S})$ is the formal sum of all $\Gamma(U_\alpha, \mathcal{S})$ over $\mathcal{U}$.

If f and g are any two functions defined on the neighborhoods U_α and U_β respectively such that $U_\alpha \cap U_\beta \neq \emptyset$, then f and g are equivalent at a point $p \in U_\alpha \cap U_\beta$ if $f(z) = g(z)$ for all z in a neighborhood of p contained in $U_\alpha \cap U_\beta$. The *germ* of a function at p is the equivalence class of local functions at p and is denoted by f_p. The set of all germs of functions at each point p in $\mathbf{M}$ forms an abelian groups under addition and multiplication. As such the following sheaves can be defined:

$\mathcal{C}^\infty$, *the sheaf of germs of infinitely differentiable functions*;
ϑ, *the sheaf of germs of holomorphic functions*;
ϑ^*, *the sheaf of germs of non-vanishing holomorphic functions*;
$\mathcal{M}$, *the sheaf of germs of meromorphic functions*;
$\mathcal{M}^*$, *the sheaf of germs of non-vanishing meromorphic functions.*

These sheaves may be ordered by:

$$\mathcal{C}^\infty \supset \mathcal{M} \supset \mathcal{M}^* \supset \vartheta \supset \vartheta^*, \tag{3.1}$$

In addition, the quotient sheaf $\mathbb{D} \equiv \mathcal{M}^*/\vartheta^*$ is the *sheaf of divisors* and is of principle interest to us.

Consider next a function $f \in \mathcal{M}^*$ defined on U_α. The order of f at a point $p \in U_\alpha$, denoted by $\nu_p(f)$, is the order of the first non-zero coefficient of the Laurent expansion of f about the point $z = p$. Furthermore, for each $f, g \in \mathcal{M}^*$, $\nu_p(fg) = \nu_p(f) + \nu_p(g)$. If g is also an element of ϑ^*, then $\nu_p(g) = 0$ and $\nu_p(gf) = \nu_p(f)$. The *divisor* is the quotient map $\mathfrak{d} : \mathcal{M}^* \to \mathbb{D}$ defined as

$$\mathfrak{d}(f) \equiv \sum_p \nu_p(f) p, \tag{3.2}$$

and there is an exact sequence of sheaves

$$O \;\rightarrow\; \vartheta^* \;\underset{\rightarrow}{\iota}\; \mathcal{M}^* \;\underset{\rightarrow}{\mathfrak{d}}\; \mathbb{D} \;\rightarrow\; O, \tag{3.3}$$

where ι is the inclusion map and O is the trivial sheaf. A sheaf is trivial if for each $p \in \mathbf{M}$, O is the trivial group.

Cohomology classes of a sheaf $\mathcal{S}$ are defined in the usual way. To each open covering $\mathcal{U}$ of $\mathbf{M}$ is associated a simplicial complex $N(\mathcal{U})$ called the nerve of $\mathcal{U}$ with each vertex of $N(\mathcal{U})$ contained in one U_α. The vertices $U_0, \ldots, U_q$ span a q-complex $\sigma = (U_0, \ldots, U_q)$ if and only if $U_0 \cap \cdots \cap U_q \neq \emptyset$. $|\sigma| = U_0 \cap \cdots \cap U_q$ is the support of σ. For any sheaf of abelian groups $\mathcal{S}$, a q-cochain of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$ is a function f which associates every q-simplex $\sigma \in N(\mathcal{U})$ a section $f(\sigma) \in \Gamma(|\sigma|, \mathcal{S})$. The set of all q-chains is $C^q(\mathcal{U}, \mathcal{S})$. Since $f, g \in C^q(\mathcal{U}, \mathcal{S}) \Rightarrow f + g \in C^q(\mathcal{U}, \mathcal{S})$, $C^q(\mathcal{U}, \mathcal{S})$ forms an abelian group.

The coboundary operator $\delta : C^q(\mathcal{U}, \mathcal{S}) \rightarrow C^{q+1}(\mathcal{U}, \mathcal{S})$ is defined as follows. For $f \in C^q(\mathcal{U}, \mathcal{S})$, and a $q+1$-simplex $q = (U_0, \ldots .U_{q+1}) \in N(\mathcal{U})$,

$$(\delta f)(U_0, \ldots, U_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \rho_{|\sigma|} f(U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{q+1}), \tag{3.4}$$

where $\rho_{|\sigma|}$ is the restriction of $f(U_0, \ldots, U_{i-1}, U_{i+1}, \ldots, U_{q+1}) \in \Gamma(U_0 \cap \cdots \cap U_{i-1} \cap U_{i+1} \cap \cdots \cap U_{q+1})$ to $|\sigma|$. δ is a group homomorphism where $\delta^2 = 0$. The subset $Z^q(\mathcal{U}, \mathcal{S}) = \{f \in C^q(\mathcal{U}, \mathcal{S}) \,|\, \delta f = 0\}$ is the group of q-cocycles while the image $\delta C^{q-1}(\mathcal{U}, \mathcal{S})$ is the group of q-coboundaries. Since $C^q(\mathcal{U}, \mathcal{S}) \subset \delta C^{q-1}(\mathcal{U}, \mathcal{S})$ and $Z^q(\mathcal{U}, \mathcal{S}) \subset \delta C^{q-1}(\mathcal{U}, \mathcal{S})$, the quotient group for $q > 0$ is

$$H^q(\mathcal{U}, \mathcal{S}) = Z^q(\mathcal{U}, \mathcal{S}) / \delta C^{q-1}(\mathcal{U}, \mathcal{S}), \tag{3.5}$$

while $H^0 = Z^0(\mathcal{U}, \mathcal{S})$. This is the q^{th} cohomology group of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$. Although this definition of $H^q(\mathcal{U}, \mathcal{S})$ depends explicitly on the choice of covering $\mathcal{U}$ of $\mathbf{M}$, the following limit [8],

$$H^q(\mathbf{M}, \mathcal{S}) = \text{dir. lim.}\ {}_{\mathcal{U}} H^q(\mathcal{U}, \mathcal{S}), \tag{3.6}$$

gives a $H^q(\mathbf{M}, \mathcal{S})$ that is independent of any specific choice of $\mathcal{U}$.

It is known that for any paracompact space $\mathbf{M}$, $H^q(\mathbf{M}, \mathcal{S}) = 0$. Furthermore, if $\mathbf{M}$ is also non-compact then $H^q(\mathbf{M}, \vartheta) = 0$ for all $q > 0$.

For a compact manifold, on the other hand, $H^0(\mathbf{M}, \mathcal{S}) = \Gamma(\mathbf{M}, \vartheta) = C$, while $H^q(\mathbf{M}, \vartheta) = 0$ for $q \geq 2$. The only non-trivial cohomology class remaining is

$$H^1(\mathbf{M}, \vartheta) \approx \Gamma(\mathbf{M}, \mathcal{C}^\infty) \Big/ \frac{\partial \Gamma(\mathbf{M}, \mathcal{C}^\infty)}{\partial z}. \tag{3.7}$$

Furthermore, $H^1(\mathbf{M}, \mathcal{M}) = 0$, which leads to the fundamental existence theorem of Riemann surfaces: every line bundle on $\mathbf{M}$ has a non-trivial meromorphic cross-section, hence every line bundle is the bundle of a divisor. Instead of studying the structure of $\mathfrak{d}$, we need only consider the structure of complex line bundles of $\mathbf{M}$.

To study the structure of $\mathbf{M}$ itself, we consider the subgroup $H^1(\mathbf{M}, \vartheta^*)$, the group of complex line bundles over $\mathbf{M}$. For every $\xi \in H^1(\mathbf{M}, \vartheta^*)$, choose a basis $\mathcal{U} = \{U_\alpha\}$ of the open covering of $\mathbf{M}$ and a cocycle $(\xi_{\alpha\beta}) \in Z^1(\mathbf{M}, \vartheta^*)$. $\xi_{\alpha\beta}$ is a holomorphic, nowhere vanishing function on $U_\alpha \cap U_\beta$. The cocycle condition insures that for $p \in U_\alpha \cap U_\beta \cap U_\gamma$, $\xi_{\alpha\beta}(p) \cdot \xi_{\beta\gamma}(p) = \xi_{\alpha\gamma}(p)$. For each U_α the group $\mathcal{S}_\alpha = \Gamma(U_\alpha, \vartheta)$ is associated with a group homomorphism $\tau_{\beta\alpha} : \mathcal{S}_\alpha \to \mathcal{S}_\beta$ defined on the inclusion $U_\alpha \supset U_\beta$ by:

$$(\tau_{\beta\alpha} f)(p) = \xi_{\beta\alpha} \cdot f(p), \tag{3.8}$$

for $p \in U_\alpha \subset U_\beta$, $f \in \mathcal{S}_\alpha$ and $\tau_{\beta\alpha}(f) \in \mathcal{S}_\beta$. On the triple overlap $U_\alpha \subset U_\beta \subset U_\gamma$ and for $f \in \mathcal{S}_\alpha$, $(\tau_{\gamma\beta}(\tau_{\beta\alpha}(f)))(p) = (\tau_{\gamma\alpha} f)(p)$ for all $p \in U_\gamma$. $\{\mathcal{U}, \mathcal{S}_\alpha, \tau_{\alpha\beta}\}$ forms a pre-sheaf and the associated sheaf $\vartheta(\xi)$ is the sheaf of germs of holomorphic cross-sections of the line bundle ξ. Similarly, taking $\mathcal{S}_\alpha = \Gamma(\mathcal{U}_\alpha, \mathcal{M}^*)$ results in $\mathcal{M}^*(\xi)$, the sheaf of germs of not identically vanishing meromorphic cross-sections of the line bundle ξ. Clearly for $\xi = 1$ we recover $\vartheta(1) = \vartheta$ and $\mathcal{M}^*(1) = \mathcal{M}^*$.

Among all the possible bundles formed from members of $H^1(\mathbf{M}, \vartheta^*)$ there is the canonical bundle κ which contains information on the structure of $\mathbf{M}$ itself. Because $\mathbf{M}$ is a Riemann surface, given an open covering $\mathcal{U}$ of $\mathbf{M}$ and charts z_α defined on U_α, there are local holomorphic transition functions $f_{\alpha\beta} : z_\alpha \to z_\beta$ where $z_\alpha(p) = f_{\alpha\beta}(z_\beta(p))$ for all $p \in U_\alpha \cap U_\beta$. These transition functions form the canonical bundle by defining

$$\kappa_{\alpha\beta}(p) = \left[f'_{\alpha\beta}(z_\beta(p))\right]^{-1}, \tag{3.9}$$

For $p \in U_\alpha \cap U_\beta \cap U_\gamma$, $z_\alpha(p) = f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma(p)))$, and using the chain rule,

$$\kappa_{\alpha\gamma}(p) \equiv \left[f'_{\alpha\gamma}(z_\gamma(p))\right]^{-1} = \left[f'_{\alpha\beta}(f_{\beta\gamma}(z_\gamma(p)))\right]^{-1} \left[f'_{\beta\gamma}(z_\gamma(p))\right]^{-1} = \kappa_{\alpha\beta}(p)\kappa_{\beta\gamma}(p) \tag{3.10}$$

κ satisfies the cocycle condition. Clearly $\{\kappa_{\alpha\beta}\}$ are just the transitions functions of the tangent space of $\mathbf{M}$.

The canonical bundle is used to study the geometry of $\mathbf{M}$ by considering the exact sequence of sheaves

$$\mathcal{O} \;\to\; \mathbf{Z} \;\to\; \vartheta \;\xrightarrow{e}\; \vartheta^* \;\to\; \mathcal{O}, \tag{3.11}$$

with $e(f) \equiv exp(2\pi i f)$ for $f \in \vartheta$. Corresponding to this sequence there is the exact cohomology sequence

$$\mathcal{O} \;\to\; H^1(\mathbf{M}, \vartheta)/H^1(\mathbf{M}, \mathbf{Z}) \;\to\; H^1(\mathbf{M}, \vartheta^*) \;\to\; H^2(\mathbf{M}, \mathbf{Z}) \;\to\; \mathcal{O}, \tag{3.12}$$

$c : H^1(\mathbf{M}, \vartheta^*) \to H^2(\mathbf{M}, \mathbf{Z})$ is the characteristic homomorphism and for each line bundle $\xi \in H^1(\mathbf{M}, \vartheta^*)$ the image $c(\xi)$ is the chern class of the line bundle ξ. A specific representation of $c(\xi)$ can be formed by taking considering the C^∞ function r_α defined on U_α such that for $p \in U_\alpha \cap U_\beta$, $r_\alpha(p) = r_\beta(p)|\xi_{\alpha\beta}|^2$. Then

$$c(\xi) = \frac{1}{2\pi i} \int_{\mathbf{M}} \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}} \log r_\alpha \, dz d\bar{z}, \tag{3.13}$$

Using this, one finds that for any function $f \in \Gamma(\mathbf{M}, \mathcal{M}^*)$, the chern class of that bundle will be

$$c(\xi) = \sum_{p \in \mathbf{M}} \nu_p(f), \qquad \text{where} \qquad \mathfrak{d}(f) = \sum_{p \in \mathbf{M}} \nu_p(f) p, \tag{3.14}$$

It is important to note that the chern class of the bundle will have the same value for *every* function f in $\Gamma(\mathbf{M}, \mathcal{M}^*(\xi))$.

As we shall see in the next section, the chern class of the bundle ξ is identified with the net charge of the vortices. Because we are primarily interested in the superfluid phase transition, we shall restrict our attention to bundles whose chern class vanishes. Referring back to the exact cohomology sequence, we find that the subset of $H^1(\mathbf{M}, \vartheta^*)$ with vanishing chern class is

$$\{\xi \in H^1(\mathbf{M}, \vartheta^*) | c(\xi) = 0\} \approx \frac{H^1(\mathbf{M}, \vartheta)}{H^1(\mathbf{M}, Z)} \approx \frac{H^1(\mathbf{M}, \mathbf{C})}{H^1(\mathbf{M}, \mathbf{Z}) + \delta\Gamma(\mathbf{M}, \vartheta^{1,0})} \tag{3.15}$$

where $\vartheta^{1,0}$ is the sheaf of germs of holomorphic 1-forms. The line bundles whose chern class vanishes are precisely those whose representative cocycles $\{\xi_{\alpha\beta}\}$ consists only of constant functions.

Finally, we note that from the Riemann–Roch theorem, dim $H^1(\mathbf{M}, \mathbf{C}) = 2g$ where g is the genus of the surface $\mathbf{M}$. A $g = 0$ surface is a sphere while a $g = 1$ surface is the complex torus. It is known that $H^1(\mathbf{M}, \mathbf{Z})$ forms a lattice subgroup of $H^1(\mathbf{M}, \mathbf{C})$. This means that any $2g$ generators of $H^1(\mathbf{M}, \mathbf{Z})$ forms a basis for $H^1(\mathbf{M}, \mathbf{C})$. The quotient group $H^1(\mathbf{M}, \mathbf{C})/H^1(\mathbf{M}, \mathbf{Z}) = (\mathbf{R}/\mathbf{Z})^{2g}$, which is the g-torus. In fact, $(\mathbf{R}/\mathbf{Z})^{2g}$ forms an abelian complex Lie group. The subgroup of $H^1(\mathbf{M}, \vartheta^*)$ with vanishing chern class is isomorphic to an abelian complex Lie group.

4 Application of sheaf theory

We have seen in the previous sections that sheaf theory provides a natural framework for describing and analyzing the behavior of vortices in two dimensional superfluids. In this section we will apply the results of sheaf theory and structure of the phenomenological theories of the superfluid phase transition in two dimensions to construct an order parameter for the transition. To do so we begin by identifying the configuration space of the Bose liquid with the Riemann surface $\mathbf{M}$. Because all the cohomology groups of a paracompact manifold vanishes, for vortices to be present $\mathbf{M}$ must be a compact manifold. Then at each time t there is a specific configuration of vortices and a corresponding constant of the motion M identified with the divisor $\mathfrak{d}$ of an equivalence class of functions $f \in \mathcal{M}^*$. The net charge of the system is identified with the chern class of the line bundle ξ, which is in turn interpreted as the internal symmetry space for the system. Since ξ must be constants, the symmetry group must be global and, as we shall see later, will be of the form $U(1)\mathrm{X}\cdots\mathrm{X}U(1)$.

This mathematical structure is rich enough to encompass any configuration of vortices. Every member of $\mathcal{M}^*$ has the same chern class; $\mathcal{M}^*$ thus contains functions representing not only all possible positions of vortices, but also all possible number of vortices. As yet, ξ is not specific enough to single out any one specific configuration of vortices. It is treated instead as a parameter that will determine the existence, and, perhaps, genesis of vortices. As with all physical parameters ξ should arise from the partition function

$$\mathcal{Z} = \int_{\Gamma(\mathbf{M}, \mathbf{C}^\infty(\xi))} d\Psi d\Psi^\dagger \exp\left(-\beta \int_{\mathbf{M}} F(\Psi, \Psi^\dagger) d^2x\right) \tag{4.1}$$

where the functional integration is over are sections $\Gamma(\mathbf{M}, \mathbf{C}^\infty(\xi))$ of the sheaf $C^\infty(\xi)$ and F is the free energy functional of an effective field $\Psi \in \Gamma(\mathbf{M}, \mathbf{C}^\infty(\xi))$ for the system. The difficulty in determining ξ lies in the observation that the domain of the functional integration and, to a lesser extent, $\mathbf{M}$ itself, is determined by ξ. To find $\mathcal{Z}$, we must find ξ. Unfortunately, to find ξ, we need $\mathcal{Z}$. To decouple the system, we expand the free energy about its minimum

$$\begin{aligned}\mathcal{Z} = \exp\left(-\beta \int_{\mathbf{M}} F_{\text{min}} d^2x\right) \int_{\Gamma(\mathbf{M},\mathbf{C}^\infty(\xi))} \mathcal{D}\Psi\mathcal{D}\Psi^\dagger \\ \times \exp\left((-\beta \int_{\mathbf{M}} \left(\frac{1}{2}\frac{\delta^2 F}{\delta\Psi^2}\bigg|_{\Psi_{\text{min}}} (\Psi - \Psi_{\text{min}})^2 + \frac{\delta^2 F}{\delta\Psi\delta\Psi^\dagger}\bigg|_{\Psi_{\text{min}}} |(\Psi - \Psi_{\text{min}})|^2 \right.\right. \\ \left.\left. + \frac{1}{2}\frac{\delta^2 F}{\delta\Psi^{\dagger 2}}\bigg|_{\Psi_{\text{min}}} (\Psi^\dagger - \Psi^\dagger_{\text{min}})^2 + \cdots\right) d^2x\right). \end{aligned} \tag{4.2}$$

where Ψ_{min} is determined by the solution of

$$\frac{\delta F}{\delta\Psi}\bigg|_{\Psi_{\text{min}}} = 0, \tag{4.3}$$

and is the order parameter.

In principle, the exact form of F can be determined by the microscopic, or "bare" fields Ψ_{bare} and the corresponding microscopic Hamiltonian. When the strong interaction between helium atoms is taken into account, Ψ_{bare} is replaced by the effective field Ψ that takes into account interactions. Consequently, the detailed form of F can be very complicated. Nevertheless, we can expand F in a power series in $|\Psi|^2$,

$$F(\Psi, \Psi^\dagger) = \frac{1}{2m}|\nabla\Psi|^2 + a(T)|\Psi|^2 + b(T)|\Psi|^4 + c(T)|\Psi|^6 + \cdots \tag{4.4}$$

where the parameters $a(T)$, $b(T)$, $c(T)$, $\ldots$ depend on the temperature T as well as the detail form of the interaction Hamiltonian between the helium atoms. Let us first cut off the expansion at the second order term; we will consider the affects of higher order terms later. We then have a Landau–Ginzberg type of free energy, and eq. (4.3) becomes

$$0 = -\frac{1}{2m}\nabla^2\Psi_{\text{min}} + a(T)\Psi_{\text{min}} + 2b(T)|\Psi_{\text{min}}|^2\Psi_{\text{min}}, \tag{4.5}$$

Requiring that Ψ_{min} be harmonic

$$0 = \Psi_{\text{min}}\left(a(T) + 2b(T)|\Psi_{\text{min}}|^2\right), \tag{4.6}$$

The general solution of eq. (4.5) is

$$\Psi_{\text{min}} = 0 \qquad \text{or} \qquad \Psi_{\text{min}} = \pm\left(-\frac{a(T)}{2b(T)}\bigg|_{\text{min}}\right)^{1/2}. \tag{4.7}$$

While the field Ψ does not have to be a global function, the free energy functional must be. Consequently, $|\xi| = 1$. From the Maximum Modulus Theorem, the only such holomorphic function is the constant function. We will therefore define ξ as the abelian group generated by $e^{i\Psi_{\min}}$. The exact form of ξ, as well as the phase of $\Psi_{\min}$, was chosen so that $|\xi| = 1$.

Vortices exist because of non-trivial differences in the phase of the macroscopic field Ψ in various regions of the Bose liquid. To each configuration of vortices there is an open covering of $\mathbf{M}$ and a set $\{\xi_{\alpha\beta}\}$ subordinate to the intersection $U_\alpha \cap U_\beta$. A non-trivial choice, for example

$$\xi_{\alpha\beta} = \left(e^{i\Psi}\right)^3 \tag{4.8}$$

would result in the presence of a vortex with strength 3 within this intersection. Going from U_α to U_β changes the phase of Ψ by $\xi_{\alpha\beta}$. The exact value of $\Psi_{\min}$ does not matter as long as it is non-zero.

In the phenomenological theory given in [17] the superfluid state is marked by the presence of vortices while the normal fluid state does not have vortices. Consequently, $\xi = 1$ when $T > T_c$ and when $T < T_c$, $\xi \neq 0$. From eq. (4.6), necessarily $b(T) > 0$ for $T > T_c$, corresponding to $\Psi = 0$, while for $T < T_c$, $a(T)/b(T) < 0$. This is the standard result expected from a Landau–Ginzberg theory, but now it is necessitated by the presence and role that vortices play in the superfluid transition.

We could have chosen to include a higher order term in the expansion of F, say to third order. Then it is possible that other linearly independent solutions could exist for the free energy minima, and each additional solution would enlarge the gauge group, although it will still remain abelian. For example, the gauge group for the quadratic free energy is a global $U(1)$, for the third order polynomial one could have is $U(1)\mathrm{X}U(1)$ and so on. However, each time the gauge group is enlarged, the genus of $\mathbf{M}$ necessarily increases. Going back to eq. (2.4), we see that the hamiltonian admits affine transformations of the coordinates $\{z_k\}$. The transition functions of the manifold $\mathbf{M}$ must thus be affine. A compact Riemann surface $\mathbf{M}$ admits affine transformations if and only if the chern class of the canonical bundle vanishes. As $c(\kappa) = 2(g-1)$, we find that $\mathbf{M}$ must be a 1-torus, From this argument then the only physically relevant free energy for superfluidity in two dimensions can have at most one non-trivial root. Any additional terms in F of an order higher that $|\Psi|^4$ will not introduce any additional physically relevant solutions at any temperature.

Finally, for $g = 0$, dim $H^1(\mathbf{M}, \mathbf{C}) = 0$ and there are no lines bundles whose chern class vanishes. Since the total charge of the system must vanish, we conclude that vortex precipitation cannot occur on the surface of a sphere. Vortex precipitation can, however, occur on the surface of a rectangular plane which can be compactified by identifying opposite sides. In doing so, we have mapped the plane into a torus, which is exactly the structure required by the Riemann–Roch theorem.

5 Summary

In summary, we have shown that the presence of vortices in the superfluid state necessitates the existence of an order parameter for the two dimensional vortex system. Since the net charge of the system is constrained to be zero, these vortices cannot precipitate on the surface of a sphere or any other surface homeomorphic to it. This result has been seen experimentally [10] and analysed in detail theoretically [19]. Furthermore, the symmetries of the hamiltonian require that the free energy be a functional of at most one macroscopic field. In addition, for the phase transition to take place, there can be at most one nontrivial solution of the equations minimizing F at any temperature.

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