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On the Bilinear Equations for Fredholm Determinants Appearing in Random Matrices

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Abstract

It is shown how the bilinear differential equations satisfied by Fredholm determinants of integral operators appearing as spectral distribution functions for random matrices may be deduced from the associated systems of nonautonomous Hamiltonian equations satisfied by auxiliary canonical phase space variables introduced by Tracy and Widom. The essential step is to recast the latter as isomonodromic deformation equations for families of rational covariant derivative operators on the Riemann sphere and interpret the Fredholm determinants as isomonodromic τ-functions.

1 Differential equations for Fredholm determinants in random matrices

In the theory of random matrices, it is known that in suitably defined double scaling limits the generating functions for spectral distributions are given by Fredholm determinants of certain integral operators [14, 17, 18, 19]. For example, in the universality class of the Gaussian Unitary Ensemble (GUE), in the bulk of the spectrum, the probability of having exactly $\{m_1, \ldots, m_n\}$ scaled eigenvalues in the sequence of disjoint intervals $\{([a_1, a_2], \ldots, [a_{2n-1}, a_{2n}]\}$ is

$$E(m_1, \ldots, m_n) = \frac{(-1)^{\bar m}}{m_1! \cdots m_n!} \frac{\partial^{\bar m} \tau^S}{\partial \lambda_1^{m_1} \cdots \partial \lambda_n^{m_n}}\Big|_{\lambda_1 = \cdots = \lambda_m = 1}, \qquad \bar m = \sum_j m_j, \tag{1.1}$$

where τ^S is the Fredholm determinant

$$\tau^S := \det\left(1 - \hat K^S\right) \tag{1.2}$$

of the integral operator $\hat K_s : L^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$ with the sine kernel

$$\left(\hat K^S v\right)(x) = \sum_{j=1}^n \lambda_j \int_{a_{2j-1}}^{a_{2j}} \frac{\sin(\pi(x-y))}{\pi(x-y)} v(y) dy. \tag{1.3}$$

Rescaling at the (soft) edge of the spectrum, the corresponding quantity is given by the Fredholm determinant

$$\tau^A := \det\left(1 - \hat{K}^A\right) \tag{1.4}$$

of the operator with the Airy kernel [18]

$$\left(\hat{K}^A v\right)(x) = \sum_{j=1}^{n} \lambda_j \int_{a_{2j-1}}^{a_{2j}} \frac{Ai(x))Ai'(y) - Ai(y)Ai'(x)}{x-y} v(y)dy, \tag{1.5}$$

where $Ai(x)$ is the Airy function. If the measure is taken to be the one associated with either the Laguerre or Jacobi orthogonal polynomials, rescaling at the (hard) edge leads to the Fredholm determinant

$$\tau^B_\alpha := \det\left(1 - \hat{K}^B_\alpha\right) \tag{1.6}$$

of the operator with Bessel kernel [6, 19]

$$\left(\hat{K}^B_\alpha v\right)(x) = \sum_{j=1}^{n} \lambda_j \int_{a_{2j-1}}^{a_{2j}} \frac{J_\alpha(\sqrt{x})\sqrt{y}J'_\alpha\left(\sqrt{y}\right) - J_\alpha\left(\sqrt{y}\right)\sqrt{x}J'_\alpha\left(\sqrt{x}\right)}{2(x-y)} v(y)dy, \tag{1.7}$$

where $J_\alpha(x)$ is the Bessel function with index α.

It was shown by Tracy and Widom [17, 18, 19], extending earlier results of the Kyoto school [11], that all these Fredholm determinants can be computed by quadratures in terms of solutions of certain associated nonautonomous Hamiltonian systems in which the end points $\{a_j\}$ play the rôle of multi-time deformation variables. Moreover, these Fredholm determinants may be interpreted as isomonodromic τ-functions [9, 16, 10, 5] in the sense of [12, 13].

More recently, Adler, Shiota and van Moerbeke [2, 3] have shown that the Fredholm determinants τ^A, τ^B_α satisfy hierarchies of bilinear differential equations with respect to the endpoint parameters. These follow from combining Virasoro constraints satisfied by certain associated KP τ-functions with the bilinear equations they also satisfy with repect to the KP flow parameters $\{t_1, t_2, \ldots\}$, evaluated at the zero values of these parameters. The approach of [2, 3] was based on the application of vertex operators, integrated over the intervals $\{[a_{2j-1}, a_{2j}]\}$, to suitable "vacuum" KP τ-functions, effecting thereby a continuous version of Darboux transformations, yielding new KP τ-functions, such that the Fredholm determinant equals the ratio of the two.

For the Airy kernel, the first equation in this hierarchy may be expressed as

$$\mathcal{D}_0^4 F^A - 4\mathcal{D}_1\mathcal{D}_0 F^A + 2\mathcal{D}_0 F^A + 6\left(\mathcal{D}_0^2 F^A\right)^2 = 0, \tag{1.8}$$

where

$$F^A := \ln \tau^A \tag{1.9}$$

and

$$\mathcal{D}_m := \sum_{j=1}^{2n} a_j^m \frac{\partial}{\partial a_j}, \qquad m \in \mathbb{N}, \tag{1.10}$$

while for the Bessel kernel, it is

$$\mathcal{D}_1^4 F_\alpha^B - 2\mathcal{D}_1^4 F_\alpha^B + \left(1-\alpha^2\right)\mathcal{D}_1^2 F_\alpha^B + \mathcal{D}_2\mathcal{D}_1 F_\alpha^B - \frac{1}{2}\mathcal{D}_2 F_\alpha^B - 4\left(\mathcal{D}_1 F_\alpha^B\right)\left(\mathcal{D}_1^2 F_\alpha^B\right) + 6\left(\mathcal{D}_1^2 F_\alpha^B\right)^2 = 0, \tag{1.11}$$

with

$$F_\alpha^B := \ln \tau_\alpha^B. \tag{1.12}$$

No analogous equations were derived for the sine kernel, although in the special case where the intervals $[a_{2j-1}, a_{2j}]$ are chosen symmetrically about the origin, the Fredholm determinant τ^S may be expressed [14, 19] as a product $\tau^B_{\frac{1}{2}}\tau^B_{-\frac{1}{2}}$ of two Bessel kernel determinants.

In the case of a single interval, it is easy to see that equations (1.8) and (1.11) just give the τ-function form of the Painlevé equations P_{II} and P_V, respectively, to which the Tracy–Widom systems reduce in the case of the Airy and Bessel kernels. It seems reasonable to expect that analogous results hold for the general case, involving an arbitrary number of intervals. The purpose of this work is to show how the hierarchies of equations derived in [2, 3] can in fact be deduced directly from the Tracy–Widom Hamiltonian systems for both the Airy and Bessel cases, and to also apply this approach to the sine kernel case. The main step is to recognize that the Hamiltonian systems imply isomonodromic deformation equations for associated families of rational covariant derivative operators on the Riemann sphere. It is known [12, 13] that such isomonodromic deformations give rise to bilinear equations for indexed sets of isomonodromic τ-functions related by Schlesinger transformations. The fact that for the systems associated with the Airy and Bessel kernels such equations may be written in terms of a single scalar τ-function is due to the presence of a pair of conserved quantities, allowing the elimination of the additional variables by fixing the level sets of these invariants. In the sine kernel case this is not possible, and the associated bilinear equations therefore involve coupled systems for τ^S together with a pair of additional variables (τ_+^S, τ_-^S).

In Section 2, equations (1.8) and (1.11) are first derived directly from the Hamiltonian systems of [18, 19]. In Section 3, it is shown how the isomonodromic deformation equations following from the associated Hamiltonian systems may be used to derive the full hierarchy of τ-function equations for all these cases. In section 4, these results are related to the rational classical R-matrix approach to isomonodromic and isospectral systems developed in [1, 8].

2 Deduction of τ-function equations from the Hamiltonian systems

To establish notation, following [17, 18, 19], we define the quantities:

$$x_{2j} := 2i\sqrt{\lambda_j}(\mathbb{I}-\hat{K})^{-1}\phi(a_{2j}), \qquad x_{2j+1} := 2\sqrt{\lambda_j}(\mathbb{I}-\hat{K})^{-1}\phi(a_{2j+1}), \tag{2.1a}$$

$$y_{2j} := i\sqrt{\lambda_j}(\mathbb{I}-\hat{K})^{-1}\psi(a_{2j}), \qquad y_{2j+1} := \sqrt{\lambda_j}(\mathbb{I}-\hat{K})^{-1}\psi(a_{2j+1}), \tag{2.1b}$$

$$x_0 := 2\sum_{j=1}^{n}\lambda_j\int_{a_{2j-1}}^{a_{2j}}\phi(x)(\mathbb{I}-\hat{K})^{-1}\psi(x)dx, \tag{2.1c}$$

$$y_0 := \sum_{j=1}^{n} \lambda_j \int_{a_{2j-1}}^{a_{2j}} \phi(x)(\mathbb{I} - \hat{K})^{-1} \phi(x) dx, \tag{2.1d}$$

where, for the case of the sine kernel $\hat{K} = \hat{K}^S$,

$$\phi(x) := \frac{\sin(\pi x)}{\pi}, \qquad \psi(x) := \cos(\pi x). \tag{2.2}$$

while for the Airy kernel $\hat{K} = \hat{K}^A$,

$$\phi(x) := Ai(x), \qquad \psi(x) := \frac{dAi(x)}{dx}, \tag{2.3}$$

and for the Bessel kernel $\hat{K} = \hat{K}^B_\alpha$,

$$\phi(x) := J_\alpha(\sqrt{x}), \qquad \psi(x) := x \frac{dJ_\alpha(\sqrt{x})}{dx}. \tag{2.4}$$

(An odd number of variables may also occur if we set one of the a_j's equal to some fixed constant, say 0 or ∞, and eliminate the corresponding pair (q_j, p_j).) As shown in [17, 18, 19], the logarithmic derivatives of the associated Fredholm determinants are given by:

$$G_j^S := \frac{\partial F^S}{\partial a_j} = \frac{\pi^2}{4} x_j^2 + y_j^2 - \frac{1}{4} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{(x_j y_k - y_j x_k)^2}{a_j - a_k} \tag{2.5}$$

for the sine kernel,

$$G_j^A := \frac{\partial F^A}{\partial a_j} = y_j^2 + \frac{1}{4}(x_0 - a_j) x_j^2 - y_0 x_j y_j - \frac{1}{4} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{(x_j y_k - y_j x_k)^2}{a_j - a_k} \tag{2.6}$$

for the Airy kernel, and

$$\begin{aligned} a_j G^B_{\alpha,j} := a_j \frac{\partial F^B_\alpha}{\partial a_j} &= y_j^2 - \frac{1}{16} \left(\alpha^2 - a_j + x_0 \right) x_j^2 \\ &\quad + \frac{1}{4} y_0 x_j y_j - \frac{1}{4} \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{a_k (x_j y_k - y_j x_k)^2}{a_j - a_k} \end{aligned} \tag{2.7}$$

for the Bessel kernel.

For use in what follows, we also define the quantities

$$R_m^S := \mathcal{D}_m F^S = \sum_{j=1}^{2n} a_j^m G_j^S, \qquad m \in \mathbb{N} \tag{2.8}$$

for the sine kernel case,

$$R_m^A := \mathcal{D}_m F^A = \sum_{j=1}^{2n} a_j^m G_j^A, \qquad m \in \mathbb{N} \tag{2.9}$$

for the Airy case and

$$R^B_{\alpha,m} := \mathcal{D}_m F^B_\alpha = \sum_{j=1}^{2n} a_j^m G^B_{\alpha,j}, \qquad m \in \mathbb{N} \tag{2.10}$$

for the Bessel case. For all three cases, we define the following sequence of bilinear forms

$$P_m := \sum_{j=1}^{2n} a_j^m y_j^2, \qquad Q_m := \sum_{j=1}^{2n} a_j^m x_j^2, \qquad S_m := \sum_{j=1}^{2n} a_j^m x_j y_j, \qquad m \in \mathbb{N}. \tag{2.11}$$

As explained below, the $\{G^A_j\}$'s and $\{G^B_{\alpha,j}\}$'s may be viewed as sets of Poisson commuting, nonautonomous Hamiltonians on an auxiliary phase space with canonical coordinates $\{x_0, y_0, x_j, y_j\}$, such that the quantities defined in (2.1) satisfy the corresponding systems of Hamiltonian equations. These equations will then be shown to imply equations (1.8) and (1.11).

2.1 The Airy kernel system

The system of dynamical equations for this case is given [18] by

$$\frac{\partial x_j}{\partial a_k} = -\frac{1}{2}\frac{(x_j y_k - y_j x_k)x_k}{a_j - a_k}, \qquad j \neq k, \tag{2.12a}$$

$$\frac{\partial y_j}{\partial a_k} = -\frac{1}{2}\frac{(x_j y_k - y_j x_k)y_k}{a_j - a_k}, \qquad j \neq k, \tag{2.12b}$$

$$\frac{\partial x_j}{\partial a_j} = \frac{1}{2}\sum_{\substack{k=1\\k\neq j}}^{n} \frac{(x_j y_k - y_j x_k)x_k}{a_j - a_k} + 2y_j - y_0 x_j, \tag{2.12c}$$

$$\frac{\partial y_j}{\partial a_j} = \frac{1}{2}\sum_{\substack{k=1\\k\neq j}}^{n} \frac{(x_j y_k - y_j x_k)y_k}{a_j - a_k} + \frac{1}{2}(a_j - x_0)x_j + y_0 x_j y_j, \tag{2.12d}$$

$$\frac{\partial x_0}{\partial a_j} = -x_j y_j, \qquad \frac{\partial y_0}{\partial a_j} = -\frac{1}{4}x_j^2. \tag{2.12e}$$

Viewing the a_j's as multi-time parameters, and the quantities $\{x_0, y_0, x_j, y_j\}$ as canonical coordinates, this is a compatible system of nonautonomous Hamiltonian equations generated by the Poisson commuting Hamiltonians $\{G^A_j\}$ defined in (2.6). There is an additional functionally independent Hamiltonian, defined by

$$G^A_0 := y_0^2 - x_0 - \frac{1}{4}Q_0, \tag{2.13}$$

which also Poisson commutes with all the G^A_j's. Since G^A_0 is not explicitly dependent on the parameters $\{a_j\}$, it follows that it is a conserved quantity. Since all the quantities $\{x_0, y_0, x_j, y_j\}$ defined in (2.1) vanish in the limit $\{a_j \to \infty,\ \forall\, j\}$, the invariant G^A_0 must vanish on this particular solution. Therefore we may express x_0 in terms of the other variables as

$$x_0 = y_0^2 - \frac{1}{4}Q_0. \tag{2.14}$$

The quantity R_0^A defined in (2.9) will just be denoted

$$R := R_0^A = \sum_{j=1}^{2n} G_j^A = P_0 - \frac{1}{4}Q_1 + \frac{1}{4}y_0^2 Q_0 - y_0 S_0 - \frac{1}{16}Q_0^2, \tag{2.15}$$

where (2.14) has been used. In terms of R, equation (1.8) becomes

$$\mathcal{D}_0^3 R - 4\mathcal{D}_1 R + 2R + 6(\mathcal{D}_0 R)^2 = 0. \tag{2.16}$$

It follows from the Poisson commutativity of the Hamiltonians $\{G_j^A\}_{j=1,\dots,2n}$ that their Hamiltonian vector fields applied as derivations to R give zero, and hence along any integral surface of eqs. (2.12), the derivatives of R with respect to the a_j's are just given by its *explicit* dependence on these parameters. This just comes from the Q_1 term in expression (2.15), and therefore we have

$$\frac{\partial R}{\partial a_j} = -\frac{1}{4}x_j^2 \tag{2.17}$$

Comparing with (2.12e), this implies that

$$G_\infty^A := y_0 - R \tag{2.18}$$

is a second conserved quantity. Since in the limit $\{a_j \to \infty,\ \forall\, j\}$, both y_0 and R vanish, G_∞^A must vanish for all values of the parameters, and therefore the invariant relation

$$y_0 = R \tag{2.19}$$

is satisfied by this solution. Applying the operators $\mathcal{D}_0$, $\mathcal{D}_1$ to R, it follows from (2.17) that

$$\mathcal{D}_0 R = -\frac{1}{4}Q_0, \tag{2.20a}$$

$$\mathcal{D}_1 R = -\frac{1}{4}Q_1. \tag{2.20b}$$

Eqs. (2.12) also imply that application of $\mathcal{D}_0$ to $\{Q_0, S_0, x_0, y_0, Q_1\}$ gives

$$\mathcal{D}_0 Q_0 = 4S_0 - 2y_0 Q_0, \qquad \mathcal{D}_0 S_0 = \frac{1}{2}Q_1 - \frac{1}{2}x_0 Q_0 + 2P_0, \tag{2.21a}$$

$$\mathcal{D}_0 x_0 = -S_0, \qquad \mathcal{D}_0 y_0 = -\frac{1}{4}Q_0, \tag{2.21b}$$

$$\mathcal{D}_0 Q_1 = Q_0 + 4S_1 - 2y_0 Q_1. \tag{2.21c}$$

Further application of $\mathcal{D}_0$ and $\mathcal{D}_1$, using (2.20a), (2.21) and (2.14), therefore gives

$$\mathcal{D}_0^2 R = \frac{1}{2}y_0 Q_0 - S_0, \tag{2.22a}$$

$$\mathcal{D}_0^3 R = -\frac{1}{2}Q_1 + 2y_0 S_0 - \frac{1}{2}y_0^2 Q_0 - \frac{1}{4}Q_0^2 - 2P_0. \tag{2.22b}$$

Substituting (2.15), (2.20), (2.22b), into (2.16) and using (2.14) shows that all terms cancel, verifying the equation.

2.2 The Bessel kernel system

In this case, the system of dynamical equations is given [19] by

$$\frac{\partial x_j}{\partial a_k} = -\frac{1}{2}\frac{(x_j y_k - y_j x_k)x_k}{a_j - a_k}, \qquad j \neq k, \tag{2.23a}$$

$$\frac{\partial y_j}{\partial a_k} = -\frac{1}{2}\frac{(x_j y_k - y_j x_k)y_k}{a_j - a_k}, \qquad j \neq k, \tag{2.23b}$$

$$a_j\frac{\partial x_j}{\partial a_j} = \frac{1}{2}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{a_k(x_j y_k - y_j x_k)x_k}{a_j - a_k} + 2y_j + \frac{1}{4}y_0 x_j, \tag{2.23c}$$

$$a_j\frac{\partial y_j}{\partial a_j} = \frac{1}{2}\sum_{\substack{k=1\\k\neq j}}^{n}\frac{a_k(x_j y_k - y_j x_k)y_k}{a_j - a_k} + \frac{1}{8}(\alpha^2 - a_j + x_0)x_j - \frac{1}{4}y_0 y_j, \tag{2.23d}$$

$$\frac{\partial x_0}{\partial a_j} = -x_j y_j, \qquad \frac{\partial y_0}{\partial a_j} = -\frac{1}{4}x_j^2. \tag{2.23e}$$

This is again a compatible system of nonautonomous Hamiltonian equations generated by the Poisson commuting Hamiltonians $a_j G^B_{\alpha,j}$ defined in (2.7), provided the Poisson brackets are defined by

$$\{x_j,\ y_k\} = \frac{1}{a_j}\delta_{jk}, \qquad \{x_0,\ y_0\} = -4. \tag{2.24}$$

There again exist two additional conserved quantities for this case. The first is defined by

$$G^B_0 := x_0 + \frac{1}{4}y_0^2 + y_0 + \frac{1}{4}Q_1, \tag{2.25}$$

as may be seen directly by differentiating with respect to the a_j's, using (2.23). Since all the quantities appearing in (2.25) vanish in the limit $\{a_j \to 0,\ \forall\ j\}$, this difference must vanish, and therefore the invariant relation

$$x_0 = -\frac{1}{4}y_0^2 - y_0 - \frac{1}{4}Q_1 \tag{2.26}$$

is satisfied for this solution. The second conserved quantity is

$$\begin{aligned} G^B_\infty &:= y_0 + 4\sum_{j=1}^{2n} a_j G^B_{\alpha,j} = y_0 + 4R^B_{\alpha,1} \\ &= y_0 - \frac{1}{4}\left(\alpha^2 + x_0\right)Q_0 + \frac{1}{4}Q_1 + y_0 S_0 + 4P_0 + Q_0 P_0 - S_0^2, \end{aligned} \tag{2.27}$$

Again, due to the Poisson commutativity of the Hamiltonians defined in (2.7), the Hamiltonian vector fields generating the a_j deformations when applied to the term $R^B_{\alpha,1}$ give zero, and therefore only the explicit dependence of this term upon the parameters need be taken into account when verifying that differentiation of the sum gives zero. Since all the quantities appearing in (2.27) vanish in the limit $\{a_j \to 0,\ \forall\ j\}$, the invariant G^B_∞ must also vanish on this particular solution, and we therefore have the relation

$$\begin{aligned} y_0 &= -4R^B_{\alpha,1} = -4\mathcal{D}_1 F^B_\alpha \\ &= \frac{1}{4}\left(\alpha^2 + x_0\right) Q_0 - \frac{1}{4} Q_1 - y_0 S_0 - 4P_0 - Q_0 P_0 + S_0^2. \end{aligned} \tag{2.28}$$

The quantities $R^B_{\alpha,1}$, $R^B_{\alpha,2}$ are given by

$$\begin{aligned} R^B_{\alpha,1} &= \mathcal{D}_1 F^B_\alpha = \sum_{j=1}^{2n} a_j G^B_{\alpha,j} \\ &= -\frac{1}{16}\left(\alpha^2 + x_0\right) Q_0 + \frac{1}{16} Q_1 + \frac{1}{4} y_0 S_0 + P_0 + \frac{1}{4} Q_0 P_0 - \frac{1}{4} S_0^2, \end{aligned} \tag{2.29a}$$

$$\begin{aligned} R^B_{\alpha,2} &= \mathcal{D}_2 F^B_\alpha = \sum_{j=1}^{2n} a_j^2 G^B_{\alpha,j} \\ &= -\frac{1}{16}\left(\alpha^2 + x_0\right) Q_1 + \frac{1}{16} Q_2 + \frac{1}{4} y_0 S_1 + P_1. \end{aligned} \tag{2.29b}$$

It again follows from the Poisson commutativity of the Hamiltonians $\{G^B_{\alpha,j}\}$ that the derivatives of $R^B_{\alpha,1}$ and $R^B_{\alpha,2}$ with respect to the parameters are given by their explicit dependence on these parameters, and hence

$$\mathcal{D}_1^2 F^B_\alpha = \mathcal{D}_1 R^B_{\alpha,1} = \frac{1}{16} Q_1, \tag{2.30a}$$

$$\mathcal{D}_2 \mathcal{D}_1 F^B_\alpha = \mathcal{D}_2 R^B_{\alpha,1} = \frac{1}{16} Q_2. \tag{2.30b}$$

From (2.23), application of $\mathcal{D}_1$ to $\{Q_1, S_1, x_0, y_0\}$ gives

$$\mathcal{D}_1 Q_1 = Q_1 + 4S_1 + \frac{1}{2} y_0 Q_1, \qquad \mathcal{D}_1 S_1 = S_1 + \frac{1}{8}\left(\alpha^2 + x_0\right) Q_1 - \frac{1}{8} Q_2 + 2P_1, \tag{2.31a}$$

$$\mathcal{D}_1 x_0 = -S_1, \qquad \mathcal{D}_1 y_0 = -\frac{1}{4} Q_1. \tag{2.31b}$$

Further application of $\mathcal{D}_1$, using (2.20a), (2.31), and (2.26) therefore gives

$$\mathcal{D}_1^3 F^B_\alpha = \frac{1}{16}\left(1 + \frac{y_0}{2}\right) Q_1 + \frac{1}{4} S_1, \tag{2.32a}$$

$$\begin{aligned} \mathcal{D}_1^4 F^B_\alpha = \frac{1}{16}&\left(1 + \frac{\alpha^2}{2} + \frac{y_0}{2} + \frac{y_0^2}{8}\right) Q_1 \\ &+ \frac{1}{2} P_1 + \left(\frac{1}{2} + \frac{y_0}{8}\right) S_1 - \frac{1}{64} Q_1^2 - \frac{1}{32} Q_2. \end{aligned} \tag{2.32b}$$

Substitution of (2.29b), (2.30), (2.32) into (1.11), and use of (2.28) to replace the term $-4\mathcal{D}_1 F^B_\alpha$ by y_0, and (2.26) to eliminate x_0, shows that all the terms cancel, verifying the equation.

3 Deduction of the τ-function equations from isomonodromic deformations

In this section, we show how the full hierarchies of equations derived in [2, 3] may be deduced from the Hamiltonian systems (2.12), (2.23) and also how the corresponding hierarchy is deduced for the case of the sine kernel. The key step is to recast these systems as isomonodromic deformation equations for an associated differential operator in an auxiliary spectral variable $z \in \mathbb{P}^1$, having rational coefficients with poles at the points $\{z = a_j\}$, and to interpret the Fredholm determinants τ^S, τ^A and τ^B_α as isomonodromic τ-functions.

3.1 The Airy kernel isomonodromic system

The Hamiltonian system (2.12) implies that the compatibility conditions

$$\frac{\partial A_j}{\partial a_k} = \frac{[A_j, A_k]}{a_j - a_k}, \qquad j \neq k, \tag{3.1a}$$

$$\frac{\partial A_j}{\partial a_j} = [a_j B + C, A_j] - \sum_{\substack{k=1\\k\neq j}}^{2n} \frac{[A_j, A_k]}{a_j - a_k}, \tag{3.1b}$$

$$\frac{\partial C}{\partial a_j} = [B, A_j] \tag{3.1c}$$

are satisfied for the following overdetermined system [9]

$$\frac{\partial \Psi^A}{\partial z} = X^A(z)\Psi^A, \tag{3.2a}$$

$$\frac{\partial \Psi^A}{\partial a_j} = -\frac{A_j}{z - a_j}\Psi^A, \qquad j = 1, \dots 2n, \tag{3.2b}$$

$$X^A(z) := zB + C + \sum_{j=1}^{2n} \frac{A_j}{z - a_j}, \tag{3.2c}$$

where $\Psi^A(z, a_1, \dots a_{2n})$ is a 2×2 matrix, invertible where defined, and

$$A_j := -\frac{1}{2}\begin{pmatrix} x_j y_j & y_j^2 \\ -x_j^2 & -x_j y_j \end{pmatrix}, \tag{3.3a}$$

$$B := \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}, \qquad C := \begin{pmatrix} y_0 & \frac{x_0}{2} \\ -2 & -y_0 \end{pmatrix}. \tag{3.3b}$$

This implies the invariance of the monodromy of the operator $\frac{\partial}{\partial z} - X^A(z)$ under changes in the parameters $\{a_j\}$. In view of eq. (2.6), according to the constructions of [12, 13], the Fredholm determinant τ^A is just the isomonodromic τ-function of the system (3.1)–(3.2).

Now define the sequence of 2×2 matrices

$$B_m := \sum_{j=1}^{2n} a_j^m A_j = -\frac{1}{2}\begin{pmatrix} S_m & P_m \\ -Q_m & -S_m \end{pmatrix}, \qquad m \in \mathbb{N}, \tag{3.4}$$

where the quantities P_m, Q_m, S_m were defined in (2.11). Expanding $X^A(z)$ for large z gives

$$X^A(z) = zB + C + \sum_{m=0}^{\infty} \frac{B_m}{z^{m+1}}. \tag{3.5}$$

Since

$$G_j^A = \frac{1}{2} \operatorname{res}_{z=a_j} \operatorname{tr} \left(\left(X^A\right)^2 (z) \right), \tag{3.6}$$

and

$$G_0^A = \frac{1}{2} \operatorname{res}_{z=\infty} \frac{1}{z} \operatorname{tr} \left(\left(X^A\right)^2 (z) \right), \tag{3.7}$$

we have

$$\frac{1}{2} \operatorname{tr} \left(\left(X^A\right)^2 (z) \right) = z + G_0^A + \sum_{m=0}^{\infty} \frac{R_m^A}{z^{m+1}}, \tag{3.8}$$

where

$$R_m^A := \sum_{j=1}^{2n} a_j^m G_j^A = \operatorname{tr}\left(BB_{m+1} + CB_m\right) + \frac{1}{2} \operatorname{tr} \sum_{k=0}^{m-1} B_k B_{m-k-1} \tag{3.9}$$

(with the last term absent if $m = 0$) are the quantities defined in (2.9).

Using the fact that the Hamiltonian vector fields generating the a_j deformations give zero when applied to the G_j^A's, and hence also the R_m^A's, it follows that the effect of applying the operators $\mathcal{D}_k$ to R_m^A gives just the explicit derivatives,

$$\mathcal{D}_k R_m^A = (m+1) \operatorname{tr}\left(BB_{m+k}\right) + m \operatorname{tr}\left(CB_{m+k-1}\right) + \sum_{l=1}^{m-1} l \operatorname{tr}\left(B_{l+k-1} B_{m-l-1}\right) \tag{3.10}$$

(with the sum in the last term absent if $m = 0$ and the second term absent if $m + k = 0$).

Applying the operator $\mathcal{D}_m$ to Ψ^A, using (3.2b) and (3.4) gives the sequence of equations

$$\mathcal{D}_m \Psi^A = -\sum_{k=0}^{\infty} \frac{B_{m+k}}{z^{k+1}} \Psi^A, \qquad m \in \mathbb{N}. \tag{3.11}$$

The compatibility of these equations with (3.2a) implies the following equations for the matrices $\{B_m, C\}$.

$$\mathcal{D}_k B_m = m B_{m+k-1} + [C, B_{m+k}] + [B, B_{m+k+1}] + \sum_{l=0}^{m-1} [B_l, B_{m+k-l-1}], \tag{3.12a}$$

$$\mathcal{D}_k C = [B, B_k], \qquad k, m \in \mathbb{N} \tag{3.12b}$$

(where the first term of (3.12a) is absent if $m+k = 0$ and the last term is absent if $m = 0$).

The strategy for deriving the hierarchy of equations for τ^A is to now choose a k-value (k_1) in (3.10), (3.12) and use these equations, together with (3.9) to express all the relevant matrix elements of the B_m's for $m \leq k$ in terms of the R_k's for $k < k_1$ and the corresponding $\mathcal{D}_k$'s applied repeatedly to them. Equations (3.12), for $k = k_1$ may then be expressed entirely in terms of these quantities, and hence in terms of repeated applications of the operators $\mathcal{D}_k$ to $F^A = \ln \tau^A$. An essential step in this procedure is to also eliminate the additional variables x_0, y_0 from the equations through use of the invariant conditions (2.14), (2.19).

For example, choosing $k_1 = 1$, we note that for $m = 0$, eq. (3.9) reduces to (2.15) while for $k = 0, 1$ and $m = 0$, (3.10) reduces to (2.20) and for $k = 0, m = 0$, eqs. (3.12) give (2.21a), (2.21b). Combining these with the invariant relations (2.14), (2.19) allows us to express the relevant matrix elements of C, B_0 and B_1 as

$$x_0 = \mathcal{D}_0 R + R^2, \qquad y_0 = R, \tag{3.13a}$$

$$Q_0 = -4\mathcal{D}_0 R, \qquad S_0 = -2R\mathcal{D}_0 R - \mathcal{D}_0^2 R, \tag{3.13b}$$

$$P_0 = \frac{1}{2}R - \frac{1}{4}\mathcal{D}_0^3 R - R\mathcal{D}_0^2 R - \frac{1}{2}(\mathcal{D}_0 R)^2 - R^2\mathcal{D}_0 R, \tag{3.13c}$$

$$Q_1 = -2R - 6(\mathcal{D}_0 R)^2 - \mathcal{D}_0^3 R. \tag{3.13d}$$

Substituting these in eq. (3.12b) for $k = 1$ gives (2.16). Similarly, eq. (3.12a) for $k = 1$, $m = 0$ and eq. (3.9) for $m = 1$ produce the following expressions for the relevant matrix elements of B_1 and B_2.

$$S_1 = -\mathcal{D}_1\mathcal{D}_0 R - R^2 - 3R(\mathcal{D}_0 R)^2 - R\mathcal{D}_0^3 R, \tag{3.14a}$$

$$\begin{aligned} Q_2 = {} & -2R_1 - \mathcal{D}_1\mathcal{D}_0^2 R - 2(\mathcal{D}_0 R)(\mathcal{D}_1 R) - R\mathcal{D}_0 R - \frac{3}{2}R\mathcal{D}_1\mathcal{D}_0 R - \frac{3}{2}(\mathcal{D}_0 R)\left(\mathcal{D}_0^3\right) R \\ & + \frac{1}{2}\left(\mathcal{D}_0^2 R\right)^2 - \frac{3}{2}R^3 - \frac{1}{2}R^2\mathcal{D}_0^3 R - 7(\mathcal{D}_0 R)^3 - \frac{9}{2}R^2(\mathcal{D}_0 R)^2. \end{aligned} \tag{3.14b}$$

Substitution of (3.14b) in eq. (3.12a) (or (3.10)) for $k = 2, m = 0$, thus gives

$$\begin{aligned} & 4\mathcal{D}_1 R - 2R_1 - \mathcal{D}_1\mathcal{D}_0^2 R - 2(\mathcal{D}_0 R)(\mathcal{D}_1 R) - R\mathcal{D}_0 R - \frac{3}{2}R\mathcal{D}_1\mathcal{D}_0 R - \frac{3}{2}(\mathcal{D}_0 R)\left(\mathcal{D}_0^3\right) R \\ & + \frac{1}{2}\left(\mathcal{D}_0^2 R\right)^2 - \frac{3}{2}R^3 - \frac{1}{2}R^2\mathcal{D}_0^3 R - 7(\mathcal{D}_0 R)^3 - \frac{9}{2}R^2(\mathcal{D}_0 R)^2 = 0. \end{aligned} \tag{3.15}$$

as the next equation of the hierarchy. The remaining equations may similarly be expressed in terms of the derivations $\mathcal{D}_k$ acting upon F^A.

3.2 The Bessel kernel isomonodromic system

The Bessel kernel case is so similar to the above that only the pertinent equations will be given, without repeating any details of the procedure. Define for this case, the matrices

$$X^B(z) := \widetilde{B} + \frac{C_\alpha - \sum_{j=1}^{2n} A_j}{z} + \sum_{j=1}^{2n} \frac{A_j}{z - a_j}, \tag{3.16a}$$

$$\widetilde{B} := \begin{pmatrix} 0 & \frac{1}{8} \\ 0 & 0 \end{pmatrix}, \qquad C_\alpha := -\frac{1}{4}\begin{pmatrix} y_0 & \frac{1}{2}\left(x_0 + \alpha^2\right) \\ 8 & -y_0 \end{pmatrix}. \tag{3.16b}$$

where the A_j's are again defined as in (3.3a).

The Hamiltonian system (2.23) implies that the compatibility conditions

$$\frac{\partial A_j}{\partial a_k} = \frac{[A_j, A_k]}{a_j - a_k}, \qquad j \neq k, \tag{3.17a}$$

$$a_j \frac{\partial A_j}{\partial a_j} = [C_\alpha + a_j \widetilde{B}, A_j] - \sum_{\substack{k=1 \\ k\neq j}}^{2n} \frac{a_k [A_j, A_k]}{a_j - a_k}, \tag{3.17b}$$

$$\frac{\partial C_\alpha}{\partial a_j} = [\widetilde{B}, A_j] \tag{3.17c}$$

are satisfied for the system

$$\frac{\partial \Psi^B}{\partial z} = X^B(z) \Psi^B, \tag{3.18a}$$

$$\frac{\partial \Psi^B}{\partial a_j} = -\frac{A_j}{z - a_j} \Psi^B, \qquad j = 1, \ldots 2n, \tag{3.18b}$$

where $\Psi^B(z, a_1, \ldots, a_{2n})$ is again a 2×2 matrix, invertible where defined. This again implies the invariance of the monodromy of the operator $\frac{\partial}{\partial z} - X^B(z)$ under changes in the parameters $\{a_j\}$. In view of eq. (2.7), the Fredholm determinant τ_α^B is again an isomonodromic τ-function for the system (3.17)–(3.18).

Defining the sequence of 2×2 matrices $\{B_m,\ m \in \mathbb{N}\}$ as in (3.4), and expanding $X^B(z)$ for large z gives

$$X^B(z) = \widetilde{B} + \frac{C_\alpha}{z} + \sum_{m=1}^{\infty} \frac{B_m}{z^{m+1}}, \tag{3.19}$$

and

$$\frac{1}{2} \operatorname{tr} \left(\left(X^B \right)^2 (z) \right) = -\frac{1}{4} z + \frac{G_0^B - G_\infty^B + \alpha^2}{4z^2} + \sum_{m=1}^{\infty} \frac{R_{\alpha,m}^B}{z^{m+1}}, \tag{3.20}$$

where

$$R_{\alpha,1}^B = \frac{1}{4} \left(G_\infty^B - G_0^B - \alpha^2 \right) + \frac{1}{2} \operatorname{tr} \left(C_\alpha^2 + 2 \widetilde{B} B_1 \right), \tag{3.21a}$$

$$R_{\alpha,m}^B = \widetilde{tr} \left(\widetilde{B} B_m + C_\alpha B_{m-1} \right) + \frac{1}{2} \operatorname{tr} \sum_{k=1}^{m-2} B_k B_{m-k-1}, \qquad m \geq 2 \tag{3.21b}$$

are the quantities defined in (2.10) and G_0^B, G_∞^B are the conserved quatities defined in (2.25), (2.27), which vanish on the particular solutions defined by (2.1).

The fact that the Hamiltonian vector fields generating the a_j deformations give zero when applied to the $G_{\alpha,j}^B$'s, and $R_{\alpha,m}^B$'s again implies that the effect of applying the operators $\mathcal{D}_k$ to the $R_{\alpha,m}^B$'s is to evaluate only explicit derivatives with respect to the parameters, giving

$$\mathcal{D}_k R^B_{\alpha,1} = \frac{1}{2}\operatorname{tr}\left(\widetilde{B}B_k\right),$$

$$\mathcal{D}_k R^B_{\alpha,m} = m\operatorname{tr}\left(\widetilde{B}B_{m+k-1}\right) + (m-1)\operatorname{tr}\left(C_\alpha B_{m+k-2}\right) + \sum_{l=1}^{m-2} l\operatorname{tr}(B_{l+k-1}B_{m-l-1}), \qquad m \geq 2 \tag{3.22}$$

(with the sum in the last term absent if $m = 2$).

Applying the operator $\mathcal{D}_m$ to Ψ^B, using (3.4) and (3.18b), again gives the sequence of equations

$$\mathcal{D}_m \Psi^B = -\sum_{k=0}^{\infty} \frac{B_{m+k}}{z^{k+1}}\Psi^B, \qquad m \in \mathbb{N}. \tag{3.23}$$

whose compatibility with (3.18a) implies the following equations for the matrices $\{B_m, C_\alpha\}$,

$$\mathcal{D}_k B_m = mB_{m+k-1} + [C_\alpha, B_{m+k-1}] + [\widetilde{B}, B_{m+k}] + \sum_{l=1}^{m-1}[B_l, B_{m+k-l-1}], \tag{3.24a}$$

$$\mathcal{D}_k C_\alpha = [\widetilde{B}, B_k], \qquad k, m \in \mathbb{N}, \quad m \geq 1. \tag{3.24b}$$

The hierarchy of equations for τ^B_α is derived in the same way as for the Airy case. For example, eqs. (3.21) for $k = 2$ reduce to (2.29), while (3.22) for $k = 1, 2$, $m = 1$ reduces to (2.30), and eqs. (3.24) for $k = 1, 2$, $m = 1$ give (2.31). Combining these with the invariant relations (2.26), (2.28) allows us to express the relevant matrix elements of C_α, B_1 and B_2 as

$$x_0 = -4\left(\mathcal{D}_1 R^B_{\alpha,1} + \left(R^B_{\alpha,1}\right)^2 - 4R^B_{\alpha,1}\right), \qquad y_0 = -4R^B_{\alpha,1}, \tag{3.25a}$$

$$Q_1 = 16\mathcal{D}_1 R^B_{\alpha,1}, \tag{3.25b}$$

$$S_1 = 8R^B_{\alpha,1}\mathcal{D}_1 R^B_{\alpha,1} - 4\mathcal{D}_1 R^B_{\alpha,1} + 4\mathcal{D}_1^2 R^B_{\alpha,1}, \tag{3.25c}$$

$$P_1 = R^B_{\alpha,2} + \alpha^2 R^B_{\alpha,1} + 4\left(R^B_{\alpha,1}\right)^2 \mathcal{D}_1 R^B_{\alpha,1} - 4\left(\mathcal{D}_1 R^B_{\alpha,1}\right)^2 + 4R^B_{\alpha,1}\mathcal{D}_1^2 R^B_{\alpha,1} - \mathcal{D}_2 R^B_{\alpha,1}, \tag{3.25d}$$

$$Q_2 = 16\mathcal{D}_2 R^B_{\alpha,1}. \tag{3.25e}$$

Substituting these in eqs. (3.24) for $k = 2$ gives (1.11). Similar calculations for higher values of k yield the further equations of the Bessel hierachy.

3.3 The sine kernel system

For this case, the quantities defined in (2.1a)–(2.1b) satisfy the system of dynamical equations defined in [11, 17]

$$\frac{\partial x_j}{\partial a_k} = -\frac{1}{2}\frac{(x_j y_k - y_j x_k)x_k}{a_j - a_k}, \qquad j \neq k, \tag{3.26a}$$

$$\frac{\partial y_j}{\partial a_k} = -\frac{1}{2}\frac{(x_j y_k - y_j x_k)y_k}{a_j - a_k}, \tag{3.26b}$$

$$\frac{\partial x_j}{\partial a_j} = \frac{1}{2}\sum_{\substack{k=1\\k\neq j}}^{n} \frac{(x_j y_k - y_j x_k)x_k}{a_j - a_k} + 2y_j, \tag{3.26c}$$

$$\frac{\partial y_j}{\partial a_j} = \frac{1}{2}\sum_{\substack{k=1\\k\neq j}}^{n} \frac{(x_j y_k - y_j x_k)y_k}{a_j - a_k} - \frac{\pi^2}{2}x_j. \tag{3.26d}$$

This is again a compatible system of nonautonomous Hamiltonian equations, generated by the Poisson commuting Hamiltonians $\{G_j^S\}$ defined in (2.5). They imply that the compatibility conditions

$$\frac{\partial A_j}{\partial a_k} = \frac{[A_j, A_k]}{a_j - a_k}, \qquad j \neq k, \tag{3.27a}$$

$$\frac{\partial A_j}{\partial a_j} = [B_S, A_j] - \sum_{\substack{k=1\\k\neq j}}^{2n} \frac{[A_j, A_k]}{a_j - a_k}, \qquad j \neq k \tag{3.27b}$$

are satisfied for the system

$$\frac{\partial \Psi^S}{\partial z} = X^S(z)\Psi^S, \tag{3.28a}$$

$$\frac{\partial \Psi^S}{\partial a_j} = -\frac{A_j}{z - a_j}\Psi^S, \qquad j = 1, \ldots, 2n, \tag{3.28b}$$

where

$$X^S(z) := B_S + \sum_{j=1}^{2n} \frac{A_j}{z - a_j}, \tag{3.29a}$$

$$B_S := \begin{pmatrix} 0 & \frac{\pi^2}{2} \\ -2 & 0 \end{pmatrix}, \tag{3.29b}$$

with the A_j's again defined as in (3.4). As in the previous cases, this implies the invariance of the monodromy of the operator $\frac{\partial}{\partial z} - X^S(z)$. In view of eq. (2.5), the Fredholm determinant τ^S is an isomonodromic τ-function for the system (3.27)–(3.28).

Expanding $X^S(z)$ for large z gives

$$X^S(z) = B_S + \sum_{m=0}^{\infty} \frac{B_m}{z^{m+1}}, \tag{3.30}$$

with the matrices $\{B_m,\ m \in \mathbb{N}\}$ again defined as in (3.4), and

$$\frac{1}{2}\operatorname{tr}\left(\left(X^S\right)^2(z)\right) = -\pi^2 + \sum_{m=0}^{\infty} \frac{R_m^S}{z^{m+1}}, \tag{3.31}$$

where

$$R_m^S := \sum_{j=1}^{2n} a_j^m G_j^S = \operatorname{tr}(B_S B_m) + \frac{1}{2}\operatorname{tr}\sum_{k=0}^{m-1} B_k B_{m-k-1}, \qquad m \in \mathbb{N}. \tag{3.32}$$

Applying the operators $\mathcal{D}_k$ to R^S_m again just differentiates explicitly with respect to the parameters, giving

$$\mathcal{D}_k R^S_m = m\,\mathrm{tr}\,(B_S B_{m+k-1}) + \sum_{l=1}^{m-1} l\,\mathrm{tr}\,(B_{l+k-1}B_{m-l-1}) \tag{3.33}$$

(with the first term absent if $k+m=0$ and the sum in the last term absent if $m=0$).

Applying $\mathcal{D}_m$ to Ψ^S, using (3.30) and (3.28b), gives the sequence of equations

$$\mathcal{D}_m \Psi^S = -\sum_{k=0}^{\infty} \frac{B_{m+k}}{z^{k+1}} \Psi^S, \qquad m \in \mathbb{N}, \tag{3.34}$$

whose compatibility with (3.28a) implies the following equations for the matrices $\{B_m\}$,

$$\mathcal{D}_k B_m = mB_{m+k-1} + [B_S, B_{m+k}] + \sum_{l=0}^{m-1} [B_l, B_{m+k-l-1}]. \tag{3.35}$$

The hierarchy of equations for τ^S is derived in the same way as for the Airy and Bessel cases, except that we no longer have two conserved quantities like $G_0^{A,B}$, $G_\infty^{A,B}$. To derive a closed system of equations, we are obliged to include two further dependent variables $\tau^S_\pm$, which we choose as the nonvanishing entries of the matrix $[B_S, B_0]\tau^S$,

$$\tau^S_+ := \left(2P_0 - \frac{\pi^2}{2}Q_0\right)\tau^S, \qquad \tau^S_- := S_0\tau^S. \tag{3.36}$$

The remaining component of B_0, which cancels in the commutator $[B_S, B_0]$, is

$$R^S_0 = \mathrm{tr}\,(B_S B_0) = P_0 + \frac{\pi^2}{4}Q_0 = 0, \tag{3.37}$$

where the first equality follows from choosing $m=0$ in (3.32). This provides a single conserved quantity that vanishes for the particular solution defined by (2.1a)–(2.1b).

To derive the hierarchy of τ-function equations, we first combine eqs. (3.36)–(3.37), which allows us to express the matrix elements of B_0 as

$$Q_0 = -\frac{\tau^S_+}{\pi^2\tau^S}, \qquad P_0 = \frac{\tau^S_+}{4\tau^S}, \qquad S_0 = \frac{\tau^S_-}{\tau^S}. \tag{3.38}$$

Eq. (3.35) for $k=0$, $m=0$ gives

$$\mathcal{D}_0 P_0 = -\pi^2 S_0, \qquad \mathcal{D}_0 Q_0 = 4S_0, \qquad \mathcal{D}_0 S_0 = 2P_0 - \frac{\pi^2}{2}Q_0, \tag{3.39}$$

and substituting (3.37), (3.38) in (3.39) gives

$$\mathcal{D}_0 \tau^S = 0, \tag{3.40a}$$

$$\mathcal{D}_0 \tau^S_- = \tau^S_+, \qquad \mathcal{D}_0 \tau^S_+ = -4\pi^2\tau^S_-. \tag{3.40b}$$

These equations are the lowest ones in the sine kernel hierarchy; note that they are linear because of the vanishing of the invariant R_0^S. To obtain higher, nonlinear equations, we first note that eq. (3.32) for $m = 1$ gives

$$R_1^S = P_1 + \frac{\pi^2}{4} Q_1 + \frac{1}{4}\left(S_0^2 - Q_0 P_0\right), \tag{3.41}$$

while (3.33) for $k = 0, 1$, $m = 1$ reduces to

$$\mathcal{D}_0 R_1^S = R_0 = 0, \tag{3.42a}$$

$$\mathcal{D}_1 R_1^S = P_1 + \frac{\pi^2}{4} Q_1. \tag{3.42b}$$

The first of these just gives the equation

$$\mathcal{D}_0 \mathcal{D}_1 \tau^S = 0, \tag{3.43}$$

which already follows from (3.40a). The second, combined with eq. (3.41) and eq. (3.35) for $k = 1$, $m = 0$ gives the further equation

$$\tau^S \mathcal{D}_1^2 \tau^S - \left(\mathcal{D}_1 \tau^S\right)^2 = \tau^S \mathcal{D}_1 \tau^S - \frac{1}{4}\left(\tau_-^S\right)^2 - \frac{1}{16\pi^2}\left(\tau_+^S\right)^2. \tag{3.44}$$

Equation (3.35) for $k = 1$, $m = 0$ gives

$$\mathcal{D}_1 S_0 = 2P_1 - \frac{\pi^2}{2} Q_1, \qquad \mathcal{D}_1 P_0 = -\pi^2 S_1, \qquad \mathcal{D}_1 Q_0 = 4 S_1. \tag{3.45}$$

Solving these, together with (3.42b), gives the following expressions for the matrix entries of B_1:

$$Q_1 = \frac{2}{\pi^2}\frac{\mathcal{D}_1 \tau^S}{\tau^S} - \frac{1}{2\pi^2}\left(\frac{\tau_-^S}{\tau^S}\right)^2 - \frac{1}{8\pi^4}\left(\frac{\tau_+^S}{\tau^S}\right)^2 - \frac{1}{\pi^2}\mathcal{D}_1\left(\frac{\tau_-^S}{\tau^S}\right), \tag{3.46a}$$

$$P_1 = \frac{1}{2}\frac{\mathcal{D}_1 \tau^S}{\tau^S} - \frac{1}{8}\left(\frac{\tau_-^S}{\tau^S}\right)^2 - \frac{1}{32\pi^2}\left(\frac{\tau_+^S}{\tau^S}\right)^2 + \frac{1}{4}\mathcal{D}_1\left(\frac{\tau_-^S}{\tau^S}\right), \tag{3.46b}$$

$$S_1 = -\frac{1}{4\pi^2}\mathcal{D}_1\left(\frac{\tau_+^S}{\tau^S}\right). \tag{3.46c}$$

Combining eq. (3.35) for $(k = 1,\ m = 1)$ and for $(k = 2,\ m = 0)$ gives

$$\mathcal{D}_2 Q_0 = \mathcal{D}_1 Q_1 - Q_1, \qquad \mathcal{D}_2 P_0 = \mathcal{D}_1 P_1 - P_1, \qquad \mathcal{D}_2 S_0 = \mathcal{D}_1 S_1 - S_1, \tag{3.47}$$

Substitution of (3.38), (3.46) into (3.47) gives the next equations of the hierarchy. Repeating this procedure for higher (k, m) values similarly generates the higher equations.

4 Classical R-matrix approach and relation to isospectral flows

In [2, 3], a key step in deriving the hierarchies of equations for the Fredholm determinants τ^A and τ_α^B was to begin with certain bilinear equations satisfied by KP τ-functions with respect to the flow parameters $\{t_1, t_2, \ldots\}$ and to then use Virasoro constraints to replace the t_m-derivations at vanishing t-values by the operators $\mathcal{D}_m$. In this section, we show how the classical R-matrix approach to the underlying isomonodromic deformation equations developed in [8] provides a direct link with commuting isospectral flows in the loop algebra $\widetilde{\mathfrak{sl}}(2)$, without the requirement that these arise as reduced KP flows. This fits into the broader framework of commutative isospectral flows in loop algebras with respect to the rational R-matrix Poisson (or Adler–Kostant–Symes) structure [15, 4, 1, 8] (and allows us to include the sine kernel case, which does not appear as a reduced KP flow).

First we recall [8, 9] that the isomonodromic deformation equations (3.1), (3.17), (3.27) may be viewed as Hamiltonian equations on the space of sets $\{A_j\}_{j=1,\ldots,2n}$ of $\mathfrak{sl}(2)$ elements, with respect to the Lie Poisson bracket, extended in the Airy and Bessel cases by the canonical variables (x_0, y_0). (The particular form (3.3a) for the A_j's just represents a canonical parametrization on the symplectic leaves for which the Casimir invariants $\{\mathrm{tr}\, A_j^2\}$ all vanish.) The formulae (3.2c), (3.16a), (3.29a) define a Poisson embedding of this space into the space $\widetilde{\mathfrak{sl}}(2)^*_R$ of rational, traceless 2×2 matrices depending rationally on the auxiliary loop variable z, with respect to the Lie Poisson bracket on $\widetilde{\mathfrak{sl}}(2)$ corresponding to the Lie bracket:

$$[X, Y]_R := \frac{1}{2}[RX, Y] + \frac{1}{2}[X, RY], \tag{4.1}$$

where

$$R := P_+ - P_- \tag{4.2}$$

is the *classical R-matrix*, given by the difference of the projection operators

$$\begin{aligned} &P_+ : \widetilde{\mathfrak{sl}}(2) \to \widetilde{\mathfrak{sl}}_+(2), \qquad P_+ : \widetilde{\mathfrak{sl}}(2) \to \widetilde{\mathfrak{sl}}_+(2), \\ &P_- : X \to X_+, \qquad P_- : X \to X_- \end{aligned} \tag{4.3}$$

to the subalgebras $\widetilde{\mathfrak{sl}}_+(2)$, $\widetilde{\mathfrak{sl}}_-(2)$ consisting respectively of the nonnegative and negative terms in the Laurent expanson of $X(z)$ for large z. The space $\widetilde{\mathfrak{sl}}(2)^*_R$ is identified as a subspace of $\widetilde{\mathfrak{sl}}(2)$ through the trace-residue pairing

$$\langle X, Y\rangle := \mathrm{res}_{z=\infty}\, \mathrm{tr}\,(X(z)Y(z)). \tag{4.4}$$

In this setting, the isomonodromic deformation equations (3.1), (3.17), (3.27) may all be expressed in the form

$$\frac{\partial X}{\partial a_j} = -[(dG_j)_-, X] + \frac{\partial (dG_j)_-}{\partial z}, \tag{4.5a}$$

$$(dG_j)_- = -\frac{A_j}{z - a_j}, \tag{4.5b}$$

where X denotes X^S, X^A or X^B, and G_j denotes G_j^S, G_j^A or $G^B_{\alpha,J}$ respectively. Viewing the Hamiltonians $\{G_j\}$ as spectral invariants defined on the space $\widetilde{\mathfrak{sl}}(2)$, eq. (4.5a) follows from the Adler–Kostant–Symes theorem, in view of the relations

$$\frac{\partial_0 X}{\partial a_j} = -\frac{\partial (dG_j)_-}{\partial z}, \tag{4.6}$$

where $\frac{\partial_0 X}{\partial a_j}$ denotes the derivative with respect to the *explicit* dependence on the parameters $\{a_j\}$ only.

Rather than using the spectral invariants $\{G_j\}$ as Hamiltonians, we consider the Hamiltonian equations generated by the linear combinations R^S_m, R^A_m or $R^B_{\alpha,m}$ defined in (2.8), (2.9), (2.10), which are all of the form

$$\mathcal{D}_m X = -[(dR_m)_-, X] + \frac{\partial (dR_m)_-}{\partial z}, \tag{4.7}$$

with the respective identifications for X and $\{R_m\}$. These are just equations (3.12), (3.24) or (3.35), depending on the identification, since

$$R_m = \frac{1}{2} \operatorname{res}_{z=\infty} z^m \operatorname{tr} X^2(z), \tag{4.8}$$

and therefore dR_m, viewed as an element of $\widetilde{\mathfrak{sl}}(2)$, is just

$$dR_m = z^m X(z) = \sum_{k=0}^{\infty} \frac{B_k}{z^{k-m+1}}. \tag{4.9}$$

implying

$$(dR_m)_- = \sum_{k=0}^{\infty} \frac{B_{m+k}}{z^{k+1}}. \tag{4.10}$$

If, instead of the nonautonomous systems occurring here because of the identifications of the a_j's as multi-time parameters, we consider the autonomous systems generated by the *same* set of Hamiltonians $\{R_0, R_1, \ldots\}$, denoting the corresponding flow parameters $\{t_0, t_1, \ldots\}$, the resulting equations have the isospectral form

$$\frac{\partial X}{\partial t_m} = \pm[(dR_m)_\pm, X], \tag{4.11}$$

where either of the projections $(dR_m)_\pm$ may be used, since the differential dR_m, given by (4.10), commutes with X. Although these systems are generated by the same Hamiltonians as the nonautonomous systems (4.7), they of course do *not* generate isomonodromic deformations of the operator $\frac{\partial}{\partial z} - X(z)$, and in fact are not even compatible with the systems (4.7); however, they are compatible amongst themselves, generating commuting isospectral Hamiltonian flows. The close relationship between the autonomous and associated nonautonomous systems implies a correspondence between the structure of the resulting hierarchies.

To see this, we substitute the expressions (3.2c), (3.16a) and (3.29a) for $X(z)$ and (4.10) for dR_m into (4.11) to obtain the systems

$$\frac{\partial B_m}{\partial t_m} = [C, B_{m+k}] + [B, B_{m+k+1}] + \sum_{l=0}^{m-1} [B_l, B_{m+k-l-1}], \tag{4.12a}$$

$$\frac{\partial C}{\partial t_m} = [B, B_k], \qquad k, m \in \mathbb{N} \tag{4.12b}$$

for $X = X^A$,

$$\frac{\partial B_m}{\partial t_m} = [C, B_{m+k-1}] + [\widetilde{B}, B_{m+k}] + \sum_{l=1}^{m-1} [B_l, B_{m+k-l-1}], \tag{4.13a}$$

$$\frac{\partial C_\alpha}{\partial t_m} = [\widetilde{B}, B_k], \qquad k, m \in \mathbb{N}, \quad m \geq 1. \tag{4.13b}$$

for $X = X^B_\alpha$ and

$$\frac{\partial B_m}{\partial t_m} = [B_S, B_{m+k}] + \sum_{l=0}^{m-1} [B_l, B_{m+k-l-1}] \tag{4.14}$$

for $X = X^S$. These only differ from the equations (3.12), (3.24) and (3.35) by the absence of the term mB_{m+k-1} in the right hand side of (4.12b), (4.13b), (4.14) and the replacement

$$\mathcal{D}_m \to \frac{\partial}{\partial t_m} \tag{4.15}$$

for the derivation on the left hand side. The procedure for deriving hierarchies for such systems is well known in the isospectral context (see, e.g. [7] for details); the recursive procedure used in Section 3 above is just the analog of this approach applied to the isomonodromic systems (3.12), (3.24) and (3.35).

As a final point, it should be noted that almost nothing in the derivation of the τ-function equations of Sections 2 and 3 depended on the fact that the specific τ-functions involved were equal to the Fredholm determinants (1.2), (1.4), (1.6). Everything just followed from the general form of the isomonodromic deformation equations (3.1), (3.17) and (3.27), the only features specific to the identifications of τ^A, τ^B_α, τ^S as Fredholm determinants being the fact that the matrix residues A_j were of rank 1 (as seen from the parametrization (3.3a)) and the invariants G^A_0, G^A_∞, G^B_0, G^B_∞ vanished. By allowing these invariants, as well as the constants $\{\det A_j\}$, to take arbitrary values, an identical procedure leads to equations for the τ-functions of the general isomonodromic systems, which only differ from the ones derived in Sections 2 and 3 by the nonzero constant values of the two additional invariants G^A_0, G^A_∞ or G^B_0 and G^B_∞. For example, eq. (2.16) is replaced in the general case by

$$\mathcal{D}_0^3 R - 4\mathcal{D}_1 R + 2R + 4\left(g_\infty^2 - g_0\right)\mathcal{D}_0 R - 2g_\infty\left(\mathcal{D}_0^2 R + 2R\mathcal{D}_0 R\right) + 6(\mathcal{D}_0 R)^2 = 0, \tag{4.16}$$

where g_0, g_∞ are the values taken by the invariants G^A_0, G^A_∞, respectively. The other equations of these hierarchies may similarly expressed in a way that allows arbitrary values for these constants.

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References

[1] Adams M R, Harnad J and Previato E, Isospectral Hamiltonian Flows in Finite and Infinite Dimensions I. Generalised Moser Systems and Moment Maps into Loop Algebras, *Commun. Math. Phys.* **117** (1988), 451–500.

[2] Adler M, Shiota T and van Moerbeke P, Random Matrices, Vertex Operators and the Virasoro Algebra, *Phys. Lett.* **A208** (1995), 67–78.

[3] Adler M, Shiota T and van Moerbeke P, Random Matrices, Virasoro Algebras and Noncommutative KP, *Duke Math. J.* **94** (1998), 379–431.

[4] Adler M and van Moerbeke P, Completely Integrable Systems, Euclidean Lie Algebras, and Curves, *Adv. Math.* **38** (1980), 267–317; Linearization of Hamiltonian Systems, Jacobi Varieties and Representation Theory, *Adv. Math.* **38** (1980), 318–379.

[5] Borodin A and Deift P, Fredholm Determinants, Jimbo–Miwa–Ueno Tau-Functions and Representation Theory, math-ph/0111007.

[6] Forrester P J, The Spectrum Edge of Random Matrix Ensembles, *Nucl. Phys.* **B402** (1993), 709–728.

[7] Flaschka H, Newell A C and Ratiu T, Kac–Moody Lie Algebras and Soliton Equations, *Physica* **D9** (1980), 300–323.

[8] Harnad J, Dual Isomonodromic Deformations and Moment Maps into Loop Algebras, *Commun. Math. Phys.* **166** (1994), 337–365.

[9] Harnad J, Tracy C A and Widom H, Hamiltonian Structure of Equations Appearing in Random Matrices, in Low Dimensional Topology and Quantum Field Theory, Editor: Osborn H, Plenum, New York, 1993, 231–245.

[10] Harnad J and Its A R, Integrable Fredholm Operators and Dual Isomonodromic Deformations, *Commun. Math. Phys.* **226** (2002), 497–530.

[11] Jimbo M, Miwa T, Môri Y and Sato M, Density Matrix of an Impenetrable Bose Gas and the Fifth Painlevé Transcendent, *Physica* **D1** (1980), 80–158.

[12] Jimbo M, Miwa T and Ueno K, Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coeefficients I, *Physica* **D2** (1981), 306–352.

[13] Jimbo M and Miwa T, Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coeefficients II, III, *Physica* **D2** (1981), 407–448; **D4** (1981), 26–46.

[14] Mehta M L, Random Matrices, 2nd edition, Academic, San Diego, 1991.

[15] Olshanetsky M M, Perelomov A M, Reyman A G and Semenov-Tian-Shansky M A, Integrable Systems II, in Encyclopaedia of Mathematical Sciences, Vol. 16, Springer–Verlag, Berlin, Heidelberg, New York, 1994.

[16] Palmer J, Deformation Analysis of Matrix Models, *Physica* **D8** (1994), 166–185.

[17] Tracy C A and Widom H, Introduction to Random Matrices, in Geometric and Quantum Methods in Integrable Systems, Springer Lecture Notes in Physics, Vol. 424, Editor: Helminck G F, Springer–Verlag, N.Y., Heidelberg, 1993, 103–130.

[18] Tracy C A and Widom H, Level Spacing Distributions and the Airy Kernel, *Commun. Math. Phys.* **159** (1994), 151–174.

[19] Tracy C A and Widom H, Level Spacing Distributions and the Bessel Kernel, *Commun. Math. Phys.* **161** (1994), 289–309.