Journal of Nonlinear Mathematical Physics Volume 9, Supplement 2 (2002), 1–9 **Birthday Issue**

Hidden Symmetries, First Integrals and Reduction of Order of Nonlinear Ordinary Differential Equations

Barbara ABRAHAM-SHRAUNER

Department of Electrical Engineering, Washington University,
St. Louis, Missouri 63130, USA
E-mail: bas@ee.wustl.edu

Received February, 2002

Abstract

The reduction of nonlinear ordinary differential equations by a combination of first integrals and Lie group symmetries is investigated. The retention, loss or even gain in symmetries in the integration of a nonlinear ordinary differential equation to a first integral are studied for several examples. The differential equations and first integrals are expressed in terms of the invariants of Lie group symmetries. The first integral is treated as a differential equation where the special case of the first integral equal to zero is examined in addition to the nonzero first integral. The inverse problem for which the first integral is the fundamental quantity enables some predictions of the change in Lie group symmetries when the differential equation is integrated. New types of hidden symmetries are introduced.

1 Introduction

The reduction of the order of a nonlinear ordinary differential equation (NLODE) is a basic procedure in the reduction of a NLODE to quadratures. The reduction of an NLODE to a linear ODE is also useful since there are many methods for solving linear ODEs. The reduction in order can be done by use of Lie group symmetries or by integration of the NLODE to a first integral. Hidden symmetries have been helpful in understanding the possible paths for reduction in order. Hidden symmetries have been defined with reference to Lie group symmetries [28, 1–9, 11, 17–19, 25]. A Type I hidden symmetry has been defined as a Lie symmetry that is lost (not inherited) in addition to the Lie symmetry that is used to reduce the ODE by one order. In any given reduction of an ODE by one order more than one Lie symmetry may be lost. A Type II hidden symmetry has been defined as a Lie symmetry that appears in addition to inherited Lie symmetries when the ODE is reduced by one order when using a Lie symmetry. A Type I hidden symmetry arises because the symmetry group that is lost is not a normal subgroup. For an abelian Lie symmetry group of an ODE no Type I hidden symmetries appear as the order of the ODE is reduced. Additional reductions in order do not produce Type I hidden symmetries and the reductions continue until the ODE is reduced to quadratures or the symmetries are depleted. Type I hidden symmetries may arise if the Lie group symmetry of an ODE is represented by a nonabelian Lie algebra. The origin of Type I hidden symmetries can be predicted by examining the commutator of the group generators of the form

$$[U_i, U_j] = C^k_{ij} U_k, \tag{1.1}$$

where U_i, U_j and U_k are local group generators and the structure constant of the group C^k_{ij} is a real number. If $C^k_{ij} \neq 0$, the reduction of an ODE by the symmetry of U_i (U_j) loses (does not inherit) the symmetry of U_j (U_i) in the reduced ODE. The origin of all Type II hidden symmetries is not so obvious. Some arise from nonlocal symmetries. If an ODE is reduced by the symmetry of U_i in the commutator above with U_i, U_j and U_k distinct, the symmetry of U_j will not be inherited in the reduced ODE but becomes a nonlocal symmetry. If the reduced ODE can be reduced further by the inherited symmetry of U_k, the nonlocal symmetry is transformed to an inherited Lie point symmetry of U_j [3].

A first integral I is here defined for an ODE

$$F\big(x, y, y', y'', \ldots, y^{(n)}\big) = 0 \tag{1.2}$$

as

$$I = f\big(x, y, y', y'', \ldots, y^{(n-1)}\big), \tag{1.3}$$

where

$$\begin{aligned} \left.\frac{dI}{dx}\right|_{F=0} &= \left.\frac{df\big(x, y, y', y'', \ldots y^{(n-1)}\big)}{dx}\right|_{F=0} \\ &= \mu\big(x, y, y', y'', \ldots, y^{(n-1)}\big) F\big(x, y, y', y'', \ldots, y^{(n)}\big)\Big|_{F=0} = 0 \end{aligned} \tag{1.4}$$

and $\mu\big(x, y, y', y'', \ldots, y^{(n-1)}\big)$ is an integrating factor. The $'$ denotes differentiation with respect to x, where for derivatives above the third order the $'$ is replaced by the order n as a superscript. Integration of the ODE (1.2) to the first integral I reduces the order by one. We consider the first integrals as ODEs in this paper [25].

First integrals are another means for reducing the order of an ODE [28, 24, 14, 16, 25]. Symmetries of first integrals of the linear ODE, $y'' = 0$, have been previously studied in terms of a basis of three first integrals [25]. The viewpoint here differs from that approach. The NLODE and its first integral are studied in terms of their Lie group invariants: the path curve that is a function of x and y [12], the differential invariants and a constant, an invariant that is often omitted in discussions. Also we concentrate on the inverse problem; that is we start with the first integral and differentiate it to find the ODE. It is more predictive to consider the inverse problem and that does suggest what may happen to the symmetries for an ODE integrated to its first integral. We consider several examples of nonlinear ODEs.

We do not consider nonlocal symmetries that are used to reduce the order of NLODEs directly [15, 26, 27] nor do we deal with potential symmetries [10] that use Bäcklund transformations. These techniques offer other paths to the reduction of order of ODEs.

2 Symmetries of NLODEs and their first integrals

We start with a second-order NLODE that has one Lie group symmetry and a first integral. If the ODE is found by an Euler–Lagrange equation, the ODE can be reduced to quadratures since the first integral retains the Lie symmetry by Noether's Theorem [28]. Consider the modified Emden equation [23]

$$\ddot{q} + \frac{5}{T}\dot{q} + q^2 = 0, \tag{2.1}$$

where the overdot denotes the time T derivative. By use of finite symmetries or by the application of the Lie classical method one finds that the group generator for the Lie symmetry of (2.1) is

$$U = T\frac{\partial}{\partial T} - 2q\frac{\partial}{\partial q}. \tag{2.2}$$

The canonical coordinates for (2.1) as found from (2.2) are

$$t = \ln T, \qquad u = qT^2. \tag{2.3}$$

The transformation of (2.1) to the new coordinates reduces it to

$$F(u_{tt}, u) = u_{tt} - 4u + u^2 = 0. \tag{2.4}$$

The path curve is u (not unique); the second differential invariant is u_{tt} for the transformed group generator $\tilde{U} = \partial/\partial t$. The first integral is found by the usual method of multiplying by the integrating factor u_t and integrating. The first integral I is

$$I = \frac{u_t^2}{2} + 2u^2 - \frac{u^3}{3}, \tag{2.5}$$

where I is a Lagrangian. It can be reduced by another order to quadratures since it is still invariant under the group denoted by $\tilde{U}$. The first integral in (2.5) can be transformed back to the first integral in the (q, T) variables [23]. Equation (2.1) is an example of a NLODE that can be reduced by two orders because it is a Euler–Lagrange equation and is invariant under one Lie point symmetry. The ODE (2.1) can be reduced to quadratures without recourse to canonical coordinates as well [20, 29]. The symmetry of the Emden equation and its first integral have been discussed previously [22]. The possible reductions in order for a NLODE with a first integral and several Lie group symmetries are quite complicated [21] and are considered in the next examples.

The second example is the nonlinear third-order ODE

$$y''' = yy'' + y'^2, \tag{2.6}$$

where $'$ again denotes differentiation with respect to x. The group generators for the Lie symmetries are $U_1 = \partial/\partial x$ and $U_2 = x\partial/\partial x - y\partial/\partial y$. A first integral I exists in addition where

$$I_1 = y'' - yy'. \tag{2.7}$$

However, the first integral (as an ODE) loses a Lie symmetry and has only $U_1 = \partial/\partial x$. We can see the structure better by looking at the invariants of the two-parameter group of (2.6). These can be found in several ways and are

$$\chi_1 = \frac{y'}{y^2}, \qquad \chi_2 = \frac{y''}{y^3}, \qquad \chi_3 = \frac{y'''}{y^4}. \tag{2.8}$$

A constant is also an invariant here. Equation (2.6) in terms of invariants becomes

$$\chi_3 = \chi_2 + \chi_1^2. \tag{2.9}$$

Equation (2.7) cannot be written in terms of these invariants but only in terms of the invariants of $U_1 = \partial/\partial x$: y, y', y''. Clearly to retain the symmetries of the original ODE, here (2.6), one needs to find the first integral as a function of two-parameter group invariants. However, the first integral I_1 can be found as a nonlocal function of the invariants of U_2: $\chi_0 = xy$ and χ_1, χ_2, χ_3 in (2.8). To see this we express (2.6) in terms of the invariants of U_2 and find the characteristic equations of the appropriate form of (1.3) in the invariants. The invariant χ_3 is eliminated in the first-order, nonlinear ordinary differential equations in the invariants. The resultant set of NLODEs is integrated to give the first integral where the term dx/x is eliminated by solving for it from χ_0 and χ_1. The first integral is

$$I_1 = (\chi_2 - \chi_1) \exp\left(3 \int \frac{\chi_1 d\chi_0}{\chi_0\chi_1 + 1}\right). \tag{2.10}$$

This first integral I_1 has the same form as I_1 in (2.7) when the invariants are expressed in terms of the original variables. We note the Blasius equation [10] has a first integral that is a nonlocal function of its invariants but both symmetries are lost.

For $I_1 \neq 0$ another first integral can be found, this time from (2.7) as

$$y' = \frac{y^2}{2} + I_1 x + J_1. \tag{2.11}$$

for J_1 a constant. This is a Riccati equation [13] and can be transformed to a linear second-order ODE. Again the first integral J_1 is a nonlocal function of the invariants of U_1. The special case of setting the first integral to zero is similar to the condition for potential symmetries but there a Bäcklund transformation is used. For the special case of $I_1 = 0$ we can write (2.7) in terms of the two-parameter group invariants as

$$\frac{y''}{y^3} - \frac{y'}{y^2} = 0, \tag{2.12}$$

where the two-parameter symmetry group is retained for this special case. For the $I_1 = 0$ case we can also substitute for $y = y''/y'$ in (2.6). Then the resultant NLODE has a Lie group symmetry represented by three group generators: $U_1 = \partial/\partial x$, $U_2 = x\partial/\partial x - y\partial/\partial y$, $U_3 = \partial/\partial y$. We have replaced the first integral and a two-parameter Lie group symmetry of (2.6) by a three-parameter Lie group symmetry for the modified NLODE.

For the NLODE in (2.6) we have different outcomes depending on whether $I_1 \neq 0$ or $I_1 = 0$. In the former case a Lie group symmetry is lost upon integration of the NLODE to the first integral. This loss of a symmetry may be viewed as a new type of Type I hidden symmetry where the stretching symmetry is blocked by the constant I_1. The Type I hidden symmetry that occurs when the order of an ODE is reduced by the use of a Lie group symmetry results in a nonlocal group generator whereas here the first integral is a nonlocal function of the invariants of the lost symmetry. For $I_1 = 0$ no symmetry is lost and in fact the equivalent NLODE is invariant under a three-parameter Lie group.

The invariants of the Lie symmetry group are crucial in our analysis of the reduction of the order of the NLODE by use of first integrals. It is more direct to consider the first integral in terms of its invariants and then differentiate the first integral to find the NLODE. Consequently, we change from the usual perspective that regards the original NLODE as the starting point and instead view the first integral as fundamental. We investigate how the symmetry groups are altered when it is differentiated to form the higher order NLODE. When I_1 in (2.7) is differentiated, the resultant (2.6) has another symmetry because there is a stretching symmetry between the terms in the ODE. A first integral with the same symmetries as (2.6) but a different form than (2.6) is

$$I_2 = \frac{y''}{y^3} - \frac{y'}{y^2}. \tag{2.13}$$

The stretching symmetry in (2.13) arises in each term singly and is not shared between terms; consequently, it is retained in the differentiation. The NLODE found by differentiating I_2 is

$$\frac{y'''}{y^4} = 3\frac{y''}{y^3}\frac{y'}{y^2} + \frac{y''}{y^3} - 2\left(\frac{y'}{y^2}\right)^2. \tag{2.14}$$

Stated another way the symmetry of NLODE of (2.14) is retained in the first integral (2.13) as compared to the loss of a symmetry of NLODE of (2.6) when it is integrated to its first integral in (2.7). In this example the symmetry of the NLODE is retained when integrated to form a first integral because the symmetries are present in each term and not shared between terms.

The third-order first integral I_3

$$I_3 = \frac{y'''}{y'^3} - K_1\frac{y''}{y'^2} - K_2 y \tag{2.15}$$

arises from a fourth-order ODE that is invariant under a three-parameter group with Lie group generators

$$U_1 = \frac{\partial}{\partial x}, \qquad U_2 = x\frac{\partial}{\partial x}, \qquad U_3 = \frac{\partial}{\partial y} \tag{2.16}$$

and integrates to this first integral. The symmetry group represented by U_3 is lost in the first integral, however. Again we find the new Type I hidden symmetry and the first integral I can be expressed as a nonlocal function of the invariants of U_3. Furthermore, one cannot eliminate y from the fourth-order NLODE since differentiation of the first integral eliminates y from the fourth-order NLODE. The first integral retains the three-parameter symmetry group of the fourth-order NLODE only by setting $K_2 = 0$. The NLODE in (2.15) has only two Lie group symmetries for $K_2 \neq 0$ so that reduction to quadratures is not expected. It can be reduced to a linear, second-order ODE in y'^2 but the solution has not been attempted. We see that when a first integral is written in terms of invariants, other symmetries may arise when it is differentiated.

The final first integral considered is

$$I_4 = 2y'y''' - 3y''^2. \tag{2.17}$$

For $I_4 = 0$ this is the Kummer–Schwarz equation [22] and is invariant under the maximum number of contact symmetries, 10, for a third-order ODE. It is invariant, however, under a six-parameter Lie group rather than the seven-parameter Lie group that is the maximum dimension Lie group for a third-order ODE. It is invariant under a three-parameter Lie group with group generators: $U_1 = \partial/\partial x$, $U_2 = \partial/\partial y$, $U_3 = x\partial/\partial y - 2y\partial/\partial y$ for $I_4 \neq 0$. The NLODE (2.17) can be reduced to quadratures by choosing the symmetries for reduction in the correct order. The NLODE found by differentiation of the first integral I_4 with respect to x gives

$$y'y^{(4)} - 2y''y''' = 0 \tag{2.18}$$

and this NLODE is invariant under a four-parameter group with group generators $U_1 = \partial/\partial x$, $U_2 = \partial/\partial y$, $\tilde{U}_3 = x\partial/\partial x$, $\tilde{U}_4 = y\partial/\partial y$. The split of the stretching group represented by U_3 to $\tilde{U}_3$ and $\tilde{U}_4$ arises because the symmetries are not present in each term separately but are shared among the two terms in (2.18). For $I_4 \neq 0$ integration of the NLODE in (2.18) to a first integral results in the loss of one symmetry where linear combinations of $\tilde{U}_3$ and $\tilde{U}_4$ can give U_3 and another group generator. Again a first integral is found for a lost symmetry as a nonlocal function of the invariants. In addition another first integral of the (2.18) exists, $I_5 = y'''/y''^2$.

For $I_4 = 0$ there are six Lie group symmetries of (2.17) represented by the group generators $U_1 = \partial/\partial x$, $U_2 = \partial/\partial y$, $\tilde{U}_3 = x\partial/\partial x$, $\tilde{U}_4 = y\partial/\partial y$, $U_5 = y\partial/\partial x$, $U_6 = x\partial/\partial y$. However, the inherited Lie group symmetries found by differentiating (2.17) remain the four of (2.18). In this case the reduction in order of (2.18) by integration produces more symmetries if $I_4 = 0$. We have a new Type II hidden symmetry, the appearance of new symmetries upon reduction of order of the NLODE. The increase in the number of symmetries in the first integral can be understood by considering the inverse problem. Express the NLODE in (2.18) for $I_4 = 0$ in the invariants of the subgroup represented by U_6 above. The first integral (2.17) must be zero as well as the (2.18) hold. This means that the NLODE (2.18) can be regarded as the differential consequences of the first integral where both must hold.

3 Conclusion

We have studied the reduction of order of nonlinear ordinary differential equations by investigating the change of their Lie point symmetries when the NLODEs are integrated to a first integral. Several examples of NLODEs have been studied in terms of the invariants of the Lie group symmetries. Investigation of the inverse problem, the change of Lie group symmetries of first integrals upon differentiation, has been useful in understanding the inheritance of Lie symmetries of the NLODEs. Two cases have been studied for the first integral I: $I \neq 0$ and $I = 0$. Although no test has emerged for the fate of all symmetries upon integration of a NLODE to a first integral, some properties have been identified. For $I \neq 0$ the symmetries may be retained upon integration of the NLODE to a first integral or the order of the symmetry group may be reduced. The latter case is identified as a new Type I hidden symmetry. For example, the stretching symmetries that hold for each term in the ODE separately can be retained but those shared between terms are reduced. The first integral I can be found as a nonlocal function of the invariants of the lost symmetry. For $I = 0$ the stretching symmetries may be retained in number even if changed in form. There is even a case where the order of the symmetry group increases upon integration of the ODE to the first integral for $I = 0$. This is identified as a new Type II hidden symmetry. The study is an effort to understand some facets of the change of symmetries of NLODEs upon integration to a first integral by concentrating on the inverse problem and analyzing the NLODE and its first integral in terms of its Lie group invariants.

References

[1] Abraham-Shrauner B, Hidden Symmetries and Linearization of the Modified Painlevé–Ince Equation, *J. Math. Phys.* **34** (1993), 4809–4816.

[2] Abraham-Shrauner B, Hidden Symmetries of Ordinary Differential Equations, in Proceedings of IMACS World Congress, Editor: Ames W F, 1994, Vol. 1, 1–4.

[3] Abraham-Shrauner B, Hidden Symmetries and Nonlocal Group Generators for Ordinary Differential Equations, *IMA J. Appl. Math.* **56** (1996), 235–252.

[4] Abraham-Shrauner B and Guo A, Hidden Symmetries Associated with the Projective Group of Nonlinear First-Order Ordinary Differential Equations, *J. Phys.* **A25** (1992), 5597–5608.

[5] Abraham-Shrauner B and Guo A, Hidden and Nonlocal Symmetries of Nonlinear Differential Equations in Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics, Editors: Ibragimov N H, Torrissi M and Valenti A, Kluwer, Dordrecht, 1993, 1–5.

[6] Abraham-Shrauner B and Guo A, Hidden Symmetries of Differential Equations, Cont. Math., Vol. 160, American Mathematical Soc., Providence, 1994, 1–13.

[7] Abraham-Shrauner B and Leach P G L, Hidden Symmetries of Nonlinear Ordinary Differential Equations in Exploiting Symmetry in Applied and Numerical Analysis, Lectures in Applied Mathematics., Vol. 29, American Mathematical Society, 1993, 1–10.

[8] Abraham-Shrauner B, Govinder K S and Leach P G L, Integration of Second Order Ordinary Differential Equations Not Possessing Lie Point Symmetries, *Phys. Lett.* **A203** (1995), 169–174.

[9] Abraham-Shrauner B, Leach P G L, Govinder K S and Ratcliff G, Hidden and Contact Symmetries of Ordinary Differential Equations, *J. Phys.* **A28** (1995), 6707–6716.

[10] Bluman G W and Kumei S, Symmetries and Differential Equations, Appl. Math. Sci., No. 81, Springer-Verlag, New York, 1989, 129–130.

[11] Coggeshall S V, Abraham-Shrauner B and Knapp C, Hidden Symmetries of Partial Differential Equations, in Proceedings of IMACS World Congress, Editor: Ames W F, 1994, Vol. 1, 102–107.

[12] Cohen A, An Introduction to the Lie Theory of One-Parameter Groups, Heath, New York, 1911.

[13] Davis H T, Introduction to Nonlinear and Differential and Integral Equations, Dover, New York, 1962.

[14] Govinder K S and Leach P G L, The Algebraic Structures of the First Integrals of Third-Order Linear Equations, *J. Math. Anal. and Appl.* **193** (1995), 114–133.

[15] Govinder K S and Leach P G L, On the Determination of Nonlocal Symmetries, *J. Phys.* **A28** (1995), 5349–5359.

[16] Govinder K S and Leach P G L, Noetherian Integrals via Nonlocal Transformation, *Phys. Lett.* **A201** (1995), 91–94.

[17] Govinder K S and Leach P G L, A Group Theoretic Approach to a Class of Second-Order Ordinary Differential Equations Not Possessing Lie Point Symmetries, *J. Phys.* **A30** (1997), 2055–2068.

[18] Guo A and Abraham-Shrauner B, Hidden Symmetries of Energy Conserving Differential Equations, *IMA J. Appl. Math.* **51** (1993), 147–153.

[19] Hartl T and Athorne C, Solvable Structures and Hidden Symmetries, *J. Phys.* **A27** (1994), 3463–3474.

[20] Ibragimov N H, Kara A H, and Mahomed F M, Lie–Bäcklund and Noether Symmetries with Applications, *Nonlinear Dynam.* **15** (1998), 115–136.

[21] Kara A H and Mahomed F M, Action of Lie–Bäcklund Symmetries on Conservation Laws, in Modern Group Analysis: Proceedings of the International Conference at the Sophus Lie Conference Center, Nordfjordeid, Norway, 1997, 175–180.

[22] Kara A H, Vawda F and Mahomed F M, Symmetries of First Integrals and Solutions of Differential Equations, *Lie Groups Appl.* **1** (1994), 27–48.

[23] Leach P G L, First Integrals for the Modified Emden Equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$, *J. Math. Phys.* **26** (1985), 2510–2514.

[24] Leach P G L, Govinder K S and Abraham-Shrauner B, Symmetries of First Integrals and Their Associated Differential Equations, *J. Math. Anal. and Appl.* **253** (1999), 58–83.

[25] Leach P G L and Mahomed F M, Maximal Subalgebra Associated with First Integral of a System Possessing $sl(3,R)$ algebra, *J. Math. Phys.* **29** (1988), 1807–1813.

[26] Muriel C and Romero J L, New Methods of Reduction for Ordinary Differential Equations, *IMA J. Appl. Math.* **66** (2001), 111–125.

[27] Muriel C and Romero J L, C^∞-Symmetries and Nonsolvable Symmetry Algebras, *IMA J. Appl. Math.* **66** (2001), 477–498.

[28] Olver P J Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1986, 148, 154, 184–185.

[29] Wafo Soh C and Mahomed F M, Noether Symmetries of $y'' = f(x)y^n$ with Applications to Non-Static Spherically Symmetric Perfect Fluid Solutions, *Classical Quantum Gravity* **16** (1999), 3553–3566.