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Four Dimensional Lie Symmetry Algebras and Fourth Order Ordinary Differential Equations

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Abstract

Realizations of four dimensional Lie algebras as vector fields in the plane are explicitly constructed. Fourth order ordinary differential equations which admit such Lie symmetry algebras are derived. The route to their integration is described.

1 Introduction

In the second half of the XIX century, Marius Sophus Lie (1842–1899), the great Norwegian mathematician, studied a class of special algebras, which he called continuous groups of transformations. In the Preface to his book [3], Luigi Bianchi (1856–1928), an Italian mathematician, who was a contemporary of Lie, wrote an eulogy about the colleague's work:

> Movendo da concetti geometrici, associati allo studio dei problemi d'integrazione, Egli riconobbe l'importanza fondamentale, per la geometria e per l'analisi, della considerazione di questi gruppi *continui*, e concepì ed attuò l'ardito disegno di costruirne la teoria generale che doveva estendere al campo continuo la teoria dei gruppi di sostituzioni e quivi compiere, per le teorie d'integrazione nell'analisi, un'opera di *classificazione* analoga a quella della teoria di GALOIS nello studio delle irrazionalità algebriche.
> E per opera di S. LIE la teoria dei gruppi continui, per quanto riguarda i gruppi *finiti* (che dipendono cioè da un numero finito di parametri), venne completamente costituita, arricchendo la scienza matematica di una delle più importanti conquiste del secolo scorso.[1]

[1]He moved from geometric concepts that are associated with the study of problems of integration in order to recognize the fundamental importance of considering those *continuous* groups in geometry and analysis, and to conceive and realize the daring plan of constructing their general theory which was going to extend the theory of substitution groups to the continuous field, and thus complete for the theories of integration in analysis a *classification* work similar to that of GALOIS' theory in the study of algebraic irrationalities. Thanks to S LIE the theory of continuous groups, for what concerns the *finite* groups (those which depend by a finite number of parameters), was completely constituted; that enriched the mathematical science of one of the most important achievements of the last century.

Upon Lie's death Bianchi wrote an obituary [2] in which he describes Lie's work. In particular, he said:

> Il LIE determinò inoltre tutti i possibili tipi di gruppi finiti continui sopra una, due o tre variabili o, se si vuole, sulla retta, nel piano o nello spazio (in quest'ultimo caso soltanto, in modo completo, pei gruppi primitivi).[2]

Today the problem of determining all finite dimensional continuous groups is formulated as the problem of determining all finite dimensional Lie algebras of vector fields up to equivalence under diffeomorphisms [6]. A recent account on the problem of classifying Lie algebras of vector fields can be found in [5]. There González-López et al. give the classification of Lie algebras of differential operators in two real variables. Also they state that "*Lie, Campbell, Bianchi, etc., never really made it clear whether they were working over the real or the complex numbers*". Let us render justice to Bianchi for he did distinguish between complex and real space. In fact in [1] Bianchi gave the classification of all the real algebras of vector fields in the real space. He based his work on Lie's classification, but stated that

> Nella classificazione di LIE non vi è luogo a distinguere il reale dall'immaginario, laddove noi vogliamo, in queste ricerche, riferirci soltanto a gruppi reali ed ai loro sottogruppi reali: dovremo perciò suddividere in più tipi qualche tipo, che dal punto di vista generale del LIE risulta unico.[3]

In particular, he introduced the Type IX three-dimensional Lie algebra which does not contain any two-dimensional subalgebra:

> Resta infine da considerare il caso in cui il gruppo G_3 non è integrabile. Per questi gruppi LIE assegna l'unico tipo
> (Tipo VIII) $(X_1\,X_2) = X_1 f$, $(X_1\,X_3) = 2X_2 f$, $(X_2\,X_3) = X_3 f$,
> ma noi dovremo aggiungervi l'altro:
> (Tipo IX) $(X_1\,X_2) = X_3 f$, $(X_2\,X_3) = X_1 f$, $(X_3\,X_1) = X_2 f$,
> il quale differisce dal precedente per ciò che in quest'ultimo non esiste alcun sottogruppo *reale* a due parametri.[4]

In [1] Bianchi also proved the following

> dimostriamo che un gruppo (transitivo) di movimenti con 6, con 5 ovvero con 4 parametri contiene necessariamente qualche sottogruppo *reale* a 3 parametri.[5]

[2] Moreover LIE determined all the possible finite continuous groups on one, two, three variables or, as one wishes, on the line, plane or space (in the latter case, only for primitive groups completely)

[3] In LIE's classification there is no place for distinguishing the real from the imaginary; whereas in the present research, we want to refer to real groups and their real subgroups only: therefore we shall have to subdivide into more types certain types, which are unique from LIE's general point of view.

[4] Finally it remains to consider the case in which group G_3 is not integrable. For these groups LIE gives a unique type (Type VIII) ..., but we will have to add another one (Type IX) ..., which differs from the other because in the latter it does not exist any *real* subgroup with two parameters.

[5] we show that a (transitive) group of motions with either 6, 5 or 4 parameters necessarily contains some *real* subgroups with 3 parameters.

In this paper we use this result to construct a (possible) realization in the plane of four-dimensional Lie algebras by considering one of their subalgebras which were listed by Patera and Winternitz in [11]. In particular we take into consideration the realizations of three-dimensional Lie algebras in the plane derived by Mahomed in [9]. Moreover we determine fourth order ordinary differential equations admitting those realizations as their Lie symmetry algebra. Finally we show the route to integration.

2 Four-dimensional Lie algebras of vector fields

Let $\mathfrak{g}$ be a real Lie algebra. A realization of $\mathfrak{g}$ as vector fields in the plane is an injective Lie algebra morphism $T : \mathfrak{g} \to \mathfrak{X}(\mathbb{R}^2)$. It is common to identify the realization T with its image. Let us remark that for any given realization of a Lie algebra one has a naturally defined effective local Lie action on $\mathbb{R}^2$.

To achieve the general goal of realizing finite dimensional Lie algebras one can, in principle, proceed by induction on the dimension of $\mathfrak{g}$. Having classified realizations up to dimension $(n-1)$ one can consider in every n-dimensional Lie algebra a maximal Lie subalgebra together with one of its realizations (there can be more than one in general) and try to extend it to a realization of the whole algebra. This means imposing the commutators on a generic vector field which translates into a system of linear partial differential equations. The explicit solutions of such system, if any, provide realizations of the algebra. However, such a programme encounters two major difficulties: neither the classification of n-dimensional Lie algebras nor the classification of maximal subalgebras of a given Lie algebra are known.

In what follows we restrict to the case of four-dimensional Lie algebras, a situation in which both classifications are well known and explicit. We rely upon [11] from which we have borrowed notations as well: $A_{p,q}$ denotes a Lie algebra of dimension p and isomorphism type q. The classification results are recollected in Table 1. Realizations of three-dimensional Lie algebras, which are our building brick, can be found in [9].

If we try to find a realization of a four-dimensional algebra, then we need to consider a realization of one of its three-dimensional subalgebras. Thus we have to determine only the remaining operator

$$e_j = a_j(x,y)\frac{\partial}{\partial x} + b_j(x,y)\frac{\partial}{\partial y} \qquad (j = 1,\ldots,q \leq 4) \tag{1}$$

with a_j and b_j arbitrary functions of (x,y). Imposing the commutation relations which characterize the four-dimensional algebra generates an overdetermined linear system of partial differential equations in the two unknowns a_j and b_j. The solution of this system may lead to a realization of the algebra. We give an example.

Algebra $A_{4,12}$. Consider the three-dimensional subalgebra, $A_{3,3}^{\mathrm{II}}$: $(e_3; e_1, e_2)$, which has the following realization [9]

$$e_1 = \frac{\partial}{\partial y}, \qquad e_2 = x\frac{\partial}{\partial y}, \qquad e_3 = y\frac{\partial}{\partial y}. \tag{2}$$

Let e_4 be an operator of type (1). If we require that the operators (2) and (1) satisfy the commutation relations of $A_{4,12}$ (see Table 1), then we obtain

$$e_1 = \frac{\partial}{\partial y}, \qquad e_2 = x\frac{\partial}{\partial y}, \qquad e_3 = y\frac{\partial}{\partial y}, \qquad e_4 = -\left(1+x^2\right)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}. \tag{3}$$

This procedure can be repeated in each case and all the results are listed in Table 2. Due to the complexity of such systems the computations were carried out using REDUCE 3.7, a computer algebra software. However, the algebras of type $A_2\oplus 2A_1$, $A_{3,1}\oplus A_1$, $A_{3,3}\oplus A_1$, $A_{3,9}\oplus A_1$, $A^1_{4,2}$, $A^{a,a}_{4,5}$, $A^{a,1}_{4,5}$, $A_{4,10}$, $A^a_{4,11}$ yield incompatible systems. Thus we infer that they cannot be realized in the plane.

3 Fourth order equations admitting a four-dimensional Lie symmetry algebra

Having a realization of a four-dimensional real Lie algebra we can construct a fourth order ordinary differential equation (ODE) which admits such an algebra as its Lie symmetry algebra by finding the differential invariants of the Lie algebra up to fourth order. We consider the most general form of a fourth order ODE

$$\Phi\left(x, y, y', y'', y''', y^{\text{iv}}\right) = 0. \tag{4}$$

We prolong the operators as given in Table 2 up to the fourth order. Then the solution of the system

$$e_i^4(\Phi)\big|_{\Phi=0} = 0 \qquad (i = 1, 2, 3, 4) \tag{5}$$

yields the differential invariants and, obviously, the corresponding fourth order ODE. We use *ad hoc* interactive REDUCE programs developed by one of the authors [10] to perform this lengthy task. All the equations we have found are listed in Table 3. We show a detailed example.

Algebra $A_{4,12}$. Consider the realization (3). The prolongations of those operators up to fourth order yield

$$\begin{aligned}
e_1^4 &= \frac{\partial}{\partial y},\\
e_2^4 &= x\frac{\partial}{\partial y} + \frac{\partial}{\partial y'},\\
e_3^4 &= y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''} + y'''\frac{\partial}{\partial y'''} + y^{\text{iv}}\frac{\partial}{\partial y^{\text{iv}}},\\
e_4^4 &= -\left(1+x^2\right)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} + (xy'-y)\frac{\partial}{\partial y'} + 3xy''\frac{\partial}{\partial y''} + (3y''+5xy''')\frac{\partial}{\partial y'''}\\
&\quad + \left(7xy^{\text{iv}} + 8y'''\right)\frac{\partial}{\partial y^{\text{iv}}}.
\end{aligned} \tag{6}$$

Now we solve the corresponding system (5). The first and second equations imply that (4) does not depend on y, y', i.e.

$$\Phi\left(x, y'', y''', y^{\text{iv}}\right) = 0. \tag{7}$$

In order to integrate the third equation, we must solve the following equations for the characteristics:

$$\frac{\mathrm{d}y''}{y''} = \frac{\mathrm{d}y'''}{y'''} = \frac{\mathrm{d}y^{\mathrm{iv}}}{y^{\mathrm{iv}}}. \tag{8}$$

We obtain the following differential invariants:

$$I_1 = \frac{y'''}{y''}, \qquad I_2 = \frac{y^{\mathrm{iv}}}{y''} \tag{9}$$

which force equation (7) to become

$$\Phi(x, I_1, I_2) = 0. \tag{10}$$

Finally, after the substitution of the invariants (9), the fourth equation is integrated by solving the following characteristic equations:

$$\frac{\mathrm{d}x}{-(1+x^2)} = \frac{\mathrm{d}I_1}{3+2xI_1} = \frac{\mathrm{d}I_2}{4xI_2+8I_1} \tag{11}$$

which yield the following differential invariants:

$$J_1 = \frac{y'''\left(1+x^2\right)+3xy''}{y''},$$
$$J_2 = \frac{\left(1+x^2\right)\left[\left(1+x^2\right)y^{\mathrm{iv}}+8xy'''\right]+12x^2y''}{y''}. \tag{12}$$

Then equation (10) becomes

$$\Phi(J_1, J_2) = 0, \tag{13}$$

videlicet, by Dini's theorem:

$$y^{iv} = \frac{y''F(J_1) - 8xy'''\left(1+x^2\right) - 12x^2y''}{\left(1+x^2\right)^2}, \tag{14}$$

with F an arbitrary function of J_1. Equation (14) admits $A_{4,12}$ as its Lie symmetry algebra.

4 Integration of fourth order equations by Lie's method

Lie showed that an ordinary differential equation of order n with a known n-dimensional Lie symmetry algebra can be integrated by quadratures provided that its symmetry algebra is solvable [8]. The general integrating procedure consists of n successive integrations and leads to quite lengthy calculations. Among the equations in Table 3 only $A_{3,8} \oplus A_1$ is not solvable. In [7] the integrating procedure was provided for any third order ODE which admits either a solvable or not solvable three-dimensional Lie symmetry algebra $\mathfrak{g}_3$. If $\mathfrak{g}_3$ is solvable, then we can reduce the given third order equation to a first order ODE which is integrable by quadrature and then obtain a second order ODE which can be transformed into a directly integrable form (Lie's method). If $\mathfrak{g}_3$ is not solvable, then we can still reduce the given third order equation to a first order equation; this equation is not integrable by quadrature, but can be easily reduced to a Riccati equation[6] by using a nonlocal symmetry which comes from one of the symmetries of the original third order ODE.

Here we follow a similar procedure. Consider a fourth order ODE which admits a four-dimensional solvable Lie algebra $\mathfrak{g}_4$. Firstly we reduce it to a first order ODE by using the differential invariants of an ideal $\mathfrak{h}_3 \subset \mathfrak{g}_4$. Then the first order equation can be integrated by quadrature because it admits the one-dimensional Lie algebra $\mathfrak{g}_4/\mathfrak{h}_3$. Its general solution becomes a third order ODE in the original variables. This equation admits $\mathfrak{h}_3$. Therefore it can be integrated with the procedure showed in [7]. If a fourth-order ODE admits a Lie symmetry algebra $\mathfrak{g}_4$ which is not solvable, then we can always reduce it to a first order ODE by using the differential invariants of a three-dimensional subalgebra $\mathfrak{g}_3$. Finally the first order ODE can be integrated by using a nonlocal symmetry which comes from the fourth symmetry. We show in details the case of a fourth order equation which admits a solvable Lie symmetry algebra and that of the equation which admits $A_{3,8} \oplus A_1$ as its Lie symmetry algebra.

Algebra $A_{4,12}$. Consider the realization

$$e_1 = \frac{\partial}{\partial y}, \qquad e_2 = x\frac{\partial}{\partial y}, \qquad e_3 = y\frac{\partial}{\partial y}, \qquad e_4 = -\left(1+x^2\right)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y} \tag{15}$$

and the fourth order ODE which admits $A_{4,12}$ with generators (15) as its Lie symmetry algebra

$$y^{\mathrm{iv}} = \frac{y''F(\xi) - 8xy'''\left(1+x^2\right) - 12x^2y''}{\left(1+x^2\right)^2}, \qquad \xi = \frac{y'''\left(1+x^2\right)+3xy''}{y''}. \tag{16}$$

The commutation relations are:

$$[e_1, e_3] = e_1, \qquad [e_2, e_3] = e_2, \qquad [e_1, e_4] = -e_2, \qquad [e_2, e_4] = e_1.$$

The algebra is solvable and the operators e_1, e_2, e_3 generate a three-dimensional ideal $\mathfrak{h}_3 = \langle e_1, e_2, e_3\rangle$. A basis of its differential invariants of order ≤ 3 is:

$$u = x, \qquad v = \frac{y'''}{y''} \tag{17}$$

Then equation (16) can be reduced to the first order ODE

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{F(\tilde{\xi}) - 8uv\left(1+u^2\right) - 12u^2 - v^2\left(1+u^2\right)^2}{\left(1+u^2\right)^2}, \qquad \tilde{\xi} = 3u + v\left(1+u^2\right) \tag{18}$$

which admits the one-dimensional Lie symmetry algebra generated by

$$e_4 = -\left(1+u^2\right)\frac{\partial}{\partial u} + (3+2uv)\frac{\partial}{\partial v}.$$

[6]In [4] a theoretical explanation of the appearance of a Riccati equation was given.

We write equation (18) as a linear differential form

$$\left(F(\tilde{\xi}) - 8uv\left(1+u^2\right) - 12u^2 - v^2\left(1+u^2\right)^2\right)\mathrm{d}u - \left(1+u^2\right)^2\mathrm{d}v = 0. \tag{19}$$

Its integrating factor is [8]

$$I = -\frac{1}{(1+u^2)\left(F(\tilde{\xi}) - 9u^2 + 3 - v\,(1+u^2)\,(6u+1+u^2)\right)}.$$

Therefore the general integral of equation (19) is obtained in the form $U(u,v) = c_1$ by the solution of

$$\begin{aligned} \frac{\partial U}{\partial u} &= -\frac{F(\tilde{\xi}) - 8uv\left(1+u^2\right) - 12u^2 - v^2\left(1+u^2\right)^2}{(1+u^2)\left(F(\tilde{\xi}) - 9u^2 + 3 - v\,(1+u^2)\,(6u+1+u^2)\right)}, \\ \frac{\partial U}{\partial v} &= -\frac{\left(1+u^2\right)}{F(\tilde{\xi}) - 9u^2 + 3 - v\,(1+u^2)\,(6u+1+u^2)}. \end{aligned} \tag{20}$$

Substitution of the original variables into U yields a third order ODE of the form $U\left(x, \frac{y'''}{y''}\right) = c_1$ which admits the Lie symmetry algebra generated by $e_1 = \partial_y$, $e_2 = x\partial_y$, $e_3 = y\partial_y$ and can be solved by quadrature [7].

Algebra $A_{3,8} \oplus A_1$. Consider the realization

$$e_1 = \frac{\partial}{\partial y}, \qquad e_2 = y\frac{\partial}{\partial y}, \qquad e_3 = -y^2\frac{\partial}{\partial y}, \qquad e_4 = \frac{\partial}{\partial x} \tag{21}$$

and the fourth order equation which admits $A_{3,8} \oplus A_1$ with the generators (21) as its Lie symmetry algebra[7]

$$\begin{aligned} y^{\mathrm{iv}} &= \frac{-3y''^3 + 4y'y''y''' + y'^3F(\xi)}{y'^2}, \\ \xi &= \frac{y'''}{y'} - \frac{3y''^2}{2y'^2}. \end{aligned} \tag{22}$$

The commutation relations are:

$$[e_1, e_3] = -2e_2, \qquad [e_2, e_3] = e_3, \qquad [e_1, e_2] = e_1.$$

The algebra is not solvable, but the operators e_1, e_2, e_3 generate a three-dimensional subalgebra $\mathfrak{g}_3 = \langle e_1, e_2, e_3\rangle$. A basis of its differential invariants of order ≤ 3 is:

$$u = x, \qquad v = \frac{y'''}{y'} - \frac{3}{2}\frac{y''^2}{y'^2}. \tag{23}$$

Then equation (22) can be reduced to the first order ODE

$$\frac{\mathrm{d}v}{\mathrm{d}u} = F(v) \tag{24}$$

[7] Of course it is not a surprise that ξ corresponds to the Schwarzian derivative [7].

which admits the one-dimensional Lie symmetry algebra generated by

$$e_4 = \frac{\partial}{\partial u}.$$

Therefore (24) can be easily integrated by quadrature, i.e.

$$\int \frac{\mathrm{d}v}{F(v)} = u + c_1$$

which in the original variables becomes a third order ODE which admits $\mathfrak{g}_3$ as its Lie symmetry algebra and can then be integrated [7].

5 Tables

In Table 1 we list the four-dimensional real Lie algebras as given in [11]. In the second column the nonzero commutation relations are given. In the last column the suitable three dimensional subalgebra that we have used either to generate a realization or to disprove that a realization exists — algebras marked with $(*)$ — are specified.

In Table 2 we have put the realizations that we found, with f, f_1, f_2 arbitrary functions. The generators of each algebra are orderly listed as e_1, e_2, e_3, e_4.

In Table 3 we list the fourth order ODEs which admit one of Lie algebras in Table 2 as Lie symmetry algebras. Note that F represents an arbitrary function and that we have chosen a particular form — shown in the third column — for each of the arbitrary functions listed in Table 2.

Table 1

Lie algebra	Nonzero commutation relations	Subalgebra
$4A_1$		$3A_1 : \langle e_2, e_3, e_4\rangle$
$(*)$ $A_2 \oplus 2A_1$	$[e_1, e_2] = e_2$	$3A_1 : \langle e_2, e_3, e_4\rangle$
$2A_2$	$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4$	$A_1 \oplus A_2 : \langle e_1, e_4, e_2\rangle$
$(*)$ $A_{3,1} \oplus A_1$	$[e_2, e_3] = e_1$	$3A_1 : \langle e_1, e_2, e_4\rangle$
$A_{3,2} \oplus A_1$	$[e_1, e_3] = e_1,\ [e_2, e_3] = e_1 + e_2$	$3A_1 : \langle e_1, e_2, e_4\rangle$
$(*)$ $A_{3,3} \oplus A_1$	$[e_1, e_3] = e_1,\ [e_2, e_3] = e_2$	$3A_1 : \langle e_1, e_2, e_4\rangle$
$A_{3,4} \oplus A_1$	$[e_1, e_3] = e_1,\ [e_2, e_3] = -e_2$	$3A_1 : \langle e_1, e_2, e_4\rangle$
$A^a_{3,5} \oplus A_1$ $(0 < \lvert a\rvert < 1)$	$[e_1, e_3] = e_1,\ [e_2, e_3] = ae_2$	$3A_1 : \langle e_1, e_2, e_4\rangle$
$A_{3,6} \oplus A_1$	$[e_1, e_3] = -e_2,\ [e_2, e_3] = e_1$	$3A_1 : \langle e_1, e_2, e_4\rangle$
$A^a_{3,7} \oplus A_1$ $(a > 0)$	$[e_1, e_3] = ae_1 - e_2,\ [e_2, e_3] = e_1 + ae_2$	$3A_1 : \langle e_1, e_2, e_4\rangle$
$A_{3,8} \oplus A_1$	$[e_1, e_3] = -2e_2,\ [e_2, e_3] = e_3,\ [e_1, e_2] = e_1$	$A_{3,8} : \langle e_1, e_2, e_3\rangle$
$(*)$ $A_{3,9} \oplus A_1$	$[e_1, e_3] = -e_2,\ [e_2, e_3] = e_1,\ [e_1, e_2] = e_3$	$A_{3,9} : \langle e_1, e_2, e_3\rangle$
$A_{4,1}$	$[e_2, e_4] = e_1,\ [e_3, e_4] = e_2$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$A^a_{4,2}$ $(a \neq 0, 1)$	$[e_1, e_4] = ae_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$(*)$ $A^1_{4,2}$	$[e_1, e_4] = e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_2 + e_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$A_{4,3}$	$[e_1, e_4] = e_1,\ [e_3, e_4] = e_2$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$A_{4,4}$	$[e_1, e_4] = e_1,\ [e_2, e_4] = e_1 + e_2,\ [e_3, e_4] = e_2 + e_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$A^{a,b}_{4,5}$ $(ab \neq 0)$ $(-1 \leq a < b < 1)$	$[e_1, e_4] = e_1,\ [e_2, e_4] = ae_2,\ [e_3, e_4] = be_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$(*)$ $A^{a,a}_{4,5}$ $(-1 \leq a < 1,\ a \neq 0)$	$[e_1, e_4] = e_1,\ [e_2, e_4] = ae_2,\ [e_3, e_4] = ae_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$(*)$ $A^{a,1}_{4,5}$ $(-1 \leq a < 1,\ a \neq 0)$	$[e_1, e_4] = e_1,\ [e_2, e_4] = ae_2,\ [e_3, e_4] = e_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$A^{1,1}_{4,5}$	$[e_1, e_4] = e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = e_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$A^{a,b}_{4,6}$ $(a \neq 0,\ b \geq 0)$	$[e_1, e_4] = ae_1,\ [e_2, e_4] = be_2 - e_3,\ [e_3, e_4] = e_2 + be_3$	$3A_1 : \langle e_1, e_2, e_3\rangle$
$A_{4,7}$	$[e_1, e_4] = 2e_1,\ [e_2, e_4] = e_2,$ $[e_3, e_4] = e_2 + e_3,\ [e_2, e_3] = e_1$	$A^{1/2}_{3,5} : \langle e_1, e_2, e_4\rangle$
$A_{4,8}$	$[e_2, e_3] = e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = -e_3$	$A_{3,1} : \langle e_1, e_2, e_3\rangle$
$A^b_{4,9}$ $(0 < \lvert b\rvert < 1)$	$[e_1, e_4] = (1 + b)e_1,\ [e_2, e_4] = e_2,$ $[e_3, e_4] = be_3,\ [e_2, e_3] = e_1$	$A_{3,1} : \langle e_1, e_2, e_3\rangle$
$A^1_{4,9}$	$[e_1, e_4] = 2e_1,\ [e_2, e_4] = e_2,$ $[e_3, e_4] = e_3,\ [e_2, e_3] = e_1$	$A_{3,1} : \langle e_1, e_2, e_3\rangle$
$A^0_{4,9}$	$[e_2, e_3] = e_1,\ [e_1, e_4] = e_1,\ [e_2, e_4] = e_2$	$A_{3,1} : \langle e_1, e_2, e_3\rangle$
$(*)$ $A_{4,10}$	$[e_2, e_3] = e_1,\ [e_2, e_4] = -e_3,\ [e_3, e_4] = e_2$	$A_{3,1} : \langle e_1, e_2, e_3\rangle$
$(*)$ $A^a_{4,11}$ $(0 < a)$	$[e_1, e_4] = 2ae_1,\ [e_2, e_4] = ae_2 - e_3,$ $[e_3, e_4] = e_2 + ae_3,\ [e_2, e_3] = e_1$	$A_{3,1} : \langle e_1, e_2, e_3\rangle$
$A_{4,12}$	$[e_1, e_4] = -e_2,\ [e_2, e_4] = e_1,\ [e_1, e_3] = e_1,$ $[e_2, e_3] = e_2$	$A_{3,3} : \langle e_1, e_2, e_3\rangle$

Table 2

Lie algebra	Generators
$4A_1$	$f_1(x)\partial_y,\ \partial_y,\ x\partial_y,\ f_2(x)\partial_y$
$2A_2$	$-x\partial_x,\ \partial_x,\ -y\partial_y,\ \partial_y$
$A_{3,2}\oplus A_1$	$\partial_y,\ -x\partial_y,\ \partial_x+(y+f(x))\partial_y,\ e^x\partial_y$
$A_{3,4}\oplus A_1$	$\partial_y,\ x^2\partial_y,\ x\partial_x+(y+f(x))\partial_y,\ x\partial_y$
$A^a_{3,5}\oplus A_1\ (0<\lvert a\rvert<1)$	$\partial_y,\ x^{1-a}\partial_y,\ x\partial_x+(y+f(x))\partial_y,\ x\partial_y$
$A_{3,6}\oplus A_1$	$\partial_y,\ \left(x^2-1\right)^{1/2}\partial_y,$ $-x\left(x^2-1\right)^{1/2}\partial_x+\left(f(x)-y\left(x^2-1\right)^{1/2}\right)\partial_y,\ x\partial_y$
$A^a_{3,7}\oplus A_1\ (a>0)$	$\partial_y,\ x\partial_y,\ -\left(1+x^2\right)\partial_x+((a-x)y+f(x))\partial_y,$ $\left(1+x^2\right)^{1/2}e^{a\arctan(x)}\partial_y$
$A_{3,8}\oplus A_1$	$\partial_y,\ y\partial_y,\ -y^2\partial_y,\ f(x)\partial_x$
$A_{4,1}$	$\partial_y,\ x\partial_y,\ (x^2/2)\partial_y,\ -\partial_x+f(x)\partial_y$
$A^a_{4,2}\ (a\neq 0,1)$	$e^{(1-a)x}\partial_y,\ -\partial_y,\ x\partial_y,\ \partial_x+y\partial_y$
$A_{4,3}$	$\partial_y,\ x\partial_y,\ -x\log(x)\partial_y,\ x\partial_x+(y+f(x))\partial_y$
$A_{4,4}$	$\partial_y,\ x\partial_y,\ \left(x^2/2\right)\partial_y,\ -\partial_x+(y+f(x))\partial_y$
$A^{a,b}_{4,5}\ (-1\le a<b<1,\ ab\neq 0)$	$\partial_y,\ x^{1-a}\partial_y,\ x^{1-b}\partial_y,\ x\partial_x+(x+y)\partial_y$
$A^{1,1}_{4,5}$	$f(x)\partial_y,\ \partial_y,\ x\partial_y,\ y\partial_y$
$A^{a,b}_{4,6}\ (a\neq 0,\ b\geq 0)$	$\left(1+x^2\right)^{1/2}e^{(b-a)\arctan(x)}\partial_y,\ x\partial_y,\ \partial_y,\ \left(1+x^2\right)\partial_x+(xy+by)\partial_y$
$A_{4,7}$	$\partial_y,\ x\partial_y,\ -\partial_x-x\log(x)\partial_y,\ x\partial_x+2y\partial_y$
$A_{4,8}$	$\partial_y,\ \partial_x,\ x\partial_y,\ x\partial_x$
$A^b_{4,9}\ (0<\lvert b\rvert<1)$	$\partial_y,\ \partial_x,\ x\partial_y,\ x\partial_x+(1+b)y\partial_y$
$A^1_{4,9}$	$\partial_y,\ \partial_x,\ x\partial_y,\ x\partial_x+2y\partial_y$
$A^0_{4,9}$	$\partial_y,\ \partial_x,\ x\partial_y,\ x\partial_x+y\partial_y$
$A_{4,12}$	$\partial_y,\ x\partial_y,\ y\partial_y,\ -\left(1+x^2\right)\partial_x-xy\partial_y$

Table 3

Lie algebra	Equation	Functions
$4A_1$	$y^{\rm iv}=F(x)$	$f_1(x)=x^2$ $f_2(x)=x^3$
$2A_2$	$y^{\rm iv}=\dfrac{y''^3}{y'^2}F\left(\dfrac{y'y'''}{y''^2}\right)$	
$A_{3,2}\oplus A_1$	$y^{\rm iv}=(y'''-y'')F\left((y'''-y'')e^{-x}\right)+y''$	$f(x)=0$
$A_{3,4}\oplus A_1$	$y^{\rm iv}=x^{-3}F\left(x^2y'''\right)$	$f(x)=0$
$A^a_{3,5}\oplus A_1$ $(0<\|a\|<1)$	$y^{\rm iv}=\dfrac{-(a+2)x^2y'''+F(\xi)}{x^3}$, $\xi=x^2y'''+x(a+1)y''$	$f(x)=0$
$A_{3,6}\oplus A_1$	$y^{\rm iv}=-\dfrac{\left(8x^4y'''+12x^3y''-10x^2y'''-9xy''+2y'''\right)x^2-F(\xi)}{\left(x^2-1\right)^2x^3}$, $\xi=\left(x^2-1\right)^{1/2}\left(x^2y'''+3xy''-y'''\right)x^2$	$f(x)=0$
$A^a_{3,7}\oplus A_1$ $(a>0)$	$y^{\rm iv}=-\dfrac{8x^3y'''+12x^2y''+8xy'''+3y''-a^2y''}{\left(x^2+1\right)^2}$ $+e^{-\arctan(x)a}\left(x^2+1\right)^{-7/2}F(\xi)$, $\xi=e^{\arctan(x)a}\left(x^2+1\right)^{3/2}\left(x^2y'''+3xy''+y'''+ay''\right)$	$f(x)=0$
$A_{3,8}\oplus A_1$	$y^{\rm iv}=\dfrac{-3y''^3+4y'y''y'''+y'^3F(\xi)}{y'^2},\ \xi=\dfrac{y'''}{y'}-\dfrac{3y''^2}{2y'^2}$	$f(x)=1$
$A_{4,1}$	$y^{\rm iv}=F(y'''+6x)$	$f(x)=x^3$
$A^a_{4,2}$ $(a\neq 0,1)$	$y^{\rm iv}=(a-1)^2y''+e^xF(\xi),\ \xi=\dfrac{(a-1)y''+y'''}{e^x}$	
$A_{4,3}$	$y^{\rm iv}=\dfrac{2xy''+F\left(xy''+x^2y'''\right)}{x^3}$	$f(x)=0$
$A_{4,4}$	$y^{\rm iv}=y'''F\left(y'''e^x\right)$	$f(x)=0$
$A^{a,b}_{4,5}$ $(ab\neq 0)$ $(-1\leq a<b<1)$	$y^{\rm iv}=\dfrac{-(2+a)x^2y'''-(1+b)\left(x^2y'''+(a+1)xy''\right)+F(\xi)}{x^3}$, $\xi=x^2y'''+(1+a)xy''+b(xy''+ay'-a\log(x))$	
$A^{1,1}_{4,5}$	$y^{\rm iv}=y'''F(x)$	$f(x)=x^2$
$A^{a,b}_{4,6}$ $(a\neq 0,\ b\geq 0)$	$y^{\rm iv}=\dfrac{(a-b)^2y''-\left(8x^3y'''+12x^2y''+8xy'''+3y''\right)}{\left(x^2+1\right)^2}$ $+e^{\arctan(x)b}\left(x^2+1\right)^{-7/2}F(\xi)$, $\xi=\left(x^2+1\right)^{3/2}\left((a-b)y''+x^2y'''+3xy''+y'''\right)e^{-\arctan(x)b}$	
$A_{4,7}$	$y^{\rm iv}=\dfrac{e^{2y''}-1}{x^2}F\left((xy'''-1)e^{-y''}\right)$	
$A_{4,8}$	$y^{\rm iv}=y''^2F\left(\dfrac{y'''^2}{y''^3}\right)$	
$A^b_{4,9}$ $(0<\|b\|<1)$	$y^{\rm iv}=y''^{\left(\frac{b-3}{b-1}\right)}F\left(y''^{(2-b)}y'''^{(b-1)}\right)$	
$A^1_{4,9}$	$y^{\rm iv}=y'''^2F(y'')$	
$A^0_{4,9}$	$y^{\rm iv}=y''^3F\left(\dfrac{y'''}{y''^2}\right)$	
$A_{4,12}$	$y^{\rm iv}=\dfrac{y''F(\xi)-8xy'''\left(1+x^2\right)-12x^2y''}{\left(1+x^2\right)^2},\ \xi=\dfrac{y'''\left(1+x^2\right)+3xy''}{y''}$	

References

[1] Bianchi L, Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti, *Mem. Soc. it. delle Sc. dei XL* (3) **11** (1897), 267–352.

[2] Bianchi L, Notizie sull'opera matematica di Sophus Lie, *Rendiconti della R. Accademia dei Lincei* (5) **8** (1899), 360–366.

[3] Bianchi L, Lezioni sulla Teoria dei Gruppi Continui Finiti di Trasformazioni, Enrico Spoerri Editore, Pisa, 1918.

[4] Clarkson P A and Olver P J, Symmetry and the Chazy Equation, *J. Diff. Eq.* **124** (1994), 225–246.

[5] González-López A, Kamran N and Olver P J, Lie Algebras of Vectors Fields in the Real Plane, *Proc. London Math. Soc.* **64** (1992), 339–368.

[6] Hawkins T, Jacobi and the Birth of Lie's Theory of Groups, *Arch. Hist. Exact Sciences* **42** (1991), 187–278.

[7] Ibragimov N H and Nucci M C, Integration of Third Order Ordinary Differential Equations by Lie's Method: Equations Admitting Three-Dimensional Lie Algebras, *Lie Groups Appl.* **1** (1994), 49–64.

[8] Lie S, Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen, B G Teubner, Leipzig, 1912.

[9] Mahomed F M, Symmetry Lie Algebras of n^{th} Order Ordinary Differential Equations, PhD Thesis, University of the Witwatersrand, Johannesburg, 1989.

[10] Nucci M C, Interactive REDUCE Programs for Calculating Lie Point, Non-Classical, Lie–Bäcklund, and Approximate Symmetries of Differential Equations: Manual and Floppy disk, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol. 3, New Trends in Theoretical Developments and Computational Methods, Editor: Ibragimov N H, CRC Press, Boca Raton, 1996, 415–481.

[11] Patera J and Winternitz P, Subalgebras of Real Three- and Four-Dimensional Lie Algebras, *J. Math. Phys.* **18** (1977), 1449–1455.