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A Basis of Conservation Laws for Partial Differential Equations

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Abstract

The classical generation theorem of conservation laws from known ones for a system of differential equations which uses the action of a canonical Lie–Bäcklund generator is extended to include any Lie–Bäcklund generator. Also, it is shown that the Lie algebra of Lie–Bäcklund symmetries of a conserved vector of a system is a subalgebra of the Lie–Bäcklund symmetries of the system. Moreover, we investigate a basis of conservation laws for a system and show that a generated conservation law via the action of a symmetry operator which satisfies a commutation rule is nontrivial if the system is derivable from a variational principle. We obtain the conservation laws of a class of nonlinear diffusion-convection and wave equations in $(1+1)$-dimensions. In fact we find a basis of conservation laws for the diffusion equations in the special case when it admits proper Lie–Bäcklund symmetries. Other examples are presented to illustrate the theory.

1 Introduction

The nonlinear diffusion-convection equation

$$u_t = (k(u)u_x)_x - (f(u))_x, \tag{1.1}$$

and the nonlinear wave equation

$$u_{tt} = g(x, u_x)u_{xx} + h(x, u_x), \tag{1.2}$$

have been of considerable interest in the literature.

The Lie point symmetry analysis of (1.1) was carried out in [18]. The work [3] presents a more detailed treatment of (1.1). For $f = 0$, (1.1) was investigated in [19] for its point symmetries and later the authors of [2] found that (1.1) admits nontrivial Lie–Bäcklund symmetries only in the case $k(u) = a(u+b)^{-2}$, where a and b are constants, for which linearization is possible (see [2] and refs. in [7, chap. 10]). Also for $f = 0$, the conservation laws of (1.1) are given in [7, chap. 10].

The class of equations (1.2) was studied by various authors. Lie [14], the originator of symmetry analysis, initiated the group classification of wave equations. In recent times Ames et al [1] studied the group properties of $v_{tt} = (g(v)v_x)_x$ or (1.2) if one sets $v = u_x$ with $g_x = 0$ and $h = 0$. These studies gave impetus to later investigations. The preliminary or partial group classification of (1.2) was performed in [9]. This resulted in 33 cases which need further investigations from the group-theoretic standpoint.

The outline of the paper is as follows. In Section 2 we provide the theory dealing with the action of Lie–Bäcklund symmetry generators on conservation laws to generate conservation laws. In this section it is also proved that a symmetry of a conserved vector is a symmetry of the system itself and this result is illustrated on an equation that arises in the study of Maxwellian tails as well as on the angular momentum for a central force problem. The results on the basis of conservation laws and symmetry action are also included here. Section 3 deals with a complete basis of conservation laws via action of symmetry generators on the fundamental conservation laws which we deduce for $(1+1)$ diffusion and wave equations. This includes the special equations in the class (1.1) which admit nontrivial Lie–Bäcklund symmetries and hence an infinite number of conservation laws with finite basis. Notwithstanding for the nonlinear wave equations we study the class when $h(x, u_x) = 0$, $g = g(u_x)$ and determine a basis of conservation laws.

2 Action of symmetries and related conservation laws

We utilise, amongst others, the following theoretical constructions which can simply be stated as: *action of any Lie–Bäcklund symmetry generator on a conservation law yields a conservation law*; *symmetry of a conserved vector is a symmetry of the system* and the notion of a *basis of conservation laws* of an equation.

The generation of conservation laws from known ones of a system of differential equations using symmetry properties of the system has been investigated over many years. In the case of ordinary differential equations, this result is well-known as the *related integral theorem* and has found widespread applications, for example, in classical mechanics (see, e.g., [20] and refs. therein). For a system of partial differential equations, a similar result has been established for canonical Lie–Bäcklund symmetries (see, e.g., [6, 17, 21]). We extend this result to include *any* Lie–Bäcklund symmetry. This extension has advantages. Firstly one does not have to convert to a canonical Lie–Bäcklund operator and thus a point symmetry generator which is geometrically transparent remains of a point type and the calculations are simpler. Secondly the generated conserved vectors via the canonical Lie–Bäcklund symmetry operator corresponding to a point symmetry operator are, in general, of a higher order than the starting ones whereas in the extended approach adopted here the order is preserved. Consequently it is not straightforward to recognise when the generated conservation laws via a canonical operator of a point operator are in fact new while it is easy to see this in the noncanonical situation. Furthermore the relationship between Lie–Bäcklund symmetry generators associated with a conserved form of an equation and the corresponding equation itself is not known in the canonical operator case. We show that such a relationship does exist. Indeed we find that the Lie algebra of Lie–Bäcklund symmetry generators of the conserved form of a system is a subalgebra of the Lie–Bäcklund symmetry generators of the system itself.

Consider an rth-order ($r \geq 1$) system of differential equations of n independent variables and m dependent variables

$$E^{\beta}\big(x, u, u_{(1)}, \ldots, u_{(r)}\big) = 0, \qquad \beta = 1, \ldots, \tilde{m}, \tag{2.1}$$

where $u_{(k)}$ denote the various collections of kth-order partial derivatives. The maximal order of the equations that appear in (2.1) is r. In most applications $\tilde{m} = m$. If x is a single independent variable, then (2.1) is a system of ordinary differential equations and otherwise it is a system of partial differential equations.

We use the following definitions and results. The summation convention is used where appropriate.

Definition 1 ([10]). The differential $(n-1)$-form

$$\omega = T^{i}\big(x, u, u_{(1)}, \ldots, u_{(r-1)}\big)\frac{\partial}{\partial x^{i}} \rfloor \left(\mathrm{d}x^{1} \wedge \ldots \wedge \mathrm{d}x^{n}\right) \tag{2.2}$$

is called a conserved form of

$$E^{\beta}\big(x, u, u_{(1)}, \ldots, u_{(r)}\big) = 0, \qquad \beta = 1, \ldots, \tilde{p} \leq \tilde{m} \tag{2.3}$$

if

$$D\omega = 0 \tag{2.4}$$

is satisfied on the manifold in the space of variables $x, u, u_{(1)}, \ldots, u_{(r)}$ defined by the system (2.3), where D is the total exterior derivative.

Theorem 1 ([10]). *Suppose that*

$$X = \xi^{i}\frac{\partial}{\partial x^{i}} + \eta^{\alpha}\frac{\partial}{\partial u^{\alpha}} + \zeta_{i}^{\alpha}\frac{\partial}{\partial u_{i}^{\alpha}} + \zeta_{i_1 i_2}^{\alpha}\frac{\partial}{\partial u_{i_1 i_2}^{\alpha}} + \cdots,$$

where the ξ^{i} $(i = 1, \ldots, n)$ *and* η^{α} $(\alpha = 1, \ldots, m)$ *are differential functions and the additional coefficients are*

$$\zeta_{i}^{\alpha} = D_{i}(\eta^{\alpha}) - u_{j}^{\alpha} D_{i}(\xi^{j}),$$
$$\zeta_{i_1 \ldots i_s}^{\alpha} = D_{i_s}(\zeta_{i_1 \ldots i_{s-1}}^{\alpha}) - u_{j i_1 \ldots i_{s-1}}^{\alpha} D_{i_s}(\xi^{j}), \qquad s > 1,$$

is a Lie–Bäcklund symmetry generator of the system (2.3) such that the conserved form ω *of (2.3), given by (2.2), is invariant under* X*. Then*

$$X(T^{i}) + T^{i} D_{j}(\xi^{j}) - T^{j} D_{j}(\xi^{i}) = 0, \qquad i = 1, \ldots, n. \tag{2.5}$$

Here $D_{i} = \partial/\partial x^{i} + u_{i}^{\alpha}\partial/\partial u^{\alpha} + \cdots$ *is the total derivative operator with respect to* x^{i}*.*

Note. For any differential function f, $Df = D_{i} f dx^{i}$ and for any k-form $\omega = f_{i_1 i_2 \ldots i_k}\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$, $D\omega = D_{j} f_{i_1 i_2 \ldots i_k}\mathrm{d}x^{j} \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$.

Definition 2 ([10]). A Lie–Bäcklund symmetry generator X is said to be *associated* with a conserved vector $T = \left(T^1, \ldots, T^n\right)$ (or its corresponding conserved form ω) of the system (2.3) if X and T^i satisfy the relations (2.5) (or equivalently if $X(\omega) = 0$).

It is a well established classical result (see e.g. [6, 17, 21]) that the Lie–Bäcklund operator X and the total differentiations D_j are related by the commutation rule as

$$[X, D_i] = -D_i(\xi^j)D_j, \qquad i = 1, \ldots, n. \tag{2.6}$$

For canonical Lie–Bäcklund symmetry operators, $\tilde{X} = X - \xi^j D_j$, (2.6) yields the well-known result (see e.g. [6, 17, 21]) that the canonical operator, $\tilde{X}$, commutes with the total differentiation D_i, viz.,

$$\tilde{X}(D_i(f)) = D_i\big(\tilde{X}(f)\big), \qquad i = 1, \ldots, n, \tag{2.7}$$

for any differential function f. If one sets $f = T^i$, $i = 1, \ldots, n$, where the T^is are the components of a conserved vector of a system (2.1), then

$$D_i\big(\tilde{X}\left(T^i\right)\big) = 0, \tag{2.8}$$

since $\tilde{X}\left(D_i\left(T^i\right)\right) = 0$ on the manifold in the space of variables $x, u, u_{(1)}, \ldots, u_{(r)}$ defined by the system (2.1). Hence

$$\tilde{T}^i_* = \tilde{X}\left(T^i\right), \qquad i = 1, \ldots, n, \tag{2.9}$$

constitute the components of a conserved vector of (2.1). This is the well-known generation theorem (see e.g. [6, 17, 21]) for conservation laws. In the following we extend this result to include *any* Lie–Bäcklund symmetry.

Theorem 2. *Suppose that X is any Lie–Bäcklund symmetry generator of (2.3) and T^i, $i = 1, \ldots, n$, are the components of a conserved vector of (2.3). Then*

$$T^i_* = X\left(T^i\right) + T^i D_j(\xi^j) - T^j D_j(\xi^i), \qquad i = 1, \ldots, n, \tag{2.10}$$

constitute the components of a conserved vector of (2.3), i.e.,

$$D_i T^i_* = 0$$

on the manifold in the space of variables $x, u, u_{(1)}, \ldots, u_{(r)}$ defined by the system (2.3).

Proof. If $D_i T^i = 0$ is a conservation law of (2.3), it follows from (2.6) that

$$D_i\left(X\left(T^i\right)\right) = D_i(\xi^j) D_j\left(T^i\right). \tag{2.11}$$

Now the application of D_i on T^i_* and the use of (2.11) easily result in $D_i\left(T^i_*\right) = 0$ on the manifold in the space of variables $x, u, u_{(1)}, \ldots, u_{(r)}$ given by the system (2.3) which proves the assertion. ■

Corollary. *If $\tilde{X}$ is the canonical operator of X, i.e. $\tilde{X} = X - \xi^i D_i$, and T^i_* is as in Theorem 2, then the following diagram commutes*

$$\begin{array}{ccc} X & \rightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ T_i^* & \rightarrow & \tilde{T}^i_*, \end{array}$$

where

$$\tilde{T}^i_* = T^i_* + T^k D_k(\xi^i) - D_k\big(\xi^k T^i\big). \tag{2.12}$$

Proof. The proof follows trivially by invoking (2.10) and (2.9). ∎

The Maxwellian tails model equation, viz.,

$$u_{xt} + u_x + u^2 = 0, \tag{2.13}$$

has conserved components,

$$T^1 = \frac{1}{3}u^3 \exp(3t), \qquad T^2 = \frac{1}{2}(u_t + u)^2 \exp(3t), \tag{2.14}$$

with associated symmetry $X_1 = \partial/\partial x$. Clearly equation (2.13) admits $X_2 = \partial/\partial t$. The action of X_2 on (2.14) yields a multiple of (2.14), viz. $T^1_* = 3T^1$ and $T^2_* = 3T_2$ and therefore does not produce a new conservation law. Thus the application of Theorem 2 does not guarantee new nontrivial conservation laws (we return to this aspect after Theorem 3 below). If one utilised the canonical approach, viz. $\tilde{X}_2 T^1 = \tilde{T}^1_*$ and $\tilde{X}_2 T^2 = \tilde{T}^2_*$, where $\tilde{X}_2$ is the canonical Lie–Bäcklund operator (not of point type), one would get

$$\tilde{T}^1_* = -u_t u^2 \exp(3t), \qquad \tilde{T}^2_* = -(u_t + u_{tt})(u_t + u)\exp(3t)$$

which is of higher order than T^1 and T^2. These are not new conserved components and are related to T^i_* via the above Corollary. That is

$$\tilde{T}^1_* = T^1_* + D_x T^2, \qquad \tilde{T}^2_* = T^2_* - D_t T^2.$$

Theorem 3. *A Lie–Bäcklund symmetry generator X associated with a conserved form ω (2.2) of a system (2.3) is a symmetry generator of the system (2.3).*

Proof. If

$$\omega = f_{i_1 i_2 \ldots i_k} \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

is any k-form, then

$$X\mathrm{D}\omega = \mathrm{D}X\omega$$

follows straightforwardly after the invocation of (2.6). Now let ω be the conserved form (2.2). Since X is associated with the conserved form ω, i.e., $X(\omega) = 0$, it follows that $X\mathrm{D}\omega = 0$ which implies that X is a symmetry of the system (2.3). ∎

To illustrate Theorem 3, we firstly determine the point symmetries associated with (2.14). To that end, we utilise the conditions (2.5). These yield, for T^1 and T^2 respectively, the following two symmetries

$$X_1 = \frac{\partial}{\partial x}, \qquad X_3 = \exp t \frac{\partial}{\partial t} - u \exp t \frac{\partial}{\partial u}. \tag{2.15}$$

Indeed $\{X_1, X_3\}$ forms a subalgebra of the Lie algebra of point symmetry generators of the equation (2.13), as it should, by Theorem 3.

We next provide the point symmetries associated with the magnitude of the conserved angular momentum (in polar coordinates)

$$L = r^2\dot{\theta} \tag{2.16}$$

of the radially dependent central force equation of motion

$$\ddot{\boldsymbol{r}} = \frac{f(r)}{r}\boldsymbol{r}, \tag{2.17}$$

where $f(r)$ is the magnitude of the radially dependent central force. The condition (2.5) gives the two-infinity point symmetry operators [5]

$$X_{\alpha,a} = \alpha(t)\frac{\partial}{\partial t} + \frac{1}{2}(r\dot{\alpha} - ra'(\theta))\frac{\partial}{\partial r} + a(\theta)\frac{\partial}{\partial \theta}.$$

By Theorem 3, the set $\{X_{\alpha,a}\}$ forms an infinite-dimensional subalgebra of point symmetry generators of the angular component of the central force equation, viz. $2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$.

The following definition is motivated by an equivalent one given for Lagrangian systems in [6].

Definition 3. Consider the set $\mathcal{C}$ of conserved vectors of a given system (2.3) which admits the symmetry Lie–Bäcklund algebra $\mathcal{L}$. A *basis of conservation laws* of the system (2.3) is a minimal subset of the set $\mathcal{C}$ which is obtained by the action, in the sense of (2.10), of each symmetry operator $X \in \mathcal{L}$ on the conserved vectors in $\mathcal{C}$.

Theorem 2 and its Corollary provide a mechanism to generate conservation laws from known symmetry generators and conservation laws of the system. However, the generated conserved vectors need not be nontrivial (it may be zero or a multiple of the known ones). When does this occur? The other question is as to when the generated conservation laws are nontrivial and how can one obtain a basis of conservation laws.

The answers lie precisely on the structure of the symmetry Lie algebra of the equation. Suppose that $\mathcal{L}$ is the symmetry Lie algebra of the equation. For any $Y \in \mathcal{L}$ the map $\operatorname{ad} Y : \mathcal{L} \to \mathcal{L}$ is defined by the derivations $\operatorname{ad} Y(X) = [X, Y]$. The set of elements $\operatorname{ad} Y$ for $Y \in \mathcal{L}$ is a Lie algebra which is called the *adjoint* algebra $\mathcal{L}^a$ of $\mathcal{L}$ since $[\operatorname{ad} Y_1, \operatorname{ad} Y_2] = \operatorname{ad}[Y_1, Y_2]$ for any $Y_1, Y_2 \in \mathcal{L}$.

We answer the first question.

Theorem 4. *Let* $\operatorname{ad} Y(X) = Z$ *such that* Y *is associated with the conserved vector* T *of (2.3) defined by* ω *and* X *is admitted by (2.3). Then* T_**, defined by* T_*^i *given by (2.10), is a trivial conserved vector of (2.3) if* $Z = bY$*, for any constant* b*.*

Proof. This follows simply by noting two points.

(i) If $Z = 0$, then $[X, Y]\omega = XY(\omega) - YX(\omega) = -YX(\omega) = (a_1X + a_2Y)(\omega) = 0$ for constants a_1 and a_2. Thus we have $X(\omega)$ vanishes or is a multiple of ω in which case $a_1 = 0$.

(ii) If $Z = bY$, it turns out that $X(\omega) = -b\omega$. ■

The second question is answered as follows.

Once again let $[X, Y] = Z$ such that Y is associated with the conserved vector T defined by ω of (2.3). The linear dependence, i.e. $X(\omega)$ being a multiple of ω, implies $Z = 0$ or $Z = bY$, for some constant b, which means that we require $Z \neq 0$ and $Z \neq bY$ in order for nontriviality of the generated conserved components T^i_* given in (2.10). This, however, is not a sufficient condition. The following example amply illustrates this fact.

Consider the following system that arises in porous media flow [4]

$$v_x = u, \qquad v_t = \left(u^{-1}\right)_x + cxu, \tag{2.18}$$

where c is a constant. An obvious conserved vector of (2.18) is $T = \left(-u, \left(u^{-1}\right)_x + cxu\right)$. Point symmetry generators admitted by the system (2.18) are

$$X_\alpha = \alpha(t, v)\frac{\partial}{\partial x} - \alpha_v u^2 \frac{\partial}{\partial u},$$

where α satisfies $\alpha_t + \alpha_{vv} + \alpha c = 0$. A symmetry generator associated with T is $Y = \partial/\partial t$. Now $\operatorname{ad} Y(X_\alpha) = -X_{\alpha_t}$ $(\neq 0, bY)$, where α_t satisfies $\alpha_{tt} + \alpha_{tvv} + \alpha_t c = 0$. By (2.10)

$$T_* = \left(-\alpha_v u^2 + u\alpha_v v_x, -\alpha cu + cxu^2\alpha_v - v_x\alpha_{vv} - u^{-2}u_x v_x \alpha_v - \alpha_t u - uv_t\alpha_v\right)$$

which is trivial since it vanishes on the solutions of (2.18).

Remark 1. In the case for which there is no symmetry operator associated with a conservation law, one can still act on the known conservation law with a symmetry generator of the equation by using Theorem 2.

We investigate the sufficient conditions for which the generated conservation is nontrivial. This occurs when the system (2.1) is derivable from a variational principle, i.e., when (2.1) can be written as an Euler–Lagrange equation with respect to a Lagrangian $L\left(x, u, \ldots, u_{(k)}\right)$, $k \leq r$, viz.

$$\frac{\delta L}{\delta u^\alpha} = 0, \qquad \alpha = 1, \ldots, m, \tag{2.19}$$

where $\delta/\delta u^\alpha$ is the Euler–Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\ldots i_s}}, \qquad \alpha = 1, \ldots, m. \tag{2.20}$$

A Lie–Bäcklund operator X is a *Noether symmetry generator* [16, 6, 17, 8] associated with a Lagrangian L of (2.19) if there exists a vector $B = \left(B^1, \ldots, B^n\right)$ such that

$$X(L) + LD_i(\xi^i) = D_i\left(B^i\right). \tag{2.21}$$

We also recall Noether's theorem [16].

Noether's theorem [16, 6, 8]. *For each Noether symmetry generator X associated with a given Lagrangian L of (2.19), there corresponds a vector $T = \left(T^1, \ldots, T^n\right)$ of (2.19), with T^i defined by*

$$T^i = N^i(L) - B^i, \qquad i = 1, \ldots, n, \tag{2.22}$$

which is a conserved vector of the Euler–Lagrange equations (2.19) and the Noether operator associated with X is

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1} \cdots D_{i_s}(W^\alpha) \frac{\delta}{\delta u^\alpha_{ii_1\ldots i_s}}, \qquad i = 1, \ldots, n,$$

in which the Euler–Lagrange operators with respect to derivatives of u^α are obtained from (2.20) by replacing u^α by the corresponding derivatives, e.g.,

$$\frac{\delta}{\delta u_i^\alpha} = \frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^s D_{j_1} \cdots D_{j_s} \frac{\partial}{\partial u^\alpha_{ij_1\ldots j_s}}, \qquad i = 1, \ldots, n, \quad \alpha = 1, \ldots, m.$$

Theorem 5 ([8]). *The components of the Noether conserved vector T of (2.19), T^i, given by (2.22), with respect to a Lie–Bäcklund operator X which is a generator of a Noether symmetry associated with a given Lagrangian L of (2.19), satisfy*

$$\begin{aligned} &X\left(T^i\right) + T^i D_k(\xi^k) - T^k D_k(\xi^i) \\ &\qquad = N^i\big(D_k\big(B^k\big)\big) + B^k D_k(\xi^i) - B^i D_k(\xi^k) - X\big(B^i\big). \end{aligned} \tag{2.23}$$

If (2.23) is satisfied, X is said to be associated with the Noether conserved vector T.

In view of equations (2.5), which hold irrespective of a Lagrangian, we can set the B^is to be zero in (2.23). This also follows from [8, 10].

The property $X(T) = 0$ has already been applied in the case of point symmetries to physically important ordinary differential equations by Leach [12].

Lemma. *If $\tilde{X}$ is the canonical Noether operator of X associated with a given Lagrangian L of (2.19), and T^i_* is a Noether conserved component generated by X, i.e. T^i_* is as in (2.10), then $\tilde{T}^i_*$ as in (2.12) is a Noether generated conserved component generated by the operator $\tilde{X}$.*

Theorem 6 (see Khamitova [11]). *Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be canonical Noether operators associated with a given Lagrangian L of (2.19) with $\operatorname{ad}\tilde{Y}(\tilde{X}) = \tilde{Z}$ such that to $\tilde{Y}$ there corresponds a Noether conserved vector T. Then the Noether conserved vector $\tilde{T}_*$, with components*

$$\tilde{T}^i_* = \tilde{X}\left(T^i\right), \tag{2.24}$$

corresponds to $\tilde{Z}$.

The following theorem provides the sufficient conditions for which the generated conservation law is nontrivial for a Lie–Bäcklund symmetry operator.

Theorem 7. *Suppose that X, Y and Z are Noether symmetry operators associated with a given Lagrangian L of (2.19) that satisfy $\mathrm{ad}\,Y(X) = Z$, where $Z \neq 0$ and $Z \neq bY$, such that Y is associated with T. Then the Noether conserved vector T_*, with components T_*^i given by (2.10), is associated with Z. Moreover, T_* is a nontrivial conserved vector different from T.*

Proof. Follows from the above Lemma and Theorem 6. ■

3 Applications

3.1 Nonlinear diffusion equations

The nonlinear diffusion-convection equation (1.1), viz. $u_t = (k(u)u_x)_x - (f(u))_x$, has an obvious conserved vector T with components

$$T^1 = u, \qquad T^2 = f(u) - k(u)u_x. \tag{3.1}$$

We determine all the other conservation laws, $D_tT^1 + D_xT^2 = 0$, for nonlinear (1.1), where T^1 and T^2 are up to first-order in the derivatives. The determining equations result in

$$\begin{aligned} T^1 &= A(t,x)u + B(t,x), \\ T^2 &= -k(u)u_xA(t,x) + f(u)A(t,x) + A_x \int k(u)\,du + C(t,x), \end{aligned} \tag{3.2}$$

where A, B and C satisfy

$$A_tu + f(u)A_x + A_{xx} \int k(u)\,du + B_t + C_x = 0.$$

We distinguish the following cases:

(a) $f(u) = 0$, $k(u) \neq \mathrm{const}$

$$\begin{aligned} T^1 &= (a_1x + a_2)u, \\ T^2 &= -k(u)u_x(a_1x + a_2) + a_1 \int k(u)\,du, \end{aligned} \tag{3.3}$$

where a_1 and a_2 are constants. Thus there are two independent conserved vectors $T_1 = \left(T_1^1, T_1^2\right)$ $(a_1 = 0, a_2 = 1)$ and $T_2 = \left(T_2^1, T_2^2\right)$ $(a_1 = 1, a_2 = 0)$.

The principal Lie algebra of (1.1) for $f = 0$ is spanned by $X_1 = \partial/\partial t$, $X_2 = \partial/\partial x$, $X_3 = 2t\partial/\partial t + x\partial/\partial x$. The algebra extends for three cases [19], viz.

1. $k(u) = e^u, \quad X_4 = x\dfrac{\partial}{\partial x} + 2\dfrac{\partial}{\partial u},$
2. $k(u) = u^\sigma, \quad \sigma \neq 0, -\dfrac{4}{3}, \quad X_4 = \dfrac{\sigma}{2}x\dfrac{\partial}{\partial x} + u\dfrac{\partial}{\partial u},$
3. $k(u) = u^{-4/3}, \quad X_4 = -\dfrac{2}{3}x\dfrac{\partial}{\partial x} + u\dfrac{\partial}{\partial u}, \quad X_5 = -x^2\dfrac{\partial}{\partial x} + 3xu\dfrac{\partial}{\partial u}.$

Remark 2. . In Case 3 the three symmetry generators X_2, X_4 and X_5 constitute $sl(3,\mathbb{R})$ which arises frequently in various applications. This algebra is intrinsic in the algebraic structure of linear ordinary differential equations [15] as well as other systems [13].

The conserved vector T_1 has associated symmetry generators X_1 and X_2, and T_2 has X_1. The symmetry operator X_3 has no associated conservation law. This is not unusual for scaling symmetry operators and has been observed before in the case of Noether symmetry operators (see e.g. [10]).

We determine the elements of the adjoint algebra of X_1 and X_2 in order to produce a basis of conservation laws.

We obtain $\operatorname{ad} X_1$: $\operatorname{ad} X_1(X_2) = 0$, $\operatorname{ad} X_1(X_3) = -2X_1$, $\operatorname{ad} X_1(X_4) = 0$ and $\operatorname{ad} X_1(X_5) = 0$. Hence the adjoint action does not produce the other linearly independent conserved vector by Theorem 4. Thus far a basis of conservation laws is both the vectors $\{T_1, T_2\}$.

We also have $\operatorname{ad} X_2$: $\operatorname{ad} X_2(X_1) = 0$, $\operatorname{ad} X_2(X_3) = -X_2$, $\operatorname{ad} X_2(X_4) = \alpha X_2$, $\alpha = -1, -\sigma/2, 2/3$ (for 1., 2., 3., respectively) and $\operatorname{ad}X_2(X_5) = -X_4$ for 3. Therefore by Theorem 4 only 3. allows the possibility of another linearly independent conserved vector $T_{1*} = \left(T^1_{1*}, T^2_{1*}\right)$, viz.

$$T^1_{1*} = X_5\left(T^1_1\right) + D_x\left(-x^2\right)T^1_1, \qquad T^2_{1*} = X_5\left(T^2_1\right).$$

Hence the adjoint action produces $T_{1*} = \left(xu, -xu_x u^{-4/3} - 3u^{-1/3}\right) = T_2$.

Therefore a basis of conservation laws is $\{T_1\}$.

Remark 3. If one had no prior knowledge of T_2 in 3. one could still have constructed it by the adjoint action to yield T_{1*}.

It was shown in [2] that nontrivial Lie–Bäcklund operators are admitted only for $k = a(u+b)^{-2}$, where a and b are constants. For $a = 1$ and $b = 0$, the Lie–Bäcklund operators have the form

$$X_r = U^{(r+2)}\frac{\partial}{\partial u} + \cdots \equiv \left[(D_x)^2(u^{-1})(D_x)^{-1}\right]^r D_x\left(u^{-2}u_x\right)\frac{\partial}{\partial u} + \cdots,$$

where r is a natural number. Using the conserved vectors T_1 and T_2 with $k = u^{-2}$, one can generate an infinite sequence of higher-order conservation laws. For example,

$$T^1_{1*} = U^{(r+2)}, \qquad T^2_{1*} = U^{(r+2)}\left(2u^{-3}u_x\right) - u^{-2}D_x U^{(r+2)}.$$

Likewise for T_{2*} one obtains

$$T^1_{2*} = xU^{(r+2)}, \qquad T^2_{2*} = u^{-2}U^{(r+2)} + 2xu^{-3}u_x U^{(r+2)} - xu^{-2}D_x U^{(r+2)}.$$

(b) $k(u) \neq \text{const}$, $f(u) = f_1 u + f_2$, where f_1, f_2 are constants not both zero

$$\begin{aligned} T^1 &= (a_1 x - a_1 f_1 t + a_3)u, \\ T^2 &= -k(u)u_x(a_1 x - a_1 f_1 t + a_3) + (f_1 u + f_2)(a_1 x - a_1 f_1 t + a_3) \\ &\quad + a_1\int k(u)\,du - a_1 f_2 x, \end{aligned} \tag{3.4}$$

where a_1 and a_3 are constants. The case $a_1 = 1$ and $a_3 = 0$ produces a conserved vector $T_1 = \left(T_1^1, T_1^2\right)$ which has no associated symmetry operator for arbitrary f_1 whereas $a_1 = 0$ and $a_3 = 1$ results in a vector $T_2 = \left(T_2^1, T_2^2\right)$ which is associated with $\partial/\partial t$.

According to our Remark 1, we can act on T_1 by means of the symmetry generator $\partial/\partial t$. This produces a conserved vector equivalent to T_2. Hence, a basis of a conserved vector is $\{T_1\}$.

(c) $f(u) = f_1 \int k(u)\,du + f_2 u + f_3$, where $f_1 \neq 0$, f_2, f_3 are constants and k an arbitrary nonconstant function

$$
\begin{aligned}
T^1 &= (a_1 + a_2 \exp(f_1 f_2 t - f_1 x))u,\\
T^2 &= -k(u)u_x(a_1 + a_2 \exp(f_1 f_2 t - f_1 x))\\
&\quad + \left(f_1 \int k(u)\,du + f_2 u + f_3\right)(a_1 + a_2 \exp(f_1 f_2 t - f_1 x))\\
&\quad - a_2 f_1 \exp(f_1 f_2 t - f_1 x) \int k(u)\,du - a_2 f_3 \exp(f_1 f_2 t - f_1 x),
\end{aligned}
\tag{3.5}
$$

where a_1 and a_2 are constants. If we set $a_1 = 1$ and $a_2 = 0$, this produces a conserved vector T_1 which has associated symmetry generators $\partial/\partial t$ and $\partial/\partial x$ whilst $a_1 = 0$ and $a_2 = 1$ result in a vector T_2 which has no associated symmetry generator for arbitrary f_2.

The action of $\partial/\partial t$ and $\partial/\partial x$ on T_2 yields a vector equivalent to T_2. Thus a basis of conserved vectors is $\{T_1, T_2\}$.

3.2 Nonlinear wave equations

The nonlinear wave equation (1.2) with $h = 0$ and $g = g(u_x)$, viz., $u_{tt} - g(u_x)u_{xx} = 0$, has Lagrangian

$$
L = \frac{1}{2}u_t^2 - \iint g(u_x)\mathrm{d}u_x\,\mathrm{d}u_x
$$

which admits the Noether point symmetry operators

$$
X_1 = \frac{\partial}{\partial t}, \qquad X_2 = t\frac{\partial}{\partial u}, \qquad X_3 = \frac{\partial}{\partial x}, \qquad X_4 = \frac{\partial}{\partial u}.
$$

From the determination of the elements of the adjoint algebra, a basis of conservation laws is $\{T_1, T_2, T_3\}$ (T_1, T_2 and T_3 are associated with X_1, X_2 and X_3, respectively), where

$$
\begin{aligned}
T_1 &= \left(\frac{1}{2}u_t^2 + \iint g(u_x)\mathrm{d}u_x\mathrm{d}u_x, -u_t \int g(u_x)\mathrm{d}u_x\right),\\
T_2 &= \left(tu_t - u, -t \int g(u_x)\mathrm{d}u_x\right),\\
T_3 &= \left(-u_x u_t, \frac{1}{2}u_t^2 - \iint g(u_x)\mathrm{d}u_x\mathrm{d}u_x + u_x \int g(u_x)\mathrm{d}u_x\right).
\end{aligned}
$$

The conservation law associated with X_4, obtained by the action of X_1 on T_2 (since $[X_1, X_2] = X_4$), is $\left(u_t, -\int g(u_x)\mathrm{d}u_x\right)$.

4 Conclusion

We have shown that for a system of partial differential equations, one can generate conservation laws from known ones using any Lie–Bäcklund symmetry operator of the system without having to make a conversion to a canonical Lie–Bäcklund symmetry operator. This approach, as we have seen, has distinct advantages. We did not need to convert to a canonical Lie–Bäcklund operator and thus a point symmetry generator remains of point type. Moreover the generated conservation laws using canonical Lie–Bäcklund symmetry operators are of a higher order than the original ones. In the approach here, for point symmetries, the order is preserved. Furthermore the relationship between Lie–Bäcklund symmetry generators associated with a conserved form of a system and the corresponding system itself is not known in the canonical case while we have shown that such a relation does exist if one does not transform to a canonical operator. Indeed we found that the Lie algebra of Lie–Bäcklund symmetry generators of the conserved form is a subalgebra of the symmetries of the system itself. We have also proved that the generated conserved vectors for the canonical and the straightforward cases are related by means of a formula.

We also investigated a basis of conservation laws and have shown that a generated conservation law via the action of a Lie–Bäcklund symmetry operator which satisfies a commutation rule is nontrivial if the system is derivable from a Lagrangian formulation.

We have given applications of nonlinear diffusion-convection and wave equations as well as presented other illustrative examples.

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