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Basis of Joint Invariants for $(1 + 1)$ Linear Hyperbolic Equations

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Abstract

We obtain a basis of joint or proper differential invariants for the scalar linear hyperbolic partial differential equation in two independent variables by the infinitesimal method. The joint invariants of the hyperbolic equation consist of combinations of the coefficients of the equation and their derivatives which remain invariant under equivalence transformations of the equation and are useful for classification purposes. We also derive the operators of invariant differentiation for this type of equation. Furthermore, we show that the other differential invariants are functions of the elements of this basis via their invariant derivatives. Applications to hyperbolic equations that are reducible to their Lie canonical forms are provided.

1 Introduction

The second-order scalar linear partial differential equation (PDE) in two independent variables (t, x) is of the form

$$Au_{tt} + 2Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu = G, \tag{1}$$

where A, B, C, D, E, F and G can be constants or given differentiable functions of t and x.

All linear PDEs similar to equation (1) are parabolic, hyperbolic or elliptic. Parabolic equations describe heat flow and diffusion processes and satisfy the property $B^2 - AC = 0$. Hyperbolic equations describe vibrating systems and wave motion and satisfy the property $B^2 - AC > 0$. Elliptic equations describe steady-state phenomena and satisfy the property $B^2 - AC < 0$.

Moreover, one can further simplify the equation (1) by introducing new coordinates, i.e. characteristics coordinates, see e.g. [8]. When this PDE (1) is written in terms of the new coordinates, it takes on one of three canonical forms (depending on whether $B^2 - AC$ is positive, zero, or negative respectively). We consider the hyperbolic canonical form.

The linear hyperbolic equation has a variety of applications in the physical and biological sciences [1, 2, 3, 4, 8, 14, 15, 16]. For example, population dynamics, tides and waves, chemical reactors, flame and combustion problems, the linearized theory of transonic aerodynamics, etc.

We are interested here in the differential invariants of the hyperbolic equation with respect to the dependent and independent variables by use of the infinitesimal method. We write it as

$$u_{tx} + a(t,x)u_t + b(t,x)u_x + c(t,x)u = 0, \tag{2}$$

where a, b and c are differentiable functions of t and x.

Lie [10] was the first to classify the general linear second-order PDE in two independent and one dependent variables. He obtained seven canonical forms according to their symmetries and classes of equations. Of these four belonged to the hyperbolic class and three belonged to the parabolic class (the elliptic equation can be transformed into the hyperbolic equation by means of complex transformations [3, 8]). He also developed methods for their integration. Practical criteria for reduction of parabolic equations to the classical heat equation are given in [7].

The two semi-invariants $h = a_t + ab - c$ and $k = b_x + ab - c$, known as Laplace invariants, were discovered by Laplace [9] in 1773 for the equation (2) in his fundamental memoir [9] dedicated to the integration theory of linear PDEs. These two quantities h and k are unaltered under linear transformations of the dependent variable

$$\bar{u} = \sigma(t,x)u, \qquad \sigma(t,x) \neq 0, \tag{3}$$

where σ is a twice differentiable arbitrary function.

It was claimed in [5], [6, p. 262] that the Ovsiannikov [12] joint invariants $p = k/h$ and $q = (\partial_t \partial_x \ln h)/h$ (see [5, 6, 12]) form a basis of invariants for the equation (2) under transformations of dependent and independent variables and the other joint invariants are functions of p, q and their invariant derivatives. Notwithstanding, the invariants p and q were utilized to classify equation (2) with three non-trivial Lie point symmetries when $p = \text{const}$ and $q = 0$ or $p = \text{const}$ and $q = \text{const} \neq 0$. In the case where p and q ($\neq 0$) are constants, equation (2) is reducible to the Euler–Poisson equation [13]. We show in this paper that there are three more third-order joint invariants of equation (2) in addition to the Ovsiannikov invariants which together form a basis of joint invariants. All other higher order joint invariants are functions of these via invariant differentiations.

We outline our work in this paper as follows. In Section 2 we deal with the derivation of the joint invariants by the infinitesimal method under the transformations of dependent and independent variables. Section 3 focuses on obtaining the operators of invariant differentiation which can be used to find the invariants of higher orders. In Section 4 we compare the basis of invariants obtained in the previous Sections 2 and 3 with those of the Lie canonical forms for the equation (2). Some examples are given in Section 5 to illustrate our results obtained. Finally, concluding remarks are made in Section 6.

2 Invariants of linear hyperbolic equations

In this section, we derive the joint differential invariants by the infinitesimal method. We begin this section by stating some preliminaries.

We recall that an equivalence transformation of equation (2) is an invertible transformation belonging to the class $\bar{t} = \phi(t,x,u)$, $\bar{x} = \varphi(t,x,u)$ and $\bar{u} = \psi(t,x,u)$ which preserves the order of equation (2) as well as the properties of linearity and homogeneity. In general, though, the transformed equation has new coefficients $\bar{a}$, $\bar{b}$ and $\bar{c}$.

It is also a known fact that the set of all equivalence transformations of equation (2) is an infinite group which consists of the linear transformations of the dependent variable (3) and invertible transformations on the independent variables:

$$\bar{t} = \phi(t), \qquad \bar{x} = \varphi(x), \qquad \phi_t \neq 0, \qquad \varphi_x \neq 0, \tag{4}$$

where $\phi(t)$ and $\varphi(x)$ are arbitrary functions and $\bar{u}$ is the new dependent variable. Two equations of the form (2) are called (locally) equivalent if they can be mapped to each other by a combination of the equivalence transformations (3)–(4). The semi-invariants (Laplace invariants) of (2) are combinations of the coefficients a, b and c of (2) that remain unchanged under the transformations (3) only. The joint differential invariants of the equation (2) are combinations of the Laplace invariants (h,k) and their derivatives which are unaltered under the transformations (3)–(4).

We obtain the joint differential invariants of (2) in terms of the Laplace invariants (h,k) by the infinitesimal method (see [5, 6] for the infinitesimal approach).

Firstly we write the operator in the form

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \mu \frac{\partial}{\partial a} + \nu \frac{\partial}{\partial b} + \omega \frac{\partial}{\partial c},$$

where $\xi^1 = \xi^1(t,x,u)$, $\xi^2 = \xi^2(t,x,u)$ and μ, ν and ω are functions of t, x, a, b and c. We invoke the determining equation $X(u_{tx} + au_t + bu_x + cu)|_{(2)} = 0$. One easily arrives at

$$\xi^1 = \alpha(t), \qquad \xi^2 = \beta(x), \qquad \mu = -a\beta_x, \qquad \nu = -b\alpha_t, \qquad \omega = -(c\alpha_t + c\beta_x), \tag{5}$$

where the functions $\alpha(t)$ and $\beta(x)$ are arbitrary.

We now seek a projected generator of the form

$$X = \alpha(t) \frac{\partial}{\partial t} + \beta(x) \frac{\partial}{\partial x} - a\beta_x \frac{\partial}{\partial a} - b\alpha_t \frac{\partial}{\partial b} - (c\alpha_t + c\beta_x) \frac{\partial}{\partial c} + \mu_t \frac{\partial}{\partial a_t} + \nu_x \frac{\partial}{\partial b_x}. \tag{6}$$

Here, we utilize the following total differentiations with respect to t and x:

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + a_t \frac{\partial}{\partial a} + a_{tt} \frac{\partial}{\partial a_t} + a_{tx} \frac{\partial}{\partial a_x} + \cdots + b_t \frac{\partial}{\partial b} \\ &\quad + b_{tt} \frac{\partial}{\partial b_t} + b_{tx} \frac{\partial}{\partial b_x} + \cdots + c_t \frac{\partial}{\partial c} + c_{tt} \frac{\partial}{\partial c_t} + c_{tx} \frac{\partial}{\partial c_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + a_x \frac{\partial}{\partial a} + a_{xx} \frac{\partial}{\partial a_x} + a_{tx} \frac{\partial}{\partial a_t} + \cdots + b_x \frac{\partial}{\partial b} \\ &\quad + b_{xx} \frac{\partial}{\partial b_x} + b_{tx} \frac{\partial}{\partial b_t} + \cdots + c_x \frac{\partial}{\partial c} + c_{xx} \frac{\partial}{\partial c_x} + c_{tx} \frac{\partial}{\partial c_t} + \cdots. \end{aligned} \tag{7}$$

We calculate μ_t by use of the equations (5) and (7) and it is

$$\mu_t = D_t(\mu) - a_t D_t(\xi^1) - a_x D_t(\xi^2) = -a_t(\alpha_t + \beta_x).$$

In a similar manner, ν_x is found as $\nu_x = -b_x(\alpha_t + \beta_x)$. Then, the action of X on the Laplace invariants yields

$$Xh = -(\alpha_t + \beta_x)h, \qquad Xk = -(\alpha_t + \beta_x)k.$$

We look for an infinitesimal generator, by utilising the preceding equations, in the space of the Laplace invariants h and k:

$$X = X(h)\frac{\partial}{\partial h} + X(k)\frac{\partial}{\partial k},$$

i.e., the generator

$$X = -(\alpha_t + \beta_x)h\frac{\partial}{\partial h} - (\alpha_t + \beta_x)k\frac{\partial}{\partial k}. \tag{8}$$

The infinitesimal test $XJ = 0$ for the invariants $J(h,k)$ is

$$h\frac{\partial J}{\partial h} + k\frac{\partial J}{\partial k} = 0.$$

The solution of this PDE gives us the first-order joint differential invariant $p = k/h$ which is one of Ovsiannikov's invariants [13] who derived it using another approach.

In order to find the second-order differential invariants, i.e., the invariants of the form $J(h,k,h_t,h_x,k_t,k_x)$, one should prolong the operator (8) once. We have

$$X = \mu\frac{\partial}{\partial h} + \nu\frac{\partial}{\partial k} + \mu_t\frac{\partial}{\partial h_t} + \mu_x\frac{\partial}{\partial h_x} + \nu_t\frac{\partial}{\partial k_t} + \nu_x\frac{\partial}{\partial k_x},$$

where $\mu = -(\alpha_t + \beta_x)h$, $\nu = -(\alpha_t + \beta_x)k$, and μ_t, μ_x, ν_t and ν_x are found via the total differentiations with respect to t and x presented in (9). The total differentiations are

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + h_t\frac{\partial}{\partial h} + h_{tt}\frac{\partial}{\partial h_t} + h_{tx}\frac{\partial}{\partial h_x} + \cdots + k_t\frac{\partial}{\partial k} + k_{tt}\frac{\partial}{\partial k_t} + k_{tx}\frac{\partial}{\partial k_x} + \cdots, \\ D_x &= \frac{\partial}{\partial x} + h_x\frac{\partial}{\partial h} + h_{xx}\frac{\partial}{\partial h_x} + h_{tx}\frac{\partial}{\partial h_t} + \cdots + k_x\frac{\partial}{\partial k} + k_{xx}\frac{\partial}{\partial k_x} + k_{tx}\frac{\partial}{\partial k_t} + \cdots \end{aligned} \tag{9}$$

with the aid of which we have

$$\begin{aligned} \mu_t &= D_t(-(\alpha_t + \beta_x)h)) - h_t D_t(\alpha) - h_x D_t(\beta), \\ &= -(\alpha_{tt}h + 2\alpha_t h_t + \beta_x h_t). \end{aligned}$$

and in an analogous manner we find

$$\begin{aligned} \mu_x &= -(\alpha_t h_x + \beta_{xx}h + 2\beta_x h_x), \\ \nu_t &= -(\alpha_{tt}k + 2\alpha_t k_t + \beta_x k_t), \\ \nu_x &= -(\alpha_t k_x + \beta_{xx}k + 2\beta_x k_x). \end{aligned}$$

Therefore, the once-extended generator of (8) is

$$\begin{aligned} X = {} & -(\alpha_t + \beta_x)h\frac{\partial}{\partial h} - (\alpha_t + \beta_x)k\frac{\partial}{\partial k} - (\alpha_{tt}h + 2\alpha_t h_t + \beta_x h_t)\frac{\partial}{\partial h_t} \\ & - (\alpha_t h_x + \beta_{xx}h + 2\beta_x h_x)\frac{\partial}{\partial h_x} - (\alpha_{tt}k + 2\alpha_t k_t + \beta_x k_t)\frac{\partial}{\partial k_t} \\ & - (\alpha_t k_x + \beta_{xx}k + 2\beta_x k_x)\frac{\partial}{\partial k_x}. \end{aligned} \tag{10}$$

The equation $XJ(h,k,h_t,h_x,k_t,k_x)=0$, upon the equation to zero of the coefficients of α_{tt}, β_{xx}, α_t and β_x, provides the following system of four PDEs:

$$h\frac{\partial J}{\partial h_t}+k\frac{\partial J}{\partial k_t}=0, \qquad h\frac{\partial J}{\partial h_x}+k\frac{\partial J}{\partial k_x}=0,$$
$$h\frac{\partial J}{\partial h}+k\frac{\partial J}{\partial k}+2h_t\frac{\partial J}{\partial h_t}+h_x\frac{\partial J}{\partial h_x}+2k_t\frac{\partial J}{\partial k_t}+k_x\frac{\partial J}{\partial k_x}=0,$$
$$h\frac{\partial J}{\partial h}+k\frac{\partial J}{\partial k}+h_t\frac{\partial J}{\partial h_t}+2h_x\frac{\partial J}{\partial h_x}+k_t\frac{\partial J}{\partial k_t}+2k_x\frac{\partial J}{\partial k_x}=0. \tag{11}$$

The solution of this system of PDEs (11) gives rise to

$$J=\Phi\left(p,J_2^1\right),$$

where Φ is an arbitrary function of the Ovsiannikov invariant p and the second-order joint differential invariant J_2^1 is

$$J_2^1=\frac{(hk_t-kh_t)(hk_x-kh_x)}{h^5}=\frac{1}{h}p_tp_x.$$

We further deduce the third-order differential invariants, i.e., those of the form $J(h,h_t,h_x,h_{tt},h_{tx},h_{xx};k,k_t,k_x,k_{tt},k_{tx},k_{xx})$ under the twice-extended generator of (8), viz.

$$X=\mu\frac{\partial}{\partial h}+\nu\frac{\partial}{\partial k}+\mu_t\frac{\partial}{\partial h_t}+\mu_x\frac{\partial}{\partial h_x}+\nu_t\frac{\partial}{\partial k_t}+\nu_x\frac{\partial}{\partial k_x}+\mu_{tt}\frac{\partial}{\partial h_{tt}}$$
$$+\mu_{tx}\frac{\partial}{\partial h_{tx}}+\mu_{xx}\frac{\partial}{\partial h_{xx}}+\nu_{tt}\frac{\partial}{\partial k_{tt}}+\nu_{tx}\frac{\partial}{\partial k_{tx}}+\nu_{xx}\frac{\partial}{\partial k_{xx}},$$

where

$$\mu_{tt}=-(\alpha_{ttt}h+3\alpha_{tt}h_t+3\alpha_th_{tt}+\beta_xh_{tt}),$$
$$\mu_{tx}=-(\alpha_{tt}h_x+2\alpha_th_{tx}+\beta_{xx}h_t+2\beta_xh_{tx}),$$
$$\mu_{xx}=-(\alpha_th_{xx}+\beta_{xxx}h+3\beta_{xx}h_x+3\beta_xh_{xx}),$$
$$\nu_{tt}=-(\alpha_{ttt}k+3\alpha_{tt}k_t+3\alpha_tk_{tt}+\beta_xk_{tt}),$$
$$\nu_{tx}=-(\alpha_{tt}k_x+2\alpha_tk_{tx}+\beta_{xx}k_t+2\beta_xk_{tx}),$$
$$\nu_{xx}=-(\alpha_tk_{xx}+\beta_{xxx}k+3\beta_{xx}k_x+3\beta_xk_{xx})$$

are calculated in a similar way as explained before.

Separation of the terms with β_{xxx}, α_{ttt}, β_{xx}, α_{tt}, α_t and β_x of the equation $XJ(h,h_t,h_x,h_{tt},h_{tx},h_{xx};k,k_t,k_x,k_{tt},k_{tx},k_{xx})=0$ results in the system of six PDEs:

$$k\frac{\partial J}{\partial k_{xx}}+h\frac{\partial J}{\partial h_{xx}}=0, \quad k\frac{\partial J}{\partial k_{tt}}+h\frac{\partial J}{\partial h_{tt}}=0,$$
$$3k_x\frac{\partial J}{\partial k_{xx}}+k_t\frac{\partial J}{\partial k_{tx}}+3h_x\frac{\partial J}{\partial h_{xx}}+h_t\frac{\partial J}{\partial h_{tx}}+k\frac{\partial J}{\partial k_x}+h\frac{\partial J}{\partial h_x}=0,$$
$$k_x\frac{\partial J}{\partial k_{tx}}+3k_t\frac{\partial J}{\partial k_{tt}}+h_x\frac{\partial J}{\partial h_{tx}}+3h_t\frac{\partial J}{\partial h_{tt}}+k\frac{\partial J}{\partial k_t}+h\frac{\partial J}{\partial h_t}=0,$$
$$k_{xx}\frac{\partial J}{\partial k_{xx}}+2k_{tx}\frac{\partial J}{\partial k_{tx}}+3k_{tt}\frac{\partial J}{\partial k_{tt}}+h_{xx}\frac{\partial J}{\partial h_{xx}}+2h_{tx}\frac{\partial J}{\partial h_{tx}}$$
$$+3h_{tt}\frac{\partial J}{\partial h_{tt}}+k_x\frac{\partial J}{\partial k_x}+2k_t\frac{\partial J}{\partial k_t}+h_x\frac{\partial J}{\partial h_x}+2h_t\frac{\partial J}{\partial h_t}+k\frac{\partial J}{\partial k}+h\frac{\partial J}{\partial h}=0,$$

$$3k_{xx}\frac{\partial J}{\partial k_{xx}} + 2k_{tx}\frac{\partial J}{\partial k_{tx}} + k_{tt}\frac{\partial J}{\partial k_{tt}} + 3h_{xx}\frac{\partial J}{\partial h_{xx}} + 2h_{tx}\frac{\partial J}{\partial h_{tx}}$$
$$+ h_{tt}\frac{\partial J}{\partial h_{tt}} + 2k_x\frac{\partial J}{\partial k_x} + k_t\frac{\partial J}{\partial k_t} + 2h_x\frac{\partial J}{\partial h_x} + h_t\frac{\partial J}{\partial h_t} + k\frac{\partial J}{\partial k} + h\frac{\partial J}{\partial h} = 0. \qquad (12)$$

The solution of this system of six equations leads to

$$J = \Phi\left(p, J_2^1, J_3^1, J_3^2, J_3^3, J_3^4\right),$$

where Φ is an arbitrary function of

$$p, \qquad J_2^1,$$
$$J_3^1 = \frac{1}{h^3}(kh_{tx} + hk_{tx} - h_tk_x - h_xk_t),$$
$$J_3^2 = \frac{1}{h^9}(hk_x - kh_x)^2\left(hkh_{tt} - h^2k_{tt} - 3kh_t^2 + 3hh_tk_t\right),$$
$$J_3^3 = \frac{1}{h^9}(hk_t - kh_t)^2\left(hkh_{xx} - h^2k_{xx} - 3kh_x^2 + 3hh_xk_x\right),$$
$$J_3^4 = \frac{k}{h^4}(hh_{tx} - h_th_x) \qquad (13)$$

in which h and k are nonzero. In the event that one of them is zero, one can factorize the equation (2). The joint differential invariant J_3^4 can be written as

$$J_3^4 = \frac{k}{h}\frac{(\partial_t\partial_x \ln h)}{h} = p\frac{(\partial_t\partial_x \ln h)}{h}.$$

Since p and J_3^4 are joint invariants, then $q = (\partial_t\partial_x \ln h)/h$ must be a joint invariant. Of course, joint invariants p and q are known as the Ovsiannikov invariants (see [13]).

3 Basis elements and invariant differentiation operator

In this section, we find the operators of invariant differentiation that enable one to calculate the joint differential invariants of higher orders for (2).

Recall that an operator $\tilde{X}$ (see [13]) is said to be an operator of invariant differentiation for a group $\tilde{G}$ if for any differential invariant J of the group $\tilde{G}$, $\tilde{X}(J)$ is also a differential invariant of this group.

Let the operator $\mathcal{D}$ be defined by

$$\mathcal{D} = \lambda D_t + \kappa D_x,$$

where λ and κ are differential functions of h, k and their derivatives and D_t, D_x be given by (9). The first prolongation of the generator (8), viz. (10), and the formula for the invariant differentiation operator, viz. $\tilde{X} = X + \mathcal{D}(\xi^1\partial_\lambda + \xi^2\partial_\kappa)$ (see [13, p. 316]) result in

$$\tilde{X} = -(\alpha_t + \beta_x)h\frac{\partial}{\partial h} - (\alpha_t + \beta_x)k\frac{\partial}{\partial k} - (\alpha_{tt}h + 2\alpha_th_t + \beta_xh_t)\frac{\partial}{\partial h_t}$$
$$- (\alpha_th_x + \beta_{xx}h + 2\beta_xh_x)\frac{\partial}{\partial h_x} - (\alpha_{tt}k + 2\alpha_tk_t + \beta_xk_t)\frac{\partial}{\partial k_t}$$
$$- (\alpha_tk_x + \beta_{xx}k + 2\beta_xk_x)\frac{\partial}{\partial k_x} + \lambda\alpha_t\frac{\partial}{\partial \lambda} + \kappa\beta_x\frac{\partial}{\partial \kappa}. \qquad (14)$$

Since the functions α and β are arbitrary, upon the equation to zero of the coefficients of α_{tt}, β_{xx}, α_t and β_x in the equation $\tilde{X}J(h, h_t, h_x; k, k_t, k_x; \lambda, \kappa) = 0$ yield the following system of four PDEs:

$$\begin{aligned}
&h\frac{\partial J}{\partial h_t} + k\frac{\partial J}{\partial k_t} = 0, \qquad h\frac{\partial J}{\partial h_x} + k\frac{\partial J}{\partial k_x} = 0,\\
&h\frac{\partial J}{\partial h} + k\frac{\partial J}{\partial k} + 2h_t\frac{\partial J}{\partial h_t} + h_x\frac{\partial J}{\partial h_x} + 2k_t\frac{\partial J}{\partial k_t} + k_x\frac{\partial J}{\partial k_x} - \lambda\frac{\partial J}{\partial \lambda} = 0,\\
&h\frac{\partial J}{\partial h} + k\frac{\partial J}{\partial k} + h_t\frac{\partial J}{\partial h_t} + 2h_x\frac{\partial J}{\partial h_x} + k_t\frac{\partial J}{\partial k_t} + 2k_x\frac{\partial J}{\partial k_x} - \kappa\frac{\partial J}{\partial \kappa} = 0.
\end{aligned} \tag{15}$$

Solution of the above equations (15) results in

$$J = J\left(p, J_2^1, C_1, C_2\right),$$

where p and J_2^1 are as before and the constants C_1 and C_2 are

$$C_1 = \lambda\kappa h, \qquad C_2 = \frac{\kappa}{h^2}(hk_x - kh_x). \tag{16}$$

We have from equations (16) that

$$\lambda = \frac{hk_x - kh_x}{h^3}C_3, \qquad \kappa = \frac{h^2}{hk_x - kh_x}C_2, \tag{17}$$

where C_3 is a constant. In the cases $C_2 = 1$, $C_3 = 0$ and $C_2 = 0$, $C_3 = 1$, one obtains two independent operators of invariant differentiation

$$\tilde{X}_1 = \frac{hk_x - kh_x}{h^3}D_t, \qquad \tilde{X}_2 = \frac{h^2}{hk_x - kh_x}D_x, \tag{18}$$

respectively.

If one uses the operator of invariant differentiation $\tilde{X}_1$ on p, the joint differential invariant J_2^1 is obtained, i.e., $\tilde{X}_1(p) = J_2^1$. Similarly, $\tilde{X}_2(p) = 1$ is found, i.e., no new invariant is found. Hence, a basis of joint differential invariants is

$$\left\{p, q, J_3^1, J_3^2, J_3^3\right\}. \tag{19}$$

Now we are in a position to state the following theorem.

Theorem 1. *The basis of invariants (19) of (2) defined by (2) gives a complete set of joint differential invariants of equation (2). Any other joint differential invariant is a function of the basic invariants (19) and their invariant derivatives.*

The joint differential invariants (19) defined by (2) provide necessary conditions for local equivalence of two $(1+1)$ linear hyperbolic equations of the form (2). The sufficient conditions are obtained by the construction of the transformations that map two equations of this type to each other.

4 Lie canonical forms

Lie [11] showed that a linear second-order hyperbolic equation can be reduced to one of four canonical forms according to the non-trivial Lie point symmetries it admits. In this section, we calculate the joint differential invariants obtained in Section 2 for the Lie canonical forms of the canonical linear hyperbolic equation. This is required for the applications in the next section.

Recall that in the event of the Laplace invariants h or k being zero, the hyperbolic equation (2) is factorizable.

Consider the first Lie canonical form

$$u_{tx} + A(x)u_x + u = 0, \tag{20}$$

where $A(x)$ is an arbitrary nonzero function of x. We have the Laplace invariants $h = -1$ and $k = A' - 1$. Then the corresponding joint invariants are

$$p = 1 - A', \qquad q = 0, \qquad J_3^1 = 0, \qquad J_3^2 = 0, \qquad J_3^3 = 0,$$

provided $A' \neq 1$. Any hyperbolic equation of the form (2) having the above differential invariants is reducible to the equation (20).

The second Lie canonical form is given by

$$u_{tx} + Q(t-x)u_x + Z(t-x)u = 0, \tag{21}$$

where Q and Z are arbitrary functions of their arguments. It has the Laplace invariants $h = -Z$ and $k = -(Q' + Z)$. The basic invariants of (21) are

$$p = \frac{Q'}{Z} + 1 = c + 1, \qquad q = \frac{Z''}{Z^2} - \frac{(Z')^2}{Z^3} = \frac{(\ln Z)''}{Z},$$
$$J_3^1 = -\frac{1}{Z^3}\left[2Z'Q'' + 2(Z')^2 - Q'Z'' - 2ZZ'' - ZQ'''\right] = 2(c+1)\frac{(\ln Z)''}{Z},$$
$$J_3^2 = 0, \qquad J_3^3 = 0,$$

where $Z \neq 0$ and $Q'/Z = c = \text{const}\ (\neq -1)$. Any equation of the form (2) is transformable into the equation (21), if it has the same differential invariants as above.

Now consider the third Lie canonical form

$$u_{tx} + Axu_x + u = 0, \tag{22}$$

where A is an arbitrary constant. The equation (22) has the Laplace invariants $h = -1$, $k = A - 1$ and the basic invariants of (22) are

$$p = 1 - A, \qquad q = 0, \qquad J_3^1 = 0, \qquad J_3^2 = 0, \qquad J_3^3 = 0,$$

where $A \neq 1$.

The fourth Lie canonical form is

$$u_{tx} + \frac{A}{t-x}u_x + \frac{B}{(t-x)^2}u = 0, \tag{23}$$

where A and B are arbitrary constants. The Laplace invariants are $h = -B/(t-x)^2$ and $k = (A-B)/(t-x)^2$. The basic joint invariants for this equation (23) are

$$p = 1 - \frac{A}{B}, \qquad q = \frac{2}{B}, \qquad J_3^1 = -\frac{4}{B^2}(A - B), \qquad J_3^2 = 0, \qquad J_3^3 = 0,$$

where $B \neq 0$ or $A \neq B$.

5 Applications

Some examples are given to illustrate the results obtained in Section 3.

Example 1. Consider the telegrapher's equation [16]

$$u_{tx} + \frac{\lambda}{2}(u_t + u_x) = 0, \qquad \lambda = \text{const} \neq 0. \tag{24}$$

This equation governs the propagation of signals on telegraph lines and is of dissipative type. Equation (24) has Laplace invariants $h = \lambda^2/4 = k$ and the joint differential invariants (19) are

$$p = 1, \qquad q = 0, \qquad J_3^1 = 0, \qquad J_3^2 = 0, \qquad J_3^3 = 0.$$

Hence, equation (24) is transformable into the third canonical form (22)

$$\bar{u}_{\bar{t}\bar{x}} + \bar{u} = 0, \qquad A = 0,$$

by means of the transformation

$$\bar{t} = \frac{\lambda}{2}t, \qquad \bar{x} = -\frac{\lambda}{2}x, \qquad \bar{u} = u\exp\left\{\frac{\lambda}{2}(t+x)\right\}.$$

Example 2. We now consider another equation [4]:

$$u_{tx} + \frac{l}{t-x}(u_t - u_x) = 0, \qquad l \neq 0, -1 \tag{25}$$

which has the Laplace invariants $h = -l(l+1)/(t-x)^2 = k$. Its invariants (19) are

$$p = 1, \qquad q = \frac{2}{l(l+1)}, \qquad J_3^1 = \frac{4}{l(l+1)}, \qquad J_3^2 = 0, \qquad J_3^3 = 0.$$

Thus, equation (25) is reducible to the fourth canonical form (23)

$$\bar{u}_{\bar{t}\bar{x}} + \frac{B}{(\bar{t}-\bar{x})^2}\bar{u} = 0,$$

provided $A = 0$ and $B = l(l+1)$, by the equivalence transformation

$$\bar{t} = t, \qquad \bar{x} = x, \qquad \bar{u} = u(t-x)^{-l}.$$

Example 3. Finally, consider

$$u_{tx} + (x+2)u_t + xu_x + u = 0 \tag{26}$$

which has the Laplace invariants $h = x^2 + 2x - 1$, $k = x^2 + 2x$ and invariants (19)

$$p = \frac{x^2+2x}{x^2+2x-1}, \qquad q = 0, \qquad J_3^1 = 0, \qquad J_3^2 = 0, \qquad J_3^3 = 0.$$

Comparison of the invariants of the first canonical form (20) with those of (26) and the use of the rules of derivatives $D_t = \dot{\phi}(t)\bar{D}_{\bar{t}}$, $D_x = \varphi'(x)\bar{D}_{\bar{x}}$ give

$$\bar{A}(\bar{x}) = -x, \qquad \varphi(x) = \frac{x^3}{3} + x^2 - x.$$

Moreover,

$$\phi(t) = -t, \qquad \sigma(t,x) = \exp\left\{\frac{x^2}{2} + 2x\right\}.$$

Thus, equation (26) is reducible to the first canonical form (20)

$$\bar{u}_{\bar{t}\bar{x}} + \bar{A}(\bar{x})\bar{u}_{\bar{x}} + \bar{u} = 0$$

by means of the transformation

$$\bar{t} = -t, \qquad \bar{x} = \frac{x^3}{3} + x^2 - x, \qquad \bar{u} = u\exp\left\{\frac{x^2}{2} + 2x\right\}.$$

6 Concluding remarks

We have derived the complete set of joint differential invariants for the scalar linear hyperbolic equation of the form (2) upto third order by the infinitesimal method. This completes the Ovsiannikov invariants obtained in [12, 13]. In fact, we have found a basis of joint differential invariants for equation (2). The operators of invariant differentiation were obtained that enable one to find the joint differential invariants of higher orders for hyperbolic equations (2). Other invariants of (2) are functions of the basic invariants and their invariant derivatives. Finally, some examples were given to illustrate our results.

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