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Singularity Analysis and a Function Unifying the Painlevé and the Psi Series

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Abstract

The classical (ARS) algorithm used in the Painlevé test picks up only those functions analytic in the complex plane. We complement it with an iterative algorithm giving the leading order and the next terms in all cases. This algorithm works both for an ascending series (about a singularity at finite time) and a descending series (asymptotic expansion for $t \to \infty$). The algorithm introduces naturally the logarithmic terms when they are necessary. The calculation, given in the first place for a system possessing the two symmetries of time translation and self-similarity, is subsequently generalised to the case in which this last symmetry is broken. The algorithm enlarges the class of equations for which more explicit methods (Lie symmetries, Darboux and Carleman invariants *etc*) should be applied with a certain hope of success.

1 Introduction

A common outcome of the modelling process is a system of ordinary differential equations, the solution of which describes the behaviour of the model and, provided the modelling has been sufficiently accurate, the reality underlying the model. The precise meaning of the solution of a system of differential equations can be cast in several ways:

(*i*) A set of explicit functions describing the variation of the dependent variables with the independent variable.

(*ii*) The existence of a sufficient number of independent explicit first integrals or conservation laws.

(*iii*) The existence of a sufficient number of explicit Lie symmetries which permits the reduction of the system of differential equations to a system of algebraic equations.

A feature, central to each of these three not completely equivalent prescriptions of integrability, is the existence of explicit functions, be they solutions, first integrals or the coefficient functions of Lie symmetries.

There is another approach to the question of integrability which is not concerned with the display of explicit functions, but with the demonstration of a specific property. This is the existence of a Laurent series for each of the dependent variables. The series may not be summable to an explicit form, but does represent an analytic function. The essential feature of this Laurent series is that it is an expansion about a particular type of movable singularity, *videlicet* a pole. Consequently the existence of these Laurent series is intimately concerned with the singularity analysis of differential equations initiated about a century ago by Painlevé, Gambier and Garnier [16] and continued since by many workers including Chazy [5], Bureau [4] and Cosgrove *et al* [7].

The connection of this type of singular behaviour and the solution of partial differential equations by the method of the Inverse Scattering Transform was noticed by Ablowitz *et al* [1, 2, 3] who developed an algorithm, called the ARS algorithm, to test whether the solution of an ordinary differential equation was expressible in terms of a Laurent expansion. If this was the case, the ordinary differential equation was said to pass the Painlevé test and was conjectured to be integrable. Under more precise conditions Conte [6] has shown that the equation is integrable. The test can also be, *mutatis mutandis*, applied to partial differential equations, but, for the purposes of the present exposition, we confine our attention to ordinary differential equations.

In a recent review of integrable systems Ramani *et al* [21] provide a listing of the process of implementation of the ARS algorithm. In the positive sense of the algorithm the polelike nature of the movable singularity is identified, the appearance of arbitrary constants of integration is determined and the consistency of the proposed Laurent expansion with the ordinary differential equation identified.

Failure of the algorithm at any one of these steps leads to rejection and the equation is deemed to be nonintegrable. We emphasise that this nonintegrability is at the level of a function, represented by a Laurent series, analytic in the complex plane with the exception of singularities which are movable poles.

The manner in which the algorithm is presented by Ramani *et al* [21] is for its implementation by the practitioner who seeks a ready answer to the question of integrability in terms of analytic functions. In this paper we are concerned not so much with the answer to the question of a particular equation's integrability as with understanding the process of the analysis of an equation in terms of its singularities. To this end we present an approach, different in philosophy to that of the ARS algorithm, which provides a greater understanding of the underlying approach, explains why certain things can go wrong and leads to a better appreciation of the succinct nature of the ARS algorithm.

2 Singular and next to singular behaviour

We detect possible singular behaviour in the solution of a differential equation by means of the leading order analysis of the differential equation. Suppose that we have an autonomous ordinary differential equation (any nonautonomous ordinary differential equation can be rendered autonomous by an increase in order)

$$x^{(n)} = E\big(x, \dot{x}, \dots, x^{(n-1)}\big). \tag{2.1}$$

To determine the leading order behaviour we set

$$x = \alpha\tau^p, \tag{2.2}$$

where $\tau = t - t_0$ and t_0 is the location of the supposed movable singularity, substitute this into the ordinary differential equation and look for two or more dominant terms. The detection of which terms are dominant is identical to the determination of which terms in an equation are self-similar, *ie* invariant under the action of the symmetry

$$G_2 = -qt\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \tag{2.3}$$

where q is a parameter. As we are concerned with just this process, we assume the differential equation to be invariant under both (2.3) and time translation. In § 5 we see what happens when the conditions are relaxed. The equation is composed of the variables

$$\frac{\dot{x}}{x^{q+1}}, \quad \frac{\ddot{x}}{x^{2q+1}}, \quad \frac{\dddot{x}}{x^{3q+1}}, \quad \dots, \tag{2.4}$$

which are the simultaneous invariants of the two symmetries. The general second order ordinary differential equation of this form is

$$\ddot{x} + x^{2q+1} f\left(\frac{\dot{x}}{x^{q+1}}\right) = 0 \tag{2.5}$$

and, when (2.2) is substituted, we obtain

$$p(p-1)\tau^{p-2} + \alpha^{2q}\tau^{(2q+1)p} f\left(\frac{p\tau^{p-1}}{\alpha^q\tau^{(q+1)p}}\right) = 0 \tag{2.6}$$

from which it is evident that the terms balance for general f if

$$p = -\frac{1}{q}, \qquad \frac{1}{q}\left(1 + \frac{1}{q}\right) + \alpha^{2q} f\left(\frac{-1}{q\alpha^q}\right) = 0, \tag{2.7}$$

ie the nature of the singularity is determined by the value of q and the coefficient, α, by the functional form of the differential equation. The only exception is when q is arbitrary, *ie* the equation is separately invariant under the two homogeneity symmetries

$$G_{21} = t\frac{\partial}{\partial t}, \qquad G_{22} = x\frac{\partial}{\partial x}. \tag{2.8}$$

Then the coefficient, α, is arbitrary and the equation determines the value of p. A simple example of this is found in the generalised Kummer–Schwarz equation [8, p. 224]

$$\dot{x}\,\dddot{x} + k\ddot{x}^2 = 0. \tag{2.9}$$

The exponent, p, of the leading term is the solution of

$$p^2(p-1)(p-2)+kp^2(p-1)^2=0,$$

ie

$$p=0,1,\frac{k+2}{k+1}. \tag{2.10}$$

In the case of the usual Kummer–Schwarz equation $k=-3/2$ and we find that the leading order behaviour is a simple pole.

Once the leading order of the solution to an equation is found, we seek the behaviour of the next to leading order term. In the philosophy of the ARS algorithm we require that this, and all subsequent terms, be compatible with the Laurent series imposed by the analycity criterion. This requirement can lead to problems. Consider the equation

$$\ddot{x}+x\dot{x}+kx^3=0 \tag{2.11}$$

which is of the type of (2.5) and which has attracted attention due to its divers provenances [14, 9, 17] and interesting properties [19, 18]. The leading order behaviour is

$$x=\alpha\tau^{-1}, \qquad 2-\alpha+k\alpha^2=0, \tag{2.12}$$

ie the movable singularity is a simple pole. Evidently our Laurent series starts at τ^{-1}. Following a standard procedure we substitute

$$x=\sum_{i=0}^{\infty}a_i\tau^{i-1} \tag{2.13}$$

($a_0=\alpha$) into (2.11) to compute the a_1, a_2, $\ldots$ through the equation of coefficients of like powers of τ to zero in

$$\sum_{i=0}^{\infty}a_i(i-1)(i-2)\tau^{i-3}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_ia_j(i-1)\tau^{i+j-3} + k\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}a_ia_ja_k\tau^{i+j+k-3}=0. \tag{2.14}$$

That this is satisfied for τ^{-3} is guaranteed by (2.12). At τ^{-2} we require

$$a_1(3ka_0-1)a_0=0$$

so that either a_1 is zero or $k=1/9$ and a_1 is arbitrary. In the latter case we have the next to singular behaviour. In the former case we move to τ^{-1} and find

$$ka_0\left(a_0a_2+a_1^2\right)=0$$

which gives an arbitrary a_2 only if $k=0$ since $a_1=0$. If this not be the case, we must take $a_2=0$ and at τ^0 we find that a_3 is arbitrary only if $k=-1$.

The process outlined above can be repeated indefinitely. At each power k is required to have a specific value for that power to begin the rest of the Laurent series and to represent the next to singular behaviour. Otherwise we must continue. In other words except for a set of precise values of the parameter k in the equation we are going to obtain zero for all of the coefficients in the Laurent expansion. Clearly, if we are interested in the next to singular behaviour of the solution of the equation, the imposition of a Laurent series is not the way to determine this behaviour for general values of the parameter k.

3 Next to singular behaviour through perturbation

Instead of imposing a Laurent series commencing at the power indicated by the singularity found by the leading order analysis we can determine the next to singular behaviour by writing

$$x(\tau) = \alpha\tau^p + g(\tau). \tag{3.1}$$

We can always write $x(\tau)$ in the form (3.1). To make the process useful we require that the first term be the leading order term, *ie* $\tau^{-p}g(\tau) \to 0$ as $\tau \to 0$ ($\Leftrightarrow t \to t_0$).

We return to the specific example of (2.11) and obtain, after taking into account (2.12),

$$\ddot{g} + \alpha\tau^{-1}\dot{g} - \alpha\tau^{-2}g + g\dot{g} + k\left(3\alpha^2\tau^{-2}g + 3\alpha\tau^{-1}g^2 + g^3\right) = 0$$

which can be rearranged as

$$\tau^2\ddot{g} + \alpha\tau\dot{g} + (3k\alpha - 1)\alpha g + (\tau g)(\tau\dot{g}) + 3k\alpha(\tau g)g + k(\tau g)^2 g = 0. \tag{3.2}$$

In the neighbourhood of t_0 we can linearise (3.2) since $\tau g \approx 0$. We have

$$\tau^2\ddot{g} + \alpha\tau\dot{g} + (3k\alpha - 1)\alpha g = 0, \tag{3.3}$$

when (2.12) is taken into account, which is an equation of Euler type for g with the solution

$$g(\tau) = \begin{cases} K_1\tau^{-2} + K_2\tau^{3-\alpha}, & \alpha \neq 5, \\ (K_1 + K_2\log\tau)\tau^{-2}, & \alpha = 5, \end{cases} \tag{3.4}$$

where K_1 and K_2 are arbitrary constants. The only solution compatible with $\tau g(\tau) \to 0$ as $\tau \to 0$ is $g(\tau) = K_2\tau^{3-\alpha}$ provided $\alpha < 4$. We return to the implications of the other solution in § 4.

Now that we have the term next to the singular behaviour we could contemplate examining the following term by putting

$$x(\tau) = \alpha\tau^{-1} + K_2\tau^{3-\alpha} + g(\tau) \tag{3.5}$$

for the case $\alpha < 4$ and by imposing the condition that $\tau^{\alpha-3}g(\tau) \to 0$ as $\tau \to 0$. We substitute (3.5) into (2.11) and, after eliminating terms according to the assumed behaviour of $g(\tau)$, obtain the nonhomogeneous equation

$$\tau^2\ddot{g} + \alpha\tau\dot{g} + (3k\alpha - 1)\alpha g = -K_2^2(3 - \alpha + 3k\alpha)\tau^{7-2\alpha} \tag{3.6}$$

with particular solution

$$g = \frac{K_2^2\left(\alpha^2 - 6\alpha + 6\right)}{\alpha(2\alpha - 9)(\alpha - 4)}\tau^{7-2\alpha} \tag{3.7}$$

since $\alpha < 4$. The complementary solution is as given in (3.4). Clearly the procedure can be continued indefinitely.

We note that the powers in the expansion thus far are τ^{-1}, $\tau^{3-\alpha}$ and $\tau^{7-2\alpha}$. This suggests that

$$x(\tau) = \tau^{-1} f\left(\tau^{4-\alpha}\right). \tag{3.8}$$

Taking as new variable $\tau = \tau^{4-\alpha}$ and substituting (3.8) into (2.11) we find that only the argument τ appears which justifies the *ansatz* of (3.8). However, the second ordinary differential equation satisfied by f as a function of $\tau^{4-\alpha}$ is not so simple, being

$$(4-\alpha)^2\tau^2\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2} + (4-\alpha)(1-\alpha)\tau\frac{\mathrm{d}f}{\mathrm{d}\tau} + (4-\alpha)\tau f\frac{\mathrm{d}f}{\mathrm{d}\tau} + 2f - f^2 + kf^3 = 0. \tag{3.9}$$

In terms of $\sigma = \log(\tau^{4-\alpha})$, equation (3.9) becomes autonomous in σ and is

$$(4-\alpha)^2\frac{\mathrm{d}f}{\mathrm{d}\sigma^2} - 3(4-\alpha)\frac{\mathrm{d}f}{\mathrm{d}\sigma} + (4-\alpha)f\frac{\mathrm{d}f}{\mathrm{d}\sigma} + 2f - f^2 + kf^3 = 0. \tag{3.10}$$

We note that (3.10) is an autonomous equation in what seem to be the natural variables of the problem and is of Gambier's Type 5 [8, p 495]. It is also a member of the Riccati hierarchy as is more obvious when it is written as

$$(4-\alpha)^2\left[\frac{\mathrm{d}^2 f}{\mathrm{d}\sigma^2} + f\frac{\mathrm{d}f}{\mathrm{d}\sigma} + \frac{k}{4-\alpha}f^3\right] - 3(4-\alpha)\left[\frac{\mathrm{d}f}{\mathrm{d}\sigma} + \frac{f^2}{3(4-\alpha)}\right] + 2f = 0.$$

The procedure outlined in some detail above to the example of (2.11) can be applied generally. For an equation of type (2.5) we simply calculate the linearisation by considering the introduced g function as a perturbation about the singular solution. For the first order invariant, $\dot{x}/x^{q+1}$, we have

$$\xi = \frac{\dot{x}}{x^{q+1}} = \frac{-1}{q\alpha^q}\left[1 - \frac{q}{\alpha}\tau^{1+\frac{1}{q}}\dot{g} - \frac{(q+1)}{\alpha}\tau^{\frac{1}{q}}g\right] \tag{3.11}$$

and for the second order invariant

$$\eta = \frac{\ddot{x}}{x^{2q+1}} = \frac{(q+1)}{q^2\alpha^{2q}}\left[1 + q^2\tau^{2+\frac{1}{q}}(q+1)\alpha\ddot{g} - \frac{(2q+1)}{\alpha}\tau^{\frac{1}{q}}g\right] \tag{3.12}$$

so that (2.5) becomes

$$\frac{(q+1)}{q^2\alpha^{2q}}\left(1 + q^2\tau^{2+\frac{1}{q}}(q+1)\alpha\ddot{g} - \frac{(2q+1)}{\alpha}\tau^{\frac{1}{q}}g\right) + f\left(\frac{-1}{q\alpha^q}\left[1 - \frac{q}{\alpha}\tau^{1+\frac{1}{q}}\dot{g} - \frac{(q+1)}{\alpha}\tau^{\frac{1}{q}}g\right]\right) = 0$$

after we expand f as a Taylor series about the singular point

$$\xi_0 = \frac{-1}{q\alpha^q}, \qquad \eta_0 = \frac{q+1}{q^2\alpha^{2q}}. \tag{3.13}$$

The linearised equation is

$$\tau^2\ddot{g} + \alpha^q f'\left(\frac{-1}{q\alpha^q}\right)\tau\dot{g} - \left[\frac{(q+1)(2q+1)}{q^2} - \frac{(q+1)}{q}\alpha^q f'\left(\frac{-1}{q\alpha^q}\right)\right]g = 0 \tag{3.14}$$

which is again of Euler type and has solution

$$g(\tau) = K_1\tau^{\lambda_1} + K_2\tau^{\lambda_2}, \tag{3.15}$$

where λ_1 and λ_2 are the roots of

$$\lambda^2 + \lambda\left(\alpha^q f'\left(\frac{-1}{q\alpha^q}\right) - 1\right) - \left[\frac{(q+1)(2q+1)}{q^2} - \frac{(q+1)}{q}\alpha^q f'\left(\frac{-1}{q\alpha^q}\right)\right] = 0, \tag{3.16}$$

or

$$g(\tau) = (K_1 + K_2\log\tau)\,\tau^{\lambda} \tag{3.17}$$

in the case that $\lambda_1 = \lambda_2 = \lambda$.

Evidently the procedure may be continued to higher orders, at the price of increasing complexity of calculation. As in the preceeding specific calculation one of the roots of (3.16) is $\lambda_1 = -1 - \frac{1}{q}$ and the corresponding term in (3.15) must be discarded.

4 Asymptotic expansions

The solution of the next to singular behaviour of (2.11) given in (3.4) contained terms which were not acceptable because they were more singular than the leading order behaviour in the vicinity of the pole. However, in an equation invariant under both time translation and self-similarity the leading order behaviour could also be that as $\tau \to \infty$, *ie* the asymptotic behaviour of the solution. Since this behaviour is far from the movable singular point, there is no loss of generality in looking at the behaviour in terms of powers of t rather than $\tau = t - t_0$. We again use (2.11) to establish in a precise context our ideas about the next to asymptotic behaviour of the solution.

Let

$$x(t) = \alpha t^{-1} + g(t), \tag{4.1}$$

where we require $tg(t) \to 0$ as $t \to \infty$. The substitution of (4.1) into (2.11) gives (3.2) with τ replaced by t. When $t \to \infty$, we recover (3.3) so that

$$g(t) = \begin{cases} K_1 t^{-2} + K_2 t^{3-\alpha}, & \alpha \neq 5, \\ (K_1 + K_2\log t)t^{-2}, & \alpha = 5, \end{cases} \tag{4.2}$$

where again K_1 and K_2 are arbitrary constants. Since we require that $tg(t) \to 0$ as $t \to \infty$, both solutions are now acceptable provided $\alpha > 4$. We note that the coefficient of t^{-2} is always arbitrary. In the case $\alpha = 5$ $(k = 3/25)$ a logarithmic term must appear.

We may continue this asymptotic expansion to any desired number of terms by piecewise solution. More generally we can consider the asymptotic solution of (2.5) or of an equation of higher order with the same symmetries. The procedure should be evident from the foregoing. However, all equations are not invariant under the two symmetries of time translation and self-similarity. As we can always maintain invariance under time translation by a simple increase of order, we need only consider the effect of the loss of the self-similarity symmetry. (Also here again an increase of order could be used, in principle, to maintain this.)

5 Symmetry-breaking

For an equation invariant under time translation and (2.3) all terms contribute to the leading order behaviour. The addition of a term (or terms) which does not share the symmetry (2.3) can be considered as a symmetry-breaking term. In the ARS algorithm such terms are neglected until the final test of compatibility is undertaken. In the perturbative approach we adopt here such a term is introduced from the beginning and its rôle will develop naturally. We illustrate our approach with a variation of (2.11), *videlicet*

$$\ddot{x} + x\dot{x} + kx^s = 0 \tag{5.1}$$

for $s \neq 3$, where three is the value for which (2.3), with $q = 1$, is a symmetry. Suppose that $s = 0$. Substitution of $x = \alpha\tau^p$ gives the three exponents $p - 2$, $2p - 1$ and 0 so that the singularity remains a simple pole due to the balancing of the first two terms and the coefficient is $\alpha = 2$. Let

$$x(\tau) = 2\tau^{-1} + g(\tau). \tag{5.2}$$

Then (5.1) becomes

$$\tau^2\ddot{g} + 2\tau\dot{g} - 2g = -k\tau^2 \tag{5.3}$$

when the property $\tau g(\tau) \to 0$ as $\tau \to 0$ is invoked. The solution is

$$g(\tau) = K_1\tau^{-2} + K_2\tau + \frac{1}{4}k\tau^2. \tag{5.4}$$

The first term must be abandoned because it does not satisfy the criterion of behaviour required of $g(\tau)$. Since the nonhomogeneous contribution is at a power higher than that of the second term, we cannot have confidence in the accuracy of its coefficient and only the second term can be kept so that the solution including the first contribution next to the leading singular behaviour is

$$x(\tau) = 2\tau^{-1} + K_2\tau. \tag{5.5}$$

The next contribution is found from setting

$$x(\tau) = 2\tau^{-1} + K_2\tau + g(\tau), \tag{5.6}$$

where, in the neighbourhood of $\tau = 0$, $g(\tau)$ satisfies (5.3) and the solution is now

$$x(\tau) = 2\tau^{-1} + K_2\tau + \frac{1}{4}k\tau^2, \tag{5.7}$$

ie the $K_2\tau$ term does not affect the next level of behaviour.

If we consider the asymptotic behaviour of the solution of (5.1) with $s = 0$, there is no possibility to balance terms as the single constant term dominates the other two. The presence of this symmetry-breaking term precludes the development of an asymptotic expansion. This is the case in general as the less singular term can never be balanced in the asymptotic expansion.

If we take $s = 6$ in (5.1), the indices for the leading order behaviour are $p-2$, $2p-1$ and $6p$. The first two terms balance with $p = -1$, but then the third term is more singular in the neighbourhood of t_0. One must take the first and third terms (thereby making symmetry-breaking more like symmetry-snatching) so that $p = -2/5$ and $\alpha = (-14/25)^{1/5}$. Set

$$x(\tau) = \alpha\tau^{-\frac{2}{5}} + g(\tau), \tag{5.8}$$

where $\tau^{2/5}g(\tau) \to 0$ as $\tau \to 0$. Then the linearised equation reduces to

$$\tau^2\ddot{g} - \frac{84}{25}g = \frac{2}{5}\left(-\frac{14}{25k}\right)^{\frac{1}{5}} \tag{5.9}$$

which has the solution

$$g = K_1\tau^{-\frac{7}{5}} + K_2\tau^{\frac{12}{5}} - \frac{5}{44}\left(-\frac{14}{25k}\right)^{\frac{2}{5}}\tau^{\frac{1}{5}}. \tag{5.10}$$

The nonhomogeneous solution describes the next to singular behaviour, the term with coefficient K_1 is discarded as too singular and that with coefficient K_2 cannot yet be regarded as reliable. Further perturbations would have to be taken to see if one or more correction terms occur between $\tau^{1/5}$ and $\tau^{12/5}$. We note that as a rule of thumb we discard all terms except the one closest to the one already obtained.

If we now consider the asymptotic behaviour of the solution for $s = 6$, the first two terms must be taken and we have the leading order behaviour $2t^{-1}$ as $t \to \infty$. We set

$$x(t) = 2t^{-1} + g(t), \tag{5.11}$$

where $tg(t) \to 0$ as $t \to \infty$. The linearised equation for g is

$$t^2\ddot{g} + 2t\dot{g} - 2g = -64kt^{-4} \tag{5.12}$$

with solution

$$g = K_1t^{-2} + K_2t - \frac{64}{10}t^{-4}. \tag{5.13}$$

The second term must be discarded and the third treated with suspicion. This suspicion is justified when we set

$$x(t) = 2t^{-1} + K_1t^{-2} + g(t) \tag{5.14}$$

and find that the leading term of $g(t)$ is $-\frac{1}{2}K_1^2t^{-3}$. It is only at the next iteration that the correct coefficient of t^{-4} is found. Contrast this result with that for $s = 0$.

We see that symmetry-breaking can have several effects. One is to make our procedure unworkable, as in the case of the asymptotic expansion of (5.1) when $s = 0$. Generally the symmetry-breaking terms introduce a nonhomogeneous term (or terms) into the equation for g. The nonhomogeneous term may or may not give rise to the next to leading behaviour term. In the case of (5.1) with $s = 6$ this was the case in the neighbourhood of the finite singularity, but was not the case for the asymptotic expansion.

6 Intermediate terms for higher order equations

For second order equations with symmetry breaking terms the nonhomogeneous contribution to the g function may interpose itself between the leading order term and the second (acceptable) solution given by the homogeneous part or it may not. In fact this variation in behaviour is not confined to equations containing a symmetry-breaking term. However, one must go to a higher order equation. The general third order ordinary differential equation invariant under time translation and the self-similarity symmetry (2.3) is

$$\dddot{x} + x^{3q+1} f\left(\frac{\dot{x}}{x^{q+1}}, \frac{\ddot{x}}{x^{2q+1}}\right) = 0. \tag{6.1}$$

A specific instance of (6.1) is the generalised Chazy equation

$$\dddot{x} + |x|^q \ddot{x} + k|x|^q \frac{\dot{x}^2}{x} = 0 \tag{6.2}$$

which has, for various values of k, attracted attention over the past century [5, 4, 12]. The behaviour of the solution of (6.2) is affected by the relationship between the values of the parameters k and q. Two interesting cases are $k = q$ and $k = q + 1$ [13]. By way of example we consider $q = 2$ and $k = 3$, *ie* the equation

$$\dddot{x} + x^2\ddot{x} + 3x\dot{x}^2 = 0 \tag{6.3}$$

for which the leading order behaviour, when all terms are dominant, is $\alpha\tau^{-1/2}$ with $\alpha^2 = 5/4$. We set

$$x(\tau) = \alpha\tau^{-\frac{1}{2}} + g(\tau) \tag{6.4}$$

with $\tau^{1/2}g(\tau) \to 0$ as $\tau \to 0$. The linearised equation for g is

$$\tau^3\dddot{g} + \frac{5}{4}\tau^2\ddot{g} - \frac{15}{4}\tau\dot{g} + \frac{45}{16} = 0 \tag{6.5}$$

with solution

$$g = K_1\tau^{-\frac{3}{2}} + K_2\tau^{\frac{3}{4}} + K_3\tau^{\frac{5}{2}}, \tag{6.6}$$

where K_1, K_2 and K_3 are arbitrary constants. We reject the term in K_1 as too singular, accept that in K_2 and cannot be confident that the next correction is only at $\tau^{5/2}$ in accordance with the rule of thumb stated above. We now substitute

$$x(\tau) = \alpha\tau^{-\frac{1}{2}} + K_2\tau^{\frac{3}{4}} + g(\tau) \tag{6.7}$$

and find that the linearised equation for $g(\tau)$ is

$$\tau^3\dddot{g} + \frac{5}{4}\tau^2\ddot{g} - \frac{15}{4}\tau\dot{g} + \frac{45}{16}g = \frac{21}{16}K_2^2\alpha\tau^2 \tag{6.8}$$

so that

$$g = K_1\tau^{-\frac{3}{2}} + K_2\tau^{\frac{3}{4}} - \frac{3}{5}K_2^2\alpha\tau^2 + K_3\tau^{\frac{5}{2}}. \tag{6.9}$$

Since the last term in (6.9) is suspicious, we now have

$$x(\tau) = \alpha\tau^{-1} + K_2\tau^{\frac{3}{4}} - \frac{3}{5}K_2^2\alpha\tau^2. \tag{6.10}$$

In (6.10) we still do not have the third arbitrary constant. We can expect it at the next iteration. Given the spacing between the powers one would not anticipate the need to introduce a logarithmic term at $\tau^{5/2}$, but this would have to be checked by specific calculation. Thereafter the additional terms will always be rational powers if no logarithmic term has to be included previously. If one has, then one would expect the logarithm to propagate though the perturbation terms.

7 Conclusion

In this paper we have sought to expand upon the detail behind the procedures of the ARS algorithm. Although the wording of the algorithm, as presented in the review by Ramani *et al* [21], does not state so explicitly, the algorithm is a no frills procedure for deciding whether the solution of a given ordinary differential equation possesses a Laurent expansion about a movable singularity. We have presented a broader approach and in so doing make a statement based on our philosophy of the way differential equations should be treated. We equally recognise that the subject of singularity analysis of ordinary differential equations provides for different philosophies. These different philosophies are also connected to the understanding of what integrability is.

There are those who require that the solution of an ordinary differential equation be an analytic function. As soon as the ARS algorithm throws up a nonintegrer power, the algorithm is terminated. A concession, implied by the qualifer "weak", is to require that the solution be analytic in part of the complex plane, *ie* rational powers, either in the singularity or in the series expansion, are admitted. In stating that an equation passes the weak Painlevé test a certain amount of common sense must be used to interpret the utility of the result. The rational exponents involved cannot be permitted to have too large a denominator or else the complex plane will be divided into unworkable parts, particularly in the neighbourhood of the singularity. In a realistic scheme of things, where solutions must in the end be computed, there is no numerical difference between an irrational exponent and a rational exponent if the latter has a large demominator.

We are motivated by a need to understand the behaviour of solutions to equations which are, in the main, going to be computed along the real axis. The analysis we have presented provides information about the behaviour of the solution in the vicinity of a movable singularity. It makes obvious the need for the introduction of logarithmic terms because they arise naturally in the solution of the g equation.

Further the whole problem of compatibility, which in the ARS algorithm has somehow become disconnected from the study of resonances (see Hua *et al* [15] for a coherent approach to the determination of both as a single process), is put into a very plain perspective. In a sense compatability ceases to be a question since the g function always provides the answer. One may not be happy with that answer when logarithms necessarily intrude, but at least the procedure is transparent.

We suggest that the considerations proposed in this paper both complement and supplement the ARS algorithm by contributing to a greater understanding of what can happen at each step of the algorithm. These benefits apply both to the conventional analysis which leads to the existence of a right Painlevé series, which is a Laurent expansion in a punctured disc about the singularity, and the more recently introduced left Painlevé series [11, 20, 10] which is an asymptotic solution and may be regarded as a Laurent expansion outside a disc centred on the movable singularity.

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