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Bounds for the Threshold Amplitude for Plane Couette Flow

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Abstract

We prove nonlinear stability for finite amplitude perturbations of plane Couette flow. A bound of the solution of the resolvent equation in the unstable complex half-plane is used to estimate the solution of the full nonlinear problem. The result is a lower bound, including Reynolds number dependence, of the threshold amplitude below which all perturbations are stable. Our result is an improvement of the corresponding bound derived in [3].

1 Introduction

Plane Couette flow is stable to infinitesimal perturbations for all Reynolds numbers [9]. Finite amplitude perturbations on the other hand can induce transition to turbulence. Thus there is a threshold amplitude below which all perturbations decay eventually (there may be transient growth) and above which some perturbations lead to turbulent flow. In the present paper we derive a bound depending on the Reynolds number R, for this threshold.

The stability of shear flows, and plane Couette flow in particular, has been extensively studied, we refer to [7, 2] and [10] for more background information. Early work focused on the eigenvalues of the linearized problem. The question of the asymptotic value of the threshold amplitude for large R was starting to be investigated in the early 1990s [11]. A dependence $\mathcal{O}(R^{-\gamma})$ with $\gamma \approx 5/4$ is supported by computations in [8]. The asymptotic analysis of [1] gives the result $\gamma \approx 1$. Another approach to this question is found in [3] where an estimate of the resolvent equation is used to prove an upper bound for the exponent, $\gamma \leq 21/4$. In this paper we use the same approach and sharpen the bound of γ obtained in [3]. This is possible because we work with a norm which weighs the different coefficients of the velocity vector, the weights depend on the Reynolds number. It is numerically demonstrated in [6] that, in the weighted norm, the resolvent grows linearly with R, as opposed to quadratically which is the case of the energy norm used in [3]. Because of the weighting the exponent γ will be different for the different components of the velocity vector. The precise result is stated in Theorem 1.

The proof of the result of this paper is carried out with the same techniques as in [3]. However, the starting point is a different resolvent estimate. Also, the energy norm fits the structure of the Navier–Stokes equations, the modified norm we use does not. This leads to several technical complications.

In Section 2 we give the mathematical formulation of the problem, the Navier-Stokes equations, the linearized equation and the resolvent equation. Then we state the main result in Theorem 1. The proof of the theorem is organized as follows. In Section 3 we analyze the linear problem. Using the result of [6] mentioned above, we derive an estimate for higher derivatives of the solution of the resolvent equation (this solution is the Laplace transform of the velocity field). By inverse transformation we obtain a time dependent estimate. In Section 4 the linear estimate is used to derive the threshold value for the nonlinear problem.

In Appendix A we have collected the definition of the weighted norm and other non-standard norms used in the paper. Technical results including estimates of norms of the nonlinearity are collected in Appendix B. In Appendix C, we prove a theorem which will be applied to show that the perturbation tends to zero if the suitable a priori estimate holds. The result of Appendix C explains the choice of left hand side in the linear estimate of Theorem 2 in Section 3.

2 The main result

We choose the coordinate system so that the velocity field of Couette flow is given by

$$\boldsymbol{u}_{\mathrm{Co}} = \begin{pmatrix} x_2 \\ 0 \\ 0 \end{pmatrix}$$

in the domain

$$\Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^3 : \ -1 < x_2 < 1 \right\}.$$

We use bold letters to denote vectors and subscripts to identify the components. The functions $\boldsymbol{u}_{\mathrm{Co}}$ and $p_{\mathrm{Co}} = \mathrm{const}$ constitute a stationary solution of the nondimensionalized Navier–Stokes equations

$$\begin{aligned} &\boldsymbol{u}_t + \boldsymbol{G}(\boldsymbol{u}) + \operatorname{grad} p = \frac{1}{R}\Delta \boldsymbol{u}, \\ &\operatorname{div} \boldsymbol{u} = 0. \end{aligned} \tag{2.1}$$

Here R denotes the Reynolds number and $\boldsymbol{G}$ is the following nonlinear differential operator.

$$\boldsymbol{G}(\boldsymbol{u}) = \sum_{k=1}^{3} u_k \frac{\partial \boldsymbol{u}}{\partial x_k}.$$

We have no-slip boundary conditions, $\boldsymbol{u} = 0$ on $\partial\Omega$, and the initial condition

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_{\mathrm{Co}}(\boldsymbol{x}) + \boldsymbol{v}^{(0)}(\boldsymbol{x}),$$

where $\boldsymbol{v}^{(0)}$ is the initial perturbation. We will assume that $\boldsymbol{v}^{(0)} \in H^4$. Local existence of a classical solution of (2.1) is proven in [5] (theorem 5 together with theorem 7 on p. 161 and p. 167 respectively of [5]). Furthermore we will derive a priori estimates which (for sufficiently small initial data) allows extension of the local solution to a global solution by successive application of the above result.

We investigate the stability of Couette flow by linearization of the Navier–Stokes equations. We denote the perturbation by $\boldsymbol{v}$ which thus is related to $\boldsymbol{u}$ according to

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{Co}} + \boldsymbol{v}.$$

Below we will apply the Laplace transform and for this purpose we want a problem with homogeneous initial data. This is accomplished with the introduction of $\boldsymbol{w}$ by

$$\boldsymbol{v} = e^{-t}\boldsymbol{v}^{(0)} + \boldsymbol{w}. \tag{2.2}$$

The function $\boldsymbol{w}$ satisfies the following problem

$$\begin{aligned} &\boldsymbol{w}_t + \operatorname{grad} p = L\boldsymbol{w} + e^{-t}(L+1)\boldsymbol{v}^{(0)} - \boldsymbol{G}\left(e^{-t}\boldsymbol{v}^{(0)} + \boldsymbol{w}\right), \\ &\operatorname{div} \boldsymbol{w} = 0, \\ &\boldsymbol{w}(\boldsymbol{x}, 0) = 0, \end{aligned} \tag{2.3}$$

where

$$L = \frac{1}{R}\Delta - x_2\frac{\partial}{\partial x_1} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\boldsymbol{w}$ we also have no-slip boundary conditions. If we assume that $\boldsymbol{v}^{(0)}$ and $\boldsymbol{w}$ are small and neglect $\boldsymbol{G}$ in (2.3) then we see that $\boldsymbol{w}$ satisfies a linear problem with forcing depending on the initial perturbation. This is the linearized equation for Couette flow.

We will first consider the forcing in (2.3) as a given function, denoted by $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ and derive estimates for the solution. Later we will return to the particular form of the forcing in (2.3) and its dependence on the initial data and $\boldsymbol{w}$.

Application of the Laplace transform now give the resolvent equation

$$\begin{aligned} &s\hat{\boldsymbol{w}} + \operatorname{grad} \hat{p} = L\hat{\boldsymbol{w}} + \hat{\boldsymbol{f}}, \\ &\operatorname{div} \hat{\boldsymbol{w}} = 0, \\ &\hat{\boldsymbol{w}} = 0, \qquad \boldsymbol{x} \in \partial\Omega. \end{aligned} \tag{2.4}$$

This problem was investigated in [6] where the following estimate of the solution was obtained

$$\|\hat{\boldsymbol{w}}\|_m^2 \le CR^2\|\hat{\boldsymbol{f}}\|_m^2, \qquad \operatorname{Re} s \ge 0. \tag{2.5}$$

See Appendix A equation (A.1) for the definition of the modified (m-)norm. We have collected the definitions of all norms used in this paper in Appendix A. Throughout the paper we will use C to denote constants which appear in inequalities, we do not use subscripts to identify different constants. We emphasize that C denotes an “absolute constant” and does not depend on the forcing, the initial data or the Reynolds number, which is the only parameter in the problem.

We now state our main result in the following theorem.

Theorem 1. *The resolvent estimate (2.5) implies that there is a constant $\delta > 0$ such that*

$$\lim_{t\to\infty} \|\boldsymbol{v}(\cdot,t)\|_\infty = 0$$

if

$$\|v_k(\cdot,0)\|_{H^4} \le \frac{\delta}{R^3}, \qquad k = 1,3,$$
$$\|v_2(\cdot,0)\|_{H^4} \le \frac{\delta}{R^4}.$$

Comparing this theorem to the previously obtained bound on the threshold in [3] we see that the R-exponent for the first and third $\boldsymbol{v}$-components is improved from 5.25 (in [3]) to 3 and the exponent for the second component is improved from 5.25 to 4.

It may appear excessive to require that the initial data is as smooth as H^4. The result in [9] require the H^2-norm of the initial data to be small. However, in [9] a completely different approach is used with the application of semi-group methods and we are not aware of any results on the threshold amplitude obtained with this approach. It may be possible to weaken our smoothness requirement using an appropriate local existence theorem for the Navier–Stokes equations which incorporates the smoothing property. We also note the error in [3, p. 193] where it is stated that small H^2-norm implies stability, for their result to hold small H^4-norm is required.

If one is interested in a particular perturbation $\boldsymbol{v}^{(0)}(\boldsymbol{x}) = \alpha\boldsymbol{\varphi}(\boldsymbol{x})$, and the threshold coefficient $\alpha(R)$, then it is insignificant which H^p-norm is required since

$$\left\|\boldsymbol{v}^{(0)}\right\|_{H^p} \le \frac{1}{R^\gamma} \qquad \Rightarrow \qquad \alpha(R) \le \frac{1}{R^\gamma \|\boldsymbol{\varphi}\|_{H^p}} = \mathcal{O}(R^{-\gamma}).$$

3 The linear estimate

In this section we derive estimates for the solution of the following problem in terms of the forcing function $\boldsymbol{f}$

$$\begin{aligned}
&\boldsymbol{w}_t + \operatorname{grad} p = L\boldsymbol{w} + \boldsymbol{f},\\
&\operatorname{div} \boldsymbol{w} = 0,\\
&\boldsymbol{w}(\boldsymbol{x},0) = 0.
\end{aligned} \tag{3.1}$$

The results are collected in the inequality of the theorem below. In the statement of the theorem the $\tilde{H}_1$- and $\tilde{H}_2$-norms appear for the first time, their definition is given in Appendix A.

Theorem 2. *The resolvent estimate (2.5) implies that the solution of equation (3.1) satisfies*

$$\begin{aligned}
\|\boldsymbol{w}(\cdot,T)\|^2_{\tilde{H}_2} &+ \int_0^T \left(\|\boldsymbol{w}(\cdot,t)\|^2_{\tilde{H}_1} + \|\boldsymbol{w}_t(\cdot,t)\|^2_{\tilde{H}_1}\right) dt\\
&\le C\left[R\left\|\boldsymbol{f}^{(0)}\right\|^2_{\tilde{H}_2} + R^2\|\boldsymbol{f}(\cdot,T)\|^2 + R^2\left\|(L+1)\boldsymbol{f}^{(0)}\right\|^2_m\right]\\
&+ CR^2\int_0^T \left(\|\boldsymbol{f}(\cdot,t)\|^2_m + \|\boldsymbol{f}_t(\cdot,t)\|^2_m\right) dt.
\end{aligned} \tag{3.2}$$

For the remainder of the paper we will suppress the notation "$(\cdot, t)$" which indicates at which time a norm is evaluated, this will be clear from the context.

In Section 3.1 we analyze the Laplace transformed problem (2.4) and then in Section 3.2 we use these results to conclude the proof of Theorem 2.

3.1 Estimates of the transformed functions

The starting point for this section is inequality (2.5) which we will use to derive the following lemma.

Lemma 1. *If* $\operatorname{Re} s \geq 0$, *the solution of (2.4) satisfies*

$$\|\hat{\boldsymbol{w}}\|_{\tilde{H}_1}^2 \leq CR^2 \|\hat{\boldsymbol{f}}\|_m^2. \tag{3.3}$$

The estimate is uniform in s.

Proof. By partial integration, the following identity is easily derived

$$\operatorname{Re}(\hat{\boldsymbol{w}}, L\hat{\boldsymbol{w}}) = -\frac{1}{R}\sum_{k=1}^{3} \left\|\frac{\partial \hat{\boldsymbol{w}}}{\partial x_k}\right\|^2 + \operatorname{Re}(\hat{w}_1, \hat{w}_2). \tag{3.4}$$

Now, we take the L^2-inner product of $\hat{\boldsymbol{w}}$ and the resolvent equation (2.4). Using (3.4) and the fact that $\hat{\boldsymbol{w}}$ is solenoidal we obtain

$$\operatorname{Re} s \, \|\hat{\boldsymbol{w}}\|^2 + \frac{1}{R}\sum_{k=1}^{3} \left\|\frac{\partial \hat{\boldsymbol{w}}}{\partial x_k}\right\|^2 = \operatorname{Re}\left[(\hat{w}_1, \hat{w}_2) + (\hat{\boldsymbol{w}}, \hat{\boldsymbol{f}})\right]. \tag{3.5}$$

The right hand side in (3.5) can be estimated using the inequality $ab \leq \epsilon a^2/2 + b^2/2\epsilon$ which holds for all $\epsilon > 0$. Without explicit mention we will use this inequality several times below. Here we take $a = \|\hat{\boldsymbol{w}}\|$, $b = \|\hat{\boldsymbol{f}}\|$ and $\epsilon = 1/R$. We also use the fact that the L_2-norm is smaller than the modified norm ($R \geq 1$), and obtain

$$\operatorname{Re}\left[(\hat{w}_1, \hat{w}_2) + (\hat{\boldsymbol{w}}, \hat{\boldsymbol{f}})\right] \leq CR\|\hat{\boldsymbol{f}}\|_m^2. \tag{3.6}$$

Combining (3.6) with (3.5) and using $\operatorname{Re} s \geq 0$ we have derived

$$\sum_{k=1}^{3} \left\|\frac{\partial \hat{\boldsymbol{w}}}{\partial x_k}\right\|^2 \leq CR^2\|\hat{\boldsymbol{f}}\|_m^2. \tag{3.7}$$

It remains to estimate $J_2(\hat{\boldsymbol{w}})$ (defined in Appendix B). To do this we differentiate the resolvent equation with respect to x_1 and x_3. We give the details in the x_1-case. We obtain

$$s\frac{\partial \hat{\boldsymbol{w}}}{\partial x_1} + \operatorname{grad}\frac{\partial \hat{p}}{\partial x_1} = L\frac{\partial \hat{\boldsymbol{w}}}{\partial x_1} + \frac{\partial \hat{\boldsymbol{f}}}{\partial x_1}. \tag{3.8}$$

Note that L commutes with $\partial/\partial x_1$ and $\partial/\partial x_3$ but not $\partial/\partial x_2$. We take the inner product of (3.8) with $\partial\hat{\boldsymbol{w}}/\partial x_1$ and obtain

$$\operatorname{Re} s \left\|\frac{\partial \hat{\boldsymbol{w}}}{\partial x_1}\right\|^2 + \frac{1}{R}\sum_{k=1}^{3} \left\|\frac{\partial^2 \hat{\boldsymbol{w}}}{\partial x_1 \partial x_k}\right\|^2 = \operatorname{Re}\left[\left(\frac{\partial \hat{w}_1}{\partial x_1}, \frac{\partial \hat{w}_2}{\partial x_1}\right) + \left(\frac{\partial \hat{\boldsymbol{w}}}{\partial x_1}, \frac{\partial \hat{\boldsymbol{f}}}{\partial x_1}\right)\right]. \tag{3.9}$$

The right hand side is estimated in the following way.

$$\operatorname{Re}\left[\left(\frac{\partial \hat{w}_1}{\partial x_1},\frac{\partial \hat{w}_2}{\partial x_1}\right)+\left(\frac{\partial \hat{\boldsymbol{w}}}{\partial x_1},\frac{\partial \hat{\boldsymbol{f}}}{\partial x_1}\right)\right]$$
$$\leq \frac{1}{4R}\left\|\frac{\partial^2 \hat{w}_1}{\partial x_1^2}\right\|^2+R\|\hat{w}_2\|^2+\frac{1}{4R}\left\|\frac{\partial^2 \hat{\boldsymbol{w}}}{\partial x_1^2}\right\|^2+R\|\hat{\boldsymbol{f}}\|^2. \tag{3.10}$$

Now we insert the estimate (3.10) into (3.9) and cancel the second derivatives in the right hand side with the corresponding terms on the left hand side. Further simplification leads to

$$\frac{1}{R}\sum_{k=1}^{3}\left\|\frac{\partial^2 \hat{\boldsymbol{w}}}{\partial x_1 \partial x_k}\right\|^2 \leq C\left(\frac{1}{R}\|\hat{\boldsymbol{w}}\|_m^2+R\|\hat{\boldsymbol{f}}\|_m^2\right)\leq CR\|\hat{\boldsymbol{f}}\|_m^2.$$

The same procedure applies to the resolvent equation differentiated with respect to x_3. We have proven

$$J_2(\hat{\boldsymbol{w}}) \leq CR^2\|\hat{\boldsymbol{f}}\|_m^2. \tag{3.11}$$

A combination of (2.5), (3.7) and (3.11) gives the lemma. ■

Remark. To obtain (3.4) we used

$$(\hat{\boldsymbol{w}}, \operatorname{grad}\hat{p}) = -(\operatorname{div}\hat{\boldsymbol{w}}, \hat{p}) = 0, \tag{3.12}$$

where the first equality follows by partial integration and the second since $\hat{\boldsymbol{w}}$ is solenoidal ($\operatorname{div}\hat{\boldsymbol{w}} = 0$). It would be preferable to work with modified (m-)norms throughout the paper, the reason one cannot do this is (3.12). If one takes the modified inner product of $\hat{\boldsymbol{w}}$ and the resolvent equation then it is not possible to eliminate the pressure using (3.12). The modified inner product is defined by

$$(\boldsymbol{u}, \boldsymbol{v})_m = (u_1, v_1) + R^2(u_2, v_2) + (u_3, v_3).$$

This is the explanation of the simplifications with the use of the energy (L_2) norm mentioned in the introduction.

3.2 Estimates of the time dependent functions

Before we prove Lemma 2 below and Theorem 2 we state three inequalities which will be of use. The following is derived from equation (3.1) using (3.4)

$$\frac{d}{dt}\|\boldsymbol{w}\|^2+\frac{1}{R}\sum_{k=1}^{3}\left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|^2 \leq |(\boldsymbol{w},\boldsymbol{f})|+|(w_1,w_2)|. \tag{3.13}$$

In the same way we obtain the next inequality, for $l = 1, 3$, starting from (3.1) differentiated with respect to x_1 and x_3 respectively

$$\frac{d}{dt}\left\|\frac{\partial \boldsymbol{w}}{\partial x_l}\right\|^2+\frac{1}{R}\sum_{k=1}^{3}\left\|\frac{\partial^2 \boldsymbol{w}}{\partial x_l \partial x_k}\right\|^2 \leq \left|\left(\frac{\partial^2 \boldsymbol{w}}{\partial x_l^2},\boldsymbol{f}\right)\right|+\left|\left(\frac{\partial^2 w_1}{\partial x_l^2},w_2\right)\right|. \tag{3.14}$$

We apply Plancherel's formula for the Laplace transform to (3.3), with the imaginary axis as integration contour. This yields

$$\int_0^T \|\boldsymbol{w}\|_{\tilde{H}_1}^2 dt \le CR^2 \int_0^T \|\boldsymbol{f}\|_m^2 dt, \tag{3.15}$$

which holds for all $T > 0$, see [4, p. 235–239] for this application of Plancherel's formula.

Lemma 2. *The solution of (3.1) satisfies*

$$\|\boldsymbol{w}\|^2 + \left\|\frac{\partial \boldsymbol{w}}{\partial x_1}\right\|^2 + \left\|\frac{\partial \boldsymbol{w}}{\partial x_3}\right\|^2 \le CR \int_0^T \|\boldsymbol{f}\|_m^2 dt, \tag{3.16}$$

for all $T > 0$.

Proof. From (3.13) we obtain

$$\frac{d}{dt}\|\boldsymbol{w}\|^2 \le \frac{\epsilon}{2}\|\boldsymbol{w}\|_m^2 + \frac{1}{2\epsilon}\|\boldsymbol{f}\|_m^2 + \frac{1}{R}\|\boldsymbol{w}\|_m^2, \qquad \epsilon > 0.$$

We integrate this from $t = 0$ to T, choose $\epsilon = 1/R$ and use (3.15),

$$\|\boldsymbol{w}\|^2 \le CR \int_0^T \|\boldsymbol{f}\|_m^2 dt.$$

We estimate the right hand side in (3.14) according to

$$\left|\left(\frac{\partial^2 w_1}{\partial x_l^2}, w_2\right)\right| + \left|\left(\frac{\partial^2 \boldsymbol{w}}{\partial x_l^2}, \boldsymbol{f}\right)\right| \le \frac{1}{4R}\left\|\frac{\partial^2 w_1}{\partial x_l^2}\right\|^2 + R\|w_2\|^2 + \frac{1}{4R}\left\|\frac{\partial^2 \boldsymbol{w}}{\partial x_l^2}\right\|^2 + R\|\boldsymbol{f}\|^2,$$

recall that $l = 1, 3$. This inequality is inserted into (3.14). We move the second derivative terms to the left hand side and integrate in time to obtain

$$\left\|\frac{\partial \boldsymbol{w}}{\partial x_l}\right\|^2 \le CR \int_0^T \|\boldsymbol{f}\|_m^2 dt.$$

This completes the proof of Lemma 2. ■

Now we turn to the proof of Theorem 2. The function $\boldsymbol{w}_t$ satisfies the same PDE as $\boldsymbol{w}$ (with p and $\boldsymbol{f}$ replaced with p_t and $\boldsymbol{f}_t$ respectively) and inhomogeneous initial data,

$$\boldsymbol{w}_t(\boldsymbol{x}, 0) = \boldsymbol{f}(\boldsymbol{x}, 0) =: \boldsymbol{f}^{(0)}(\boldsymbol{x}).$$

We use the symbol := to indicate that the expression on the "colon side" is defined in terms of the expression on the other side. Now we introduce $\boldsymbol{\omega}$ by

$$\boldsymbol{w}_t = e^{-t}\boldsymbol{f}^{(0)} + \boldsymbol{\omega}. \tag{3.17}$$

The function $\boldsymbol{\omega}$ satisfies the following problem.

$$\begin{aligned}
&\boldsymbol{\omega}_t + \operatorname{grad} p_t = L\boldsymbol{\omega} + \boldsymbol{f}_t + e^{-t}(L+1)\boldsymbol{f}^{(0)},\\
&\operatorname{div} \boldsymbol{\omega} = 0,\\
&\boldsymbol{\omega}(\boldsymbol{x}, 0) = 0.
\end{aligned}$$

The only difference between this problem and equation (3.1) for $\boldsymbol{w}$ is the forcing. The inequalities (3.15) and (3.16) can thus be applied here yielding estimates for $\boldsymbol{\omega}$. Using (3.17) and the triangle inequality these can be converted to the following estimates for $\boldsymbol{w}$.

$$\int_0^T \|\boldsymbol{w}_t\|_{\tilde{H}_1}^2 dt \le C\left(\|\boldsymbol{f}^{(0)}\|_{\tilde{H}_1}^2 + R^2\|(L+1)\boldsymbol{f}^{(0)}\|_m^2 + R^2\int_0^T \|\boldsymbol{f}_t\|_m^2 dt\right) \tag{3.18}$$

and

$$\|\boldsymbol{w}_t\|^2 + \left\|\frac{\partial \boldsymbol{w}_t}{\partial x_l}\right\|^2 \le C\left(\|\boldsymbol{f}^{(0)}\|^2 + \left\|\frac{\partial \boldsymbol{f}^{(0)}}{\partial x_l}\right\|^2 + R\|(L+1)\boldsymbol{f}^{(0)}\|_m^2 + R\int_0^T \|\boldsymbol{f}_t\|_m^2 dt\right), \quad l = 1, 3. \tag{3.19}$$

With (3.15) and (3.18) we have estimated the time integral in the left hand side of the inequality of Theorem 2. The $\tilde{H}_2$-norm at time T remains. To estimate the first order derivatives in the $\tilde{H}_2$-norm we start from the following estimate which is easily obtained from (3.13)

$$\sum_{k=1}^{3}\left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|^2 \le CR\left(|(\boldsymbol{w}, \boldsymbol{f})| + |(\boldsymbol{w}, \boldsymbol{w}_t)| + |(w_1, w_2)|\right). \tag{3.20}$$

Using the Cauchy–Schwartz inequality and (3.16) and (3.19) we see that the right hand side of (3.20) can be estimated in terms of the right hand side of (3.2).

To estimate $J_2(\boldsymbol{w})$ we rearrange (3.14) and obtain

$$\sum_{k=1}^{3}\left\|\frac{\partial^2 \boldsymbol{w}}{\partial x_l \partial x_k}\right\|^2 \le CR\left[\left|\left(\frac{\partial w_1}{\partial x_l}, \frac{\partial w_2}{\partial x_l}\right)\right| + \left|\left(\frac{\partial^2 \boldsymbol{w}}{\partial x_l^2}, \boldsymbol{f}\right)\right| + \left|\left(\frac{\partial \boldsymbol{w}}{\partial x_l}, \frac{\partial \boldsymbol{w}_t}{\partial x_l}\right)\right|\right], \tag{3.21}$$

for $l = 1, 3$. For the second term in the right hand side we have

$$\left|\left(\frac{\partial^2 \boldsymbol{w}}{\partial x_l^2}, \boldsymbol{f}\right)\right| \le \frac{1}{4R}\left\|\frac{\partial^2 \boldsymbol{w}}{\partial x_l^2}\right\|^2 + R\|\boldsymbol{f}\|^2. \tag{3.22}$$

We insert (3.22) into (3.21) and move the second derivative term to the left hand side. The remaining two terms in the right hand side of (3.21) are estimated using the Cauchy–Schwartz inequality and (3.16) and (3.19). We see that $J_2(\boldsymbol{w})$ can be estimated by the expression on the right hand side of the inequality of Theorem 2. This concludes the proof of the theorem.

4 Derivation of the threshold bound

Now we return to the full nonlinear problem (2.3) for $\boldsymbol{w}$. This is the same as (3.1) if we take

$$\boldsymbol{f} = e^{-t}(L+1)\boldsymbol{v}^{(0)} + \sum_{k=1}^{3}\left(e^{-2t}v_k^{(0)}\frac{\partial \boldsymbol{v}^{(0)}}{\partial x_k} + e^{-t}v_k^{(0)}\frac{\partial \boldsymbol{w}}{\partial x_k} + e^{-t}w_k\frac{\partial \boldsymbol{v}^{(0)}}{\partial x_k} + w_k\frac{\partial \boldsymbol{w}}{\partial x_k}\right).$$

For simplicity we first consider

$$\boldsymbol{f} = e^{-t}(L+1)\boldsymbol{v}^{(0)} + \boldsymbol{G}(\boldsymbol{w}). \tag{4.1}$$

The quadratic terms in $\boldsymbol{v}^{(0)}$ and the coupling terms between $\boldsymbol{v}^{(0)}$ and $\boldsymbol{w}$ only add technical difficulties, the asymptotic value of the threshold amplitude is the same. To make the formulas below more lucid we introduce the following notation for expressions in the inequality of Theorem 2

$$\begin{aligned}
\mathcal{N}(\boldsymbol{w},T) &= \|\boldsymbol{w}\|^2_{\tilde{H}_2} + \int_0^T \left(\|\boldsymbol{w}\|^2_{\tilde{H}_1} + \|\boldsymbol{w}_t\|^2_{\tilde{H}_1}\right) dt,\\
\mathcal{M}(\boldsymbol{f},T) &= CR^2\left[\|\boldsymbol{f}\|^2 + \int_0^T \left(\|\boldsymbol{f}\|^2_m + \|\boldsymbol{f}_t\|^2_m\right) dt\right],\\
\mathcal{M}_0(\boldsymbol{f},T) &= \mathcal{M}(\boldsymbol{f},T) + C\left(R\|\boldsymbol{f}^{(0)}\|^2_{\tilde{H}_2} + R^2\|(L+1)\boldsymbol{f}^{(0)}\|^2_m\right).
\end{aligned}$$

As long as the solution $\boldsymbol{w}$ exists, the estimate of Theorem 2 is valid with forcing (4.1). Expressed with our new notation we have

$$\mathcal{N}(\boldsymbol{w},T) \le \mathcal{M}_0\left(e^{-t}(L+1)\boldsymbol{v}^{(0)},T\right) + \mathcal{M}(\boldsymbol{G}(\boldsymbol{w}),T). \tag{4.2}$$

Here we have used that $\boldsymbol{w}(\boldsymbol{x},0)=0$. The first term on the right hand side of (4.2) can be estimated in terms of $\boldsymbol{v}^{(0)}$, independently of T, as we show in Appendix B, inequality (B.1). We have

$$\mathcal{M}_0\left(e^{-t}(L+1)\boldsymbol{v}^{(0)},T\right) \le CR^2\|\boldsymbol{v}^{(0)}\|^2_{H^4,m} =: \overline{\mathcal{M}}. \tag{4.3}$$

Now we will determine the threshold value by assuming that $\boldsymbol{w}$ does not tend to zero. As a consequence of this assumption we will derive the inequality (4.13) (equivalent to (4.14) which bounds a norm of $\boldsymbol{v}^{(0)}$ from below. The inverse of this inequality gives the threshold which we thus prove by contradiction.

After the above outline, we now turn to the details of the proof. We thus assume that $\|\boldsymbol{w}(\cdot,T)\|_\infty$ does not tend to zero as $T\to\infty$. According to Theorem 3 in Appendix C (with $f=\|\boldsymbol{w}\|_{\tilde{H}_1}$) we then have

$$\lim_{T\to\infty}\mathcal{N}(\boldsymbol{w},T)=\infty.$$

In particular, there is a T_0 such that

$$\mathcal{N}(\boldsymbol{w},T_0) = 2\overline{\mathcal{M}} \tag{4.4}$$

and

$$\mathcal{N}(\boldsymbol{w},T) \le 2\overline{\mathcal{M}}, \qquad T\le T_0. \tag{4.5}$$

Using (4.5) and the Sobolev inequality (A.2) we can now estimate the three terms of $\mathcal{M}(\boldsymbol{G}(\boldsymbol{w}),T_0)$ in terms of $\overline{\mathcal{M}}$. For the first term we have the inequality (B.2) from Appendix B,

$$CR^2\|\boldsymbol{G}(\boldsymbol{w})\|^2 \le CR^2\|\boldsymbol{w}\|^2_\infty \sum_{k=1}^{3}\left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|^2. \tag{4.6}$$

The two factors in the right hand side can be estimate as follows

$$\|\boldsymbol{w}\|_\infty^2 \leq C\|\boldsymbol{w}\|_{\tilde{H}_1}^2 \leq C\overline{\mathcal{M}}, \tag{4.7}$$

$$\left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|^2 \leq C\|\boldsymbol{w}\|_{\tilde{H}_1}^2 \leq C\overline{\mathcal{M}}. \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6), we obtain

$$CR^2\|\boldsymbol{G}(\boldsymbol{w})\|^2 \leq CR^2\overline{\mathcal{M}}^2. \tag{4.9}$$

We use similar techniques and inequality (B.3) and (B.4) for the remaining two terms,

$$CR^2 \int_0^{T_0} \|\boldsymbol{G}(\boldsymbol{w})\|_m^2 dt \leq CR^2 \|\boldsymbol{w}\|_\infty^2 \int_0^{T_0} \sum_{k=1}^{3} \left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|_m^2 dt \leq CR^4\overline{\mathcal{M}}^2 \tag{4.10}$$

and

$$\begin{aligned} CR^2 \int_0^{T_0} \|\boldsymbol{G}_t(\boldsymbol{w})\|_m^2 dt &\leq CR^2 \sum_{k=1}^{3} \left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|_m^2 \int_0^{T_0} \left\|\frac{\partial \boldsymbol{w}}{\partial t}\right\|_\infty^2 dt \\ &+ CR^2\|\boldsymbol{w}\|_\infty^2 \int_0^{T_0} \sum_{k=1}^{3} \left\|\frac{\partial \boldsymbol{w}_t}{\partial x_k}\right\|_m^2 dt \leq CR^4\overline{\mathcal{M}}^2. \end{aligned} \tag{4.11}$$

Combining (4.9), (4.10) and (4.11) we obtain

$$\mathcal{M}(\boldsymbol{G}(\boldsymbol{w}), T_0) \leq CR^4\overline{\mathcal{M}}^2. \tag{4.12}$$

Now we insert (4.3), (4.4) and (4.12) into (4.2) to obtain

$$2\overline{\mathcal{M}} \leq \overline{\mathcal{M}} + CR^4\overline{\mathcal{M}}^2. \tag{4.13}$$

We divide the inequality by $\overline{\mathcal{M}}$, insert the expression (4.3) for $\overline{\mathcal{M}}$ and rearrange to

$$\|\boldsymbol{v}^{(0)}\|_{H^4,m}^2 \geq \frac{1}{CR^6}. \tag{4.14}$$

We have thus shown that an initial perturbation which leads to instability must satisfy (4.14). If the initial perturbation satisfies the inverse inequality then the instability assumption leads to a contradiction. In other words, if the initial perturbation satisfies the inverse inequality of (4.14) then $\|\boldsymbol{w}(\cdot, T)\|_\infty \to 0$ as T tends to infinity. Noting the $\boldsymbol{w}$ is related to $\boldsymbol{v}$ by (2.2), the proof of Theorem 1 is complete.

A Definition and notation for norms

In this paper we track the R-dependence of all estimates we derive. This forces us to introduce some non-standard norms which weigh the different components of the vector function with coefficients depending on R. Also we need a Sobolev inequality (A.2) with a right hand side not containing all second derivatives.

We start, however, to introduce notation for the standard norms. The L^2-norm for functions $u : \Omega \to \mathbb{C}$ is defined by

$$\|u\|^2 = \int_\Omega |u(\boldsymbol{x})|^2 d\boldsymbol{x}.$$

We use the same notation for the L^2-norm of vector functions

$$\|\boldsymbol{u}\|^2 = \sum_{k=1}^{3} \|u_k\|^2.$$

It is clear from the argument if it is the scalar or vector norm which is intended. For the standard Sobolev spaces with "integration exponent" two we use the following notation

$$\|\boldsymbol{u}\|_{H^k}^2 = \sum_{|\alpha|\le k} \left\| \frac{\partial^{|\alpha|}\boldsymbol{u}}{\partial x^\alpha} \right\|^2,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index.

We refer to the weighted norms also as modified norms and use the subscript m to identify them. Let X denote any of the norms introduced so far, then the corresponding modified version is

$$\|\boldsymbol{u}\|_{X,m}^2 = \|u_1\|_X^2 + R^2\|u_2\|_X^2 + \|u_3\|_X^2.$$

In particular we have

$$\|\boldsymbol{u}\|_m^2 = \|u_1\|^2 + R^2\|u_2\|^2 + \|u_3\|^2. \tag{A.1}$$

Before we give the Sobolev inequality (A.2) we define the semi-norm J_2 consisting of selected second order derivatives. The choice of second order derivatives is dictated by the difficulty in estimating normal derivatives (derivatives with respect to the x_2-coordinate)

$$J_2(\boldsymbol{u}) = \left\| \frac{\partial^2\boldsymbol{u}}{\partial x_1^2} \right\|^2 + \left\| \frac{\partial^2\boldsymbol{u}}{\partial x_1 \partial x_2} \right\|^2 + \left\| \frac{\partial^2\boldsymbol{u}}{\partial x_2 \partial x_3} \right\|^2 + \left\| \frac{\partial^2\boldsymbol{u}}{\partial x_3^2} \right\|^2.$$

Now we can define two similar norms

$$\|\boldsymbol{u}\|_{\tilde{H}_1}^2 = \|\boldsymbol{u}\|_m^2 + \sum_{k=1}^{3} \left\| \frac{\partial\boldsymbol{u}}{\partial x_k} \right\|^2 + J_2(\boldsymbol{u})$$

and

$$\|\boldsymbol{u}\|_{\tilde{H}_2}^2 = \|\boldsymbol{u}\|^2 + \sum_{k=1}^{3} \left\| \frac{\partial\boldsymbol{u}}{\partial x_k} \right\|^2 + J_2(\boldsymbol{u}).$$

The $\tilde{H}_1$-norm is greater than the $\tilde{H}_2$-norm ($R \ge 1$) for which we have the Sobolev inequality

$$\|\boldsymbol{u}\|_\infty^2 \le C\|\boldsymbol{u}\|_{\tilde{H}_2}^2, \tag{A.2}$$

see [4, p. 385]. By the left hand side in (A.2) we of course mean

$$\|\boldsymbol{u}\|_\infty = \max_{k\in\{1,2,3\}} \operatorname*{ess\,sup}_{\boldsymbol{x}\in\Omega} |u_k(\boldsymbol{x})|.$$

B Auxiliary estimates

To bound $\mathcal{M}_0(e^{-t}(L+1)\boldsymbol{v}^{(0)}, T)$, we must estimate the following four terms

$$I = CR^2\big\|(L+1)\boldsymbol{u}^{(0)}\big\|^2,$$
$$CR^2\int_0^T e^{-2t}\big\|(L+1)\boldsymbol{v}^{(0)}\big\|_m^2 dt = CR^2\big\|(L+1)\boldsymbol{v}^{(0)}\big\|_m^2 =: II,$$
$$III = \big\|(L+1)\boldsymbol{v}^{(0)}\big\|_{\tilde{H}_2}^2$$

and

$$IV = \big\|(L+1)^2\boldsymbol{v}^{(0)}\big\|_m^2.$$

For the first term we have

$$I \le CR\big\|\Delta\boldsymbol{v}^{(0)}\big\|^2 + CR^2\left\|\frac{\partial\boldsymbol{v}^{(0)}}{\partial x_1}\right\|^2 + CR^2\big\|v_2^{(0)}\big\|^2 \le CR^2\big\|\boldsymbol{v}^{(0)}\big\|_{H^2}^2.$$

For the second term

$$II \le CR\big\|\Delta\boldsymbol{v}^{(0)}\big\|_m^2 + CR^2\left\|\frac{\partial\boldsymbol{v}^{(0)}}{\partial x_1}\right\|_m^2 + CR^2\big\|v_2^{(0)}\big\|^2 \le CR^2\big\|\boldsymbol{v}^{(0)}\big\|_{H^2,m}^2.$$

The third and fourth terms are bounded according to

$$III \le CR^2\big\|\boldsymbol{v}^{(0)}\big\|_{H^4,m}^2, \qquad IV \le CR^2\big\|\boldsymbol{v}^{(0)}\big\|_{H^4,m}^2.$$

Combining the above results we obtain

$$\mathcal{M}_0\left(e^{-t}(L+1)\boldsymbol{v}^{(0)}, T\right) \le CR^2\big\|\boldsymbol{v}^{(0)}\big\|_{H^4,m}^2. \tag{B.1}$$

To bound $\mathcal{M}(\boldsymbol{G}(\boldsymbol{w}), T)$ we need the following estimates of the nonlinearity

$$\begin{aligned}\|\boldsymbol{G}(\boldsymbol{w})\|^2 &= \left\|\sum_{k=1}^3 w_k\frac{\partial\boldsymbol{w}}{\partial x_k}\right\|^2 \le \sum_{k=1}^3\left\|w_k\frac{\partial\boldsymbol{w}}{\partial x_k}\right\|^2\\ &\le \sum_{k=1}^3\|w_k\|_\infty^2\left\|\frac{\partial\boldsymbol{w}}{\partial x_k}\right\|^2 \le \|\boldsymbol{w}\|_\infty^2\sum_{k=1}^3\left\|\frac{\partial\boldsymbol{w}}{\partial x_k}\right\|^2,\end{aligned} \tag{B.2}$$

$$\begin{aligned}\|\boldsymbol{G}(\boldsymbol{w})\|_m^2 &= \sum_{k=1}^3\left(\left\|w_k\frac{\partial w_1}{\partial x_k}\right\|^2 + R^2\left\|w_k\frac{\partial w_2}{\partial x_k}\right\|^2 + \left\|w_k\frac{\partial w_3}{\partial x_k}\right\|^2\right)\\ &\le \sum_{k=1}^3\|w_k\|_\infty^2\left\|\frac{\partial\boldsymbol{w}}{\partial x_k}\right\|_m^2 \le \|\boldsymbol{w}\|_\infty^2\sum_{k=1}^3\left\|\frac{\partial\boldsymbol{w}}{\partial x_k}\right\|_m^2\end{aligned} \tag{B.3}$$

and

$$\|\boldsymbol{G}_t(\boldsymbol{w})\|_m^2 = \left\|\frac{\partial}{\partial t}\left(\sum_{k=1}^{3} w_k \frac{\partial \boldsymbol{w}}{\partial x_k}\right)\right\|_m^2 \leq \sum_{k=1}^{3}\left(\left\|\frac{\partial w_k}{\partial t}\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|_m^2 + \left\|w_k \frac{\partial \boldsymbol{w}_t}{\partial x_k}\right\|_m^2\right)$$
$$\leq \sum_{k=1}^{3}\left\|\frac{\partial \boldsymbol{w}_k}{\partial t}\right\|_\infty^2 \left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|_m^2 + \sum_{k=1}^{3}\|w_k\|_\infty^2 \left\|\frac{\partial \boldsymbol{w}_t}{\partial x_k}\right\|_m^2$$
$$\leq \left\|\frac{\partial \boldsymbol{w}}{\partial t}\right\|_\infty^2 \sum_{k=1}^{3}\left\|\frac{\partial \boldsymbol{w}}{\partial x_k}\right\|_m^2 + \|\boldsymbol{w}\|_\infty^2 \sum_{k=1}^{3}\left\|\frac{\partial \boldsymbol{w}_t}{\partial x_k}\right\|_m^2. \tag{B.4}$$

C A Sobolev inequality for asymptotic stability

Let $f : [0,\infty) \to \mathbb{R}$. The theorem below gives a condition which imply that $f(t) \to 0$ as the time (t) tends to infinity. In this paper we will choose $f(t) = \|\boldsymbol{w}(\cdot,t)\|_{\tilde{H}_1}$, combined with the inequality (A.2), the theorem can then be used to prove that $\|\boldsymbol{w}(\cdot,t)\|_\infty \to 0$.

Theorem 3. *If*

$$\int_0^\infty \left(|f(t)|^2 + |f'(t)|^2\right) dt < \infty, \tag{C.1}$$

then

$$\lim_{t\to\infty} f(t) = 0.$$

Proof. The required Sobolev inequality is

$$\sup_{t\geq T} |f(t)|^2 \leq C \int_T^\infty \left(|f(t)|^2 + |f'(t)|^2\right) dt, \tag{C.2}$$

where C is independent of T. For a proof, see [4, appendix 3].

Clearly

$$|f(T)|^2 \leq \sup_{t\geq T} |f(t)|^2. \tag{C.3}$$

Because of (C.1) we also have

$$\lim_{T\to\infty} \int_T^\infty \left(|f(t)|^2 + |f'(t)|^2\right) dt = 0. \tag{C.4}$$

Combining (C.2), (C.3) and (C.4) completes the proof. ■

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