A numerical method for nonlinear inverse heat conduction problem

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Abstract—**Inverse Heat Conduction Problems(IHCPs) have been extensively studied over the last 60 years. They have numerous applications in many branches of science and technology.The problem consists in determining the temperature and heat flux at inaccessible parts of the boundary of a 2- or 3-dimension al body from corresponding data–called "Cauchy data" on accessible parts of the boundary. It is well-known that IHCPs are severely ill-posed which means that small perturbations in the data may cause extremely large errors in the solution. In this contribution, I present an analysis of numerical solution of nonlinear inverse heat conduction problem in a region with moving boundaries, a regularization method is used to construct an algorithm for smoothing the experimental data in a complication of the input data for the inverse problem.**

Keywords- *inverse heat conduction problems, ill-posed, regularization, nonlinear*

1 Introduction

The classical problem in heat conduction consists in determining the interior temperature distribution of a body from data given on its surface. However, in many physical problems data are not available over the entire surface, or are available only at certain locations in the interior. In such case, the aim is to determine surface temperatures and fluxes from a measured temperature history at fixed locations inside the body or at certain parts of the surface. This problem is called the inverse heat conduction problem(IHCP).

There are many practical problems in physics and in the engineering sciences which essentiallyrely on the solution of an IHCP. Fields of applications comprise the developmentof new materials, the casting and welding in steel or polymer processing, the development of a sophisticated temperature measurement related to lifetime analysis of plans, the development of transient calorimeters, ill-posed problems in rheometry,etc.

IHCP can be efficiently employed when some governing parameters of the heat transfer equations, such as thermal conductivity or heat flux, are not known precisely. Reliability of the inversion relies on the accuracy of the identified model. In general, prediction of solution for the IHCP can be achieved via minimization of a sum of squared error function; which is focused on the difference among the values of the measured temperatures and those obtained by an efficient computational method. The unknown thermal coefficients on the mathematical model (i.e., thermal properties, boundary or initial conditions) that lead to an acceptable value for the aforementioned error function, based on the iterative regularization method, are the solution of the IHCP. In addition to textbooks [1,2] available in the literature, numerous recent published researches have discussed the estimation of boundary conditions in IHCP [3-12]. Most of the previous work was restricted to problems with constant thermophysical properties. However, the material properties can become temperature-dependent for most realistic engineering problems. As a result, the heat conduction problem will become nonlinear. The estimation of the unknown boundary conditions for the nonlinear IHCP is more complicated than those for the linear IHCP [11-12]. Among the well-known methods for boundary estimation problems, the gradient methods have received the most attention. Gradient methods are typically applied on the entire time domain and use all of the temperature data simultaneously in estimation of any components of unknowns. Two important categories of gradient methods are Conjugate Gradient Method (CGM) and Variable Metric Method (VMM). The first is used widely in the field and has been among the highly successful IHCP algorithms. A comprehensive discussion on CGM is found in the textbook of Ozicik and Orlande [2]. The second categoryhas recently employed by kowsary et al. [10] for estimation of space and time varying heat flux. Because of computationally taxing associated with nonlinearity of problem, considerable attention is devoted to use an adjoint equation approach coupled with these methods in order to reduce the computational time.

Heat flux estimation is particularly more of a challenge, when the material at the boundary reaches its phase change temperature. It is worth mentioning that a few publications available in the literature have considered the effect of moving boundary on the inverse problems [13-16]. Oliveira and Orlande [14] applied the CGMAP to estimate sum of the convection and radiation heat flux at the surface of charring ablator using simultaneously temperature measurement as well as measurement of the position of the receding surface. They used an approximate technique (i.e., heat of ablation) for thermal response calculations. In practice, this technique is very simplified in concept and straightforward to implement; however, it has several limitations. The heat conduction approximation dose not account for material decomposition, where the density, thermal conductivity, and specific heat are functions of temperature and also vary as the material is pyrolyzed going from a virgin to a charred composite. It also does not account for the pyrolysis gas percolating through the material and the energy absorbed via this process. These methods provide the systems designer with a ball-park answer, more as a starting point, but are lacking in that the steady state ablation assumption is usually not valid during ablation process and will generally over-predict recession [17]. To overcome this problem, Hakkaki-fard and Kowsary [15] employed CGMAP in order to predict sum of the convection and radiation heat flux at ablating surface; however, the effect of gas enthalpy adjacent to the wall on the boundary conditions was ignored in their work. Ignoring the gas enthalpy at wall can causes consid-erable deviation in predicting the correct temperature at heat transfer phenomenon. Both studies were assumed that the temperature of ablating surface is fixed. The selection of an ablation temperature is arbitrary and dose not represents the actual surface temperature or pyrolysis/virgin material interface temperature in application. This can result in overly conservative predictions of thermal protection thicknesses to maintain structure temperatures at acceptable limits. Accordingly, if desired accuracy can be obtained for the required input parameters, attempting to characterize the actual chemical and mechanical processes occurring in the material and modeling the boundary layer interactions may enhance the accuracy of in-depth temperature predictions and provide a means of reducing the thermal protection system weight contribution [18]. In addition, there are two main shortcomings in the model developed in aforementioned studies. First, the thermophysical properties of ablators are assumed to be constant. However, their thermophysical properties are greatly affected, because of the nature of the ablative materials and their applications (in which they are exposed to a wide temperature range). Typically, temperature-dependent properties augment the nonlinearity problem. Hence, for this kind of problems, estimation of the time-dependent heat flux is essential. Second, both studies used the fixed and constant grid spacing scheme for the solution of the governing equations. It is evident that the difference between the calculated and the measured temperature appears as a time-dependent source term at fixed distances, depends on the sensor locations, in adjoint problem. To account for this source term in the numerical solution, the fixed and constant grid spacing scheme is necessary. Employing this scheme in problems with moving boundary creates a large discontinuity in adjacent cell sizes at the receding surface, which could have negative numerical implications on the spatial discretization accuracy [19]. Thus, the lack of a highly efficient as well as accurate direct calculation schemes can be best recognized. In the present paper, we outline a numerical method for solving the inverse problem for the case of the one-dimensional quasi-linear heat-conduction equation with a continuous heat source and a convective term. This model corresponds, in particular, to the threedimensional thermal destruction of the material of an object, with the flow of the resulting gaseous products in the pores of the object. This method is based on an implicit difference scheme; the integration is carried out along the direction of the spatial variable. This approach to the solution of this problem is taken under the assumption that implicit difference schemes should have significant "viscous" properties for this choice of the direction of integration of the heat-conduction equation. This hypothesis is verified in the numerical use of the method, so that it becomes possible to obtain a regular solution of the inverse problem with short time steps, even if the input temperatures are afflicted with certain (small) errors.

2 Numerical solution and regularization method

Let it be required to determine the temperature field in the domain $\{X_1(\tau) \le x \le X_2(\tau), 0 \le \tau \le \tau_m\}$ and heat conditions at boundary $X_1(\tau)$ by a solution of the following generalized Cauchy problem:

$$C(T)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + K(T)\frac{\partial T}{\partial x} + S(T) \ ,$$
$$X_1(\tau) < x < X_2(\tau), 0 < \tau \le \tau_m$$
$$T(x,0) = \xi(x), \quad X_1(0) \le x \le X_2(0) \qquad (2.1)$$

$$-\lambda T(X_2(\tau),\tau)\frac{\partial T(X_2(\tau,\tau))}{\partial x} = q^*(\tau),$$
$$T(X_2(\tau),\tau) = T^*(\tau)$$

where $C(T)$, $\lambda(T)$ $K(T)$ $S(T)$ $\xi(x)$ $q^*(\tau)$ $T^*(\tau)$ are known functions. Using new variables

$$y = \frac{x - X_1(\tau)}{X_2(\tau) - X_1(\tau)} \ , t = \tau \ , \qquad (2.2)$$

Go to the rectangular of integration. Then we obtain the problem

$$
\begin{aligned}
&C(T)\frac{\partial T}{\partial \tau}=\frac{1}{[X_2(t)-X_1(t)]^2}\frac{\partial}{\partial y}\left(\lambda(T)\frac{\partial T}{\partial y}\right)\\
&+\frac{C(T)\{X_1'(t)+y[X_2'(t)-X_1'(t)]\}+K(T)}{X_2(T)-X_1(T)}+S(T), \quad 0<y<1, 0<\tau\le\tau_m\\
&T(y,0)=\phi(y), \qquad 0\le y\le 1,\\
&-\frac{\lambda(T(1,t))}{X_2(T)-X_1(T)}\frac{\partial T(1,t)}{\partial y}=q^*(\tau);\\
&T(1,t)=T^*(t),
\end{aligned}
\tag{2.3}
$$

Where $X_i'=\dfrac{dX_i}{dt}, i=1,2.$

Introduce a difference grid $\{yl=\Delta y\cdot l, l=1,2,..., L; t_n=\Delta t\cdot n, n=1,2,...m\}$. Perform integration in the direction of the spatial coordinate. Approximate derivatives at note $(l+1,n)$ by the following difference relationships:

$$
\begin{aligned}
&\left.\frac{\partial T}{\partial t}\right|_{l+1,n}\cong\frac{T_{l+1,n}-T_{l+1,n-1}}{\Delta t},\\
&\left.\frac{\partial T}{\partial y}\right|_{l+1,n}\cong\frac{T_{l+2,n}-T_{\ln}}{2\Delta y},\\
&\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)\Big|_{l+1,n}\cong\frac{1}{\Delta y}\left[\frac{\lambda_{l+2,n}+\lambda_{l+1,n}}{\Delta y}\right.\\
&\left.-\frac{\lambda_{l+1,n}+\lambda_{\ln}}{2}\cdot\frac{T_{l+1,n}-T_{\ln}}{\Delta y}\right]
\end{aligned}
\tag{2.4}
$$

Then a difference analog of equation (2.3) for every spatial layer with number l is

$$B_{\ln}T_{\ln}=F_{\ln}, \quad n=1,2,\cdots,m;\ l=L-2,L-3,\cdots 0, \tag{2.5}$$

Where

$$
\begin{aligned}
B_{\ln}&=\frac{\lambda_{l+1,n}+\lambda_{\ln}}{2\Delta y^2C_{l+1,n}(X_{2n}-X_{1n})^2}-\\
&\frac{C_{l+1,n}\left[X_{1n}'+\Delta y(l+1)(X_{2n}'-X_{1n}')\right]+K_{l+1,n}}{2\Delta yC_{l+1,n}(X_{2n}-X_{1n})},\\
F_{\ln}&=\frac{T_{l+1,n}-T_{l+1,n-1}}{\Delta t}-\\
&\frac{\lambda_{l+2,n}+\lambda_{l+1.n}}{2\Delta y^2C_{l+1,n}(X_{2n}-X_{1n})^2}\left(T_{l+2,n}-T_{l+1.n}\right)\\
&+\frac{\lambda_{l+1,n}+\lambda_{\ln}}{2\Delta y^2C_{l+1,n}(X_{2n}-X_{1n})^2}T_{l+1,n}-\\
&\frac{C_{l+1,n}\left[X_{1n}'+\Delta y(l+1)(X_{2n}'-X_{1n}')\right]+K_{l+1,n}}{2\Delta yC_{l+1,n}(X_{2n}-X_{1n})}T_{l+2,n}\\
&+\frac{S_{l+1,n}}{C_{l+1,n}}.
\end{aligned}
\tag{2.6}
$$

The nonlinear recurrent relation (1.5) permits the grid temperature to be restored at $l-th$ spatial layer. The temperature values at zero time are determined from the condition $(2.3): T_{l0}=\phi_l, l=1,\cdots,L.$

Relation (1.5) is a relation of one of the direct numerical methods of IHCP solution. In this case a temperature profile, received for a fixed value of index l is corrected by means of iterations through the thermal coefficients. The thermophysical characteristics computation for $l-layer$ is carried out using values of temperature computed at previous iteration. The grid solution, Obtained for the previous layer, is used as an initial approximation(for a profile of temperature). The iteration process ends when the condition is satisfied:

$$\max_n\left|T_{\ln}^{(p)}-T_{\ln}^{(p-1)}\right|\le\varepsilon, \tag{2.7}$$

where p is the iteration number; $\varepsilon>0$ is some small number.

To get an algorithm for solving the algebraic problem (1.5), not bounded by the stability constraints on step Δt ,

$$
\begin{aligned}
\Phi_\alpha[T_l]&=\sum_{n=1}^m(B_{\ln}T_{\ln}-F_{\ln})^2+\alpha_lk_1\sum_{n=1}^m(T_{\ln}-T_{l,n-1})^2\\
&+\alpha k_2\sum_{n=1}^m(T_{l,n+1}-2T_{\ln}+T_{l,n-1})^2,
\end{aligned}
\tag{2.8}
$$

where $k_1\ge 0, k_2\ge 0.$

Assuming that thermophysical characteristics λ, C, K, S in the expression (1.6) be computed for $l-th$ layer through temperature values obtained on the previous iteration, and minimizing $\Phi_\alpha[T_l]$ in all $T_{\ln}, n=1,\cdots,m$, we receive a system of algebraic equations with a symmetric five-diagonal positive definite matrix

$$\sum_{r=n-2}^{n+2}a_{nr}T_{lr}=b_n, n=1,2,\cdots m, \tag{2.9}$$

Where

$$
a_{nn}=\begin{cases}B_{\ln}^2+\alpha(2k_1+5k_2), n=1,m-1,\\ B_{\ln}^2+\alpha(2k_1+6k_2), n=2,3,\cdots,m-2\\ B_{\ln}^2+\alpha(k_1+k_2), n=m;\end{cases}
$$

$$a_{n,n+1}=a_{n+1,n}=\begin{cases}\alpha(k_1+4k_2), n=1,2,\cdots m-2,\\ \alpha(k_1+2k_2), n=m-1,\\ 0, n=m;\end{cases}$$

$$a_{n,n+2}=a_{n+2,n}=\begin{cases}\alpha k_2, n=1,2,\cdots m-2,\\ 0, n=m-1,m;\end{cases}$$

$$b_n=B_{\ln}F_{\ln}, n=1,2,\cdots,m-2;$$

$$b_{m-1}=B_{l,m-1}F_{l,m-1}-\alpha k_2 C_m;$$

$$B_m=B_{lm}F_{lm}+2\alpha k_2 C_m;$$

$$C_m=\frac{d^2T_l(\tau m)}{dt^2}.$$

System (2.9) has the unique solution which is a regularized profile of temperatures at $l-th$ spatial layer under definite value of parameter α_l. The problem (2.9) is better to solve by the square root method[20].

To choose a suitable value α_1 of the regularization parameter use the residual principle in the form of equality

$$\rho l(\alpha)=\sum_{n=1}^{m}\left(T_{Ln}^{\alpha}-T_n^{*}\right)^2=\delta^2 , \qquad (2.10)$$

Where $T_{Ln}^{\alpha}=T_{Ln}(\alpha l), n=1,\cdots,m$ is the solution of a direct heat conduction problem in the domain $yl\le y\le l, 0<t\le\tau_m$ under condition (2.2) on the right-hand boundary and with given temperature on the left-hand boundary. The later is computed by solving the system (2.6). To calculate the grid function T_{Ln}^{α} we follow an algorithm based on the same approximating relations as in system (2.4). In this case an implicit T-type scheme of approximation is used for the direct problem. The root $\alpha*l$ of equation (2.10) is determined by a chord method.

A computational algorithm for $l-th$ spatial layer is constructed as follow:

1. Set an initial value α_l^0 of the regularization parameter(for example, Assume $\alpha_l^0=\alpha_{l-1}$).

2. Compute the regularized profile of temperatures $T_{\ln}(\alpha_l^0), n=1,\cdots,m$ at $l-th$ layer, that is, solve the system (1.6) with value α_l^0 of the parameter.

3. Using the obtained regularized solution $T_{\ln}(\alpha_l^0), n=1,\cdots,m$ at $l-th$ as the grid boundary condition, solve the direct problem of heat conduction in the domain $yl\le y\le l$, $0<t\le\tau_m$ and define the grid function $T_{Ln}(\alpha_l^0), n=1,\cdots,m$

4. Compute the residual $\rho l(\alpha_l^0)$ and solve the equation by the chord method(step 2 and step 3 are used here).

Note that in this calculation we should successively correct the values of coefficients λ, C, K, S α_l^* in the grid nodes. The computation of these coefficients must be performed by using a sufficiently smooth temperature field. Otherwise, an inadmissible distortion of the solution takes place, and the iterative process of determining root α_l^* may not converge.That is why, the correction of thermophysical coefficients is carry out only when $\rho l(\alpha l)>\delta^2$ .In this case the regularized solution $T_{\ln}(\alpha_l)=1,\cdots,m$ is in advance of a smooth nature. In such a manner we perform computations of temperature field in the whole domain considered. The heat flux on the boundary is defined from a finite difference approximation of the expression

$$q(t)=-\frac{\lambda(T(0,t))}{X_2(t)-X_1(t)}\frac{\partial T(0,t)}{\partial y}. \quad (2.11)$$

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