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# Deformations of the Bihamiltonian Structures on the Loop Space of Frobenius Manifolds

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**Abstract**

We consider an important class of deformations of the genus zero bihamiltonian structure defined on the loop space of semisimple Frobenius manifolds, and present results on such deformations at the genus one and genus two approximations.

## 1 Introduction

The notion of Frobenius manifold was introduced by Boris Dubrovin in [1, 2, 3], it is a coordinate free formulation of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations of associativity which arose in the study of 2D topological field theory (TFT) [4, 5]. The initial motivation of the study of Frobenius manifolds comes from the idea to reconstruct a 2D TFT starting from its primary free energy, which is a solution of the WDVV equations of associativity. The bihamiltonian structure on the loop space of the Frobenius manifold comes into the play when one consider the coupling of the matter sector of the 2D TFT to topological gravity. It was shown in [1, 2, 3] that at the genus zero (tree level) approximation the procedure of coupling to the topological gravity of the matter sector of a 2D TFT can be described by a bihamiltonian hierarchy of integrable systems of hydrodynamic type, this hierarchy of integrable systems is call the *genus zero bihamiltonian hierarchy* [6], it is defined for any Frobenius manifold. The genus zero free energy of the 2D TFT is a particular tau-function of this hierarchy. By assuming the semisimplicity property of the Frobenius maifold, it was shown in [6] that there is also a universal procedure to construct the genus one free energy of a 2D TFT starting from its primary free energy. This procedure is also described by a bihamiltonian hierachy of integrable systems which is certain deformation of the genus zero one.

It is conjectured that there should exist certain deformation of the genus zero bihamiltonian hierarchy which controls the construction of a full 2D TFT. Such deformed bihamiltonian hierarchy of integrable systems is known for the special case of 2D topological gravity. It is shown by the theory of Witten [7, 8] and Kontsevich [9] that the full genera free energy for the 2D topological gravity is the logarithm of the tau-function of a particular solution of the (bihamiltonian) KdV hierarchy. For the case of the topological minimal models and the $CP^1$ topological sigma model such deformed hierarchies are conjectured to be the Gelfand-Dickey hierarchies and the Toda lattice hierarchy respectively [1, 10, 11, 12, 13, 14, 15].

We present here some results on an important class of deformations of the genus zero bihamiltonian structure defined on the loop space of any semisimple Frobenius manifold, they arise naturally when we study the problem of reconstruction of a 2D TFT from its primary free energy, and are called the *quasitrivial deformations* of the genus zero bihamiltonian structures. The main results of this talk is based on [14] where the notion of quasitrivial deformation was introduced. We first recall the genus zero bihamiltonian structure that is defined on the loop space of a Frobenius manifold in section 2 and then consider its deformations in section 3.

## 2 The genus zero bihamiltonian structure on the loop space of a Frobenius manifold

By the definition of Boris Dubrovin [1, 3], a smooth manifold $M$ is called a Frobenius manifold if on each of its tangent spaces $T_vM$ there exists a structure of Frobenius algebra, i.e., there is defined on $T_vM$ an operation of multiplication and a nondegenerate bilinear form $<\, ,\, >$ such that $T_vM$ forms a commutative and associative algebra with unity $e$, and the bilinear form is invariant with respect to this multiplication. The Frobenius algebra structure is required to depend smoothly on the point $v \in M$ and satisfies the following axioms:

i) The metric $<\, ,\, >$ is flat, and if we denote by $\nabla$ the Levi-Civita connection of this metric, then $\nabla e = 0$.

ii) Define the three tensor $c(\xi, \zeta, \rho) = <\xi \cdot \zeta, \rho>$ on $T_vM$, where $\xi, \zeta, \rho \in T_vM$, then the four tensor $\nabla_\sigma c(\xi, \zeta, \rho)$ is symmetric w.r.t. $\xi, \zeta, \rho, \sigma \in T_vM$.

iii) There exists a vector field $E$ on $M$, called the Euler vector field, such that $\nabla\nabla E = 0$ and

$$\mathcal{L}_E\, c^\gamma_{\alpha\beta} = c^\gamma_{\alpha\beta}, \quad \mathcal{L}_E\, \eta_{\alpha\beta} = (2-d)\, \eta_{\alpha\beta}, \tag{2.1}$$

where $c^\gamma_{\alpha\beta}$ is the structure constants of the Frobenius algebra in local flat coordinates $v^1, \dots, v^n$ and $\eta_{\alpha\beta} = <\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}>$.

A Frobenius manifold $M$ is called semisimple if there exists $v \in M$ such that the algebra defined on $T_vM$ is semisimple.

We can choose local flat coordinates $(v^1, \dots, v^n)$ of the metric $<\, ,\, >$ near any point on a Frobenius manifold such that $e = \frac{\partial}{\partial v^1}$. We denote

$$< \frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta} >= \eta_{\alpha\beta}, \quad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}\ . \tag{2.2}$$

Here $(\eta_{\alpha\beta})$ is a constant matrix. The above definition ensures the existence of a function $F = F(v^1, \dots, v^n)$ such that

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial v^1 \partial v^\alpha \partial v^\beta}, \tag{2.3}$$

and the multiplication on $T_vM$ is given by the formula

$$\frac{\partial}{\partial v^\alpha} \cdot \frac{\partial}{\partial v^\beta} = c^\gamma_{\alpha\beta} \frac{\partial}{\partial v^\gamma} \tag{2.4}$$

with

$$c^\gamma_{\alpha\beta} = \eta^{\gamma\nu} \frac{\partial^3 F}{\partial v^\nu \partial v^\alpha \partial v^\beta}. \tag{2.5}$$

Here and henceforth summation over repeated indices is assumed. Axiom (iii) implies the quasihomogeneity property of the function $F$. We assume for simplicity that the linear part of the Euler vector field is diagonalizable, then the local flat coordinates can be chosen so that the Euler vector field $E$ has the form

$$E = \sum_{\gamma=1}^n E^\gamma \frac{\partial}{\partial v^\gamma} = \sum_{\alpha=1}^n \left((1 - \frac{d}{2} - \mu_\alpha) v^\alpha + r_\alpha\right) \frac{\partial}{\partial v^\alpha}, \tag{2.6}$$

where $d, \mu_\alpha, r_\alpha$ are some constants which has the property that $r_\alpha = 0$ if $1 - \frac{d}{2} - \mu_\alpha \neq 0$, and

$$(\mu_\alpha + \mu_\beta)\eta_{\alpha\beta} = 0. \tag{2.7}$$

Then the function $F$ satisfies

$$\partial_E F = (3 - d)F + \frac{1}{2} A_{\alpha\beta} v^\alpha v^\beta + B_\alpha v^\alpha + C, \tag{2.8}$$

for some constants $A_{\alpha\beta}, B_\alpha$ and $C$. We call $F$ the potential of the Frobenius manifold, it satisfies the WDVV equations of associativity. In 2D TFT it is called the primary free energy and in quantum cohomology it is called the Gromov-Witten potential.

The genus zero bihamiltonian structure on the loop space of the Frobenius manifold

$$\mathcal{L}(M) = \{(v^1(x), \dots, v^n(x)) \big| \, x \in S^1\}$$

is induced by a linear pencil of flat metrics on the Frobenius manifold. This linear pencil of flat metrics is composed of the flat metric $< \, , \, >$ and a second flat metric given by the intersection form on the cotangent bundle of the Frobenius manifold. In the local flat coordinates of the metric $< \, , \, >$ the intersection form is given by the formula

$$g^{\alpha\beta}(v) = (dv^\alpha, dv^\beta) = E^\gamma \, c^{\alpha\beta}_\gamma \, , \tag{2.10}$$

where

$$c^{\alpha\beta}_\gamma = \eta^{\alpha\nu} \, c^\beta_{\nu\gamma} \tag{2.11}$$

and $c^\beta_{\nu\gamma}$ are defined in (2.5). The genus zero bihamiltonian structure is given by the following two compatible Poisson brackets

$$\{v^\alpha(x), v^\beta(y)\}_1 = \eta^{\alpha\beta} \, \delta'(x - y) \, , \tag{2.12a}$$

$$\{v^\alpha(x), v^\beta(y)\}_2 = g^{\alpha\beta}(v(x)) \, \delta'(x - y) + \Gamma^{\alpha\beta}_\gamma(v(x)) \, v^\gamma_x \, \delta(x - y) \, , \tag{2.12b}$$

here $\Gamma^{\alpha\beta}_\gamma$ are the coefficients of the Levi-Civita connection of the second metric ( , ) which has the following expression

$$\Gamma^{\alpha\beta}_\gamma = -g^{\alpha\nu}\,\Gamma^\beta_{\nu\gamma} = \left(\frac{1}{2}-\mu_\beta\right) c^{\alpha\beta}_\gamma \ . \tag{2.13}$$

The compatibility of the above two Poisson brackets means that for any parameter $\lambda$, the following combination of them

$$\{v^\alpha(x), v^\beta(y)\}_2 - \lambda\,\{v^\alpha(x), v^\beta(y)\}_1 \tag{2.14}$$

also defines a Poisson bracket.

**Example 1.** Let $M$ be the one dimensional Frobenius manifold, it has the potential

$$F = \frac{1}{6}\,(v^1)^3 \ . \tag{2.15}$$

The Euler vector field is given by

$$E = v^1\,\frac{\partial}{\partial v^1} \ , \tag{2.16}$$

and the genus zero bihamiltonian structure has the form

$$\{v^1(x), v^1(y)\}_1 = \delta'(x-y) \ , \tag{2.17a}$$

$$\{v^1(x), v^1(y)\}_2 = v^1(x)\,\delta'(x-y) + \frac{1}{2}\,v^1_x\,\delta(x-y) \ . \tag{2.17b}$$

**Example 2.** Let $M$ be a two dimensional Frobenius manifold with potential

$$F = \frac{1}{2}\,(v^1)^2\,v^2 + \exp(v^2) \ . \tag{2.18}$$

The Euler vector field is given by

$$E = v^1\,\frac{\partial}{\partial v^1} + 2\,\frac{\partial}{\partial v^2} \tag{2.19}$$

and the genus zero bihamiltonian structure has the form

$$\{v^1(x), v^1(y)\}_1 = \{v^2(x), v^2(y)\}_1 = 0 \ , \tag{2.20a}$$

$$\{v^1(x), v^2(y)\}_1 = \delta'(x-y) \ , \tag{2.20b}$$

$$\{v^1(x), v^1(y)\}_2 = 2\,e^{v^2(x)}\,\delta'(x-y) + v^1_x\,e^{v^2(x)}\,\delta(x-y) \ , \tag{2.20c}$$

$$\{v^1(x), v^2(y)\}_2 = v^1(x)\,\delta'(x-y) \ , \tag{2.20d}$$

$$\{v^2(x), v^2(y)\}_2 = 2\,\delta'(x-y) \ . \tag{2.20e}$$

Associated to the bihamiltonian structure (2.12) we have the following genus zero bihamiltonian hierarchy of integrable systems:

$$\frac{\partial v^\alpha}{\partial t^{\beta,q}} = K^\alpha_{\beta,q}(v; v_x) = \{v^\alpha(x), H_{\beta,q}\}_1 \ , \quad \alpha,\beta = 1,\dots,n;\ q \ge 0 \ , \tag{2.21}$$

the Hamiltonians

$$H_{\beta,q}=\frac{1}{2\pi}\int_0^{2\pi}\theta_{\beta,q+1}(v(x))dx,\quad \alpha=1,\dots,n;\ q\ge -1 \tag{2.22}$$

are defined by the recursion relations

$$\begin{aligned}&\theta_{1,0}=v^2,\ \theta_{2,0}=v^1\ ,\\ &\frac{\partial^2\theta_{\beta,q+1}}{\partial v^\gamma\partial v^\nu}=c^\xi_{\gamma\nu}\frac{\partial\theta_{\beta,q}}{\partial v^\xi},\quad \alpha,\beta=1,\dots,n;\ q\ge 0\ ,\\ &\partial_E\theta_{\beta,q}=(q+1-\frac{d}{2}+\mu_\beta)\,\theta_{\beta,q}+\sum_{k=1}^q(R_k)^\gamma_\beta\,\theta_{\gamma,q-k}\ .\end{aligned} \tag{2.23}$$

Here $R_k$ are constant matrices satisfying

$$[\mu,R_k]=k\,R_k,\quad (R_k)^\gamma_\alpha\,\eta_{\gamma\beta}=(-1)^{k+1}\,(R_k)^\gamma_\beta\,\eta_{\gamma\alpha}\ , \tag{2.24}$$

they are part of the monodromy datas of the Frobenius manifold $M$ at origin [1, 3]. For the one dimensional Frobenius manifold given in Example 1, we have $R_k=0$, $k\ge 1$ and for Frobenius manifold given in Example 2, we have $(R_1)^\alpha_\beta=2\,\delta_{\alpha,2}\,\delta_{\beta,1}$, $R_k=0$, for $k\ge 2$.

The genus zero bihamiltonian hierarchy (2.21) satisfies the bihamiltonian recursion relations

$$\begin{aligned}&\{v^\alpha(x),H_{\beta,q-1}\}_2\\ &\quad=(q+\mu_\beta+\frac{1}{2})\,\{v^\alpha(x),H_{\beta,q}\}_1+\sum_{k=1}^q(R_k)^\gamma_\beta\,\{v^\alpha(x),H_{\gamma,q-k}\}_1\ ,\\ &\quad\alpha,\beta=1,\dots,n,\ q\ge 0\ .\end{aligned} \tag{2.25}$$

Since

$$\frac{\partial v^\alpha}{\partial t^{1,0}}=\frac{\partial v}{\partial x}\ , \tag{2.26}$$

we identify $t^{1,0}$ with the spatial variable $x$.

The bihamiltonian hierarchies related to the bihamiltonian structures (2.17) and (2.20) of the above two examples are the dispersionless KdV hierarchy and the dispersionless Toda lattice hierarchy respectively. The dispersionless Toda lattice hierarchy is usually called the long wave limit of the Toda lattice hierarchy.

The genus zero bihamiltonian hierarchy (2.21) possesses an important property that the one form defined by

$$\Omega=\sum\theta_{\alpha,p}(v)dt^{\alpha,p} \tag{2.27}$$

is closed, this property together with (2.26) is called a tau-structure of the bihamiltonian hierarchy (2.21) in [14]. It ensures [3, 14] that for any solution $v(t)=(v^1(t),\dots,v^n(t))$ of the hierarchy (2.21) there exists a function $\tau(t)$, called the tau-function of this solution, such that

$$\theta_{\alpha,p}(v(t)) = \frac{\partial^2 \log \tau(t)}{\partial x \partial t^{\alpha,p}}. \tag{2.28}$$

As it was shown in [1, 2, 3], the genus zero free energy of a 2D TFT is the logarithm of the tau-function of a particular solution $v^{(0)}(t) = (v^1(t), \dots, v^n(t))$ of the hierarchy (2.21), this solution is uniquely determined by the following conditions:

$$v^\alpha(t)\Big|_{t^{\beta,q}=0,\, q\ge 1} = t^{\alpha,0}\ , \tag{2.29}$$

$$\sum t^{\beta,q}\,\frac{\partial v^\alpha}{\partial t^{\beta,q}} = \frac{\partial v^\alpha}{\partial t^{1,1}}\ . \tag{2.30}$$

We denote this tau-function by $\tau^{(0)}(t)$. In the setting of 2D TFT, the time variables $t^{\alpha,0}$ and $t^{\alpha,p},\ p \ge 1$ are the coupling constants of the primary fields and their gravitational descendents respectively. These variables constitute the *big phase space* of the 2D TFT. The genus zero free energy $\mathcal{F}_0(t) = \log \tau^{(0)}(t)$ yields the primary free energy (i.e., the potential of the Frobenius manifold) $F(v)$ if we restrict $\mathcal{F}_0(t)$ to the *small phase space* $t^{\alpha,0} = v^\alpha,\ v^{\beta,q} = 0,\ q \ge 1$. From the identity (2.28) we know that the component ${v^{(0)}}^\alpha(t)$ of the above particular solution of the genus zero bihamiltonian hierarchy coincides with the following genus zero two point correlation functions:

$$\eta^{\alpha\gamma}\frac{\partial^2 \mathcal{F}_0}{\partial t^{1,0}\partial t^{\gamma,0}}\ . \tag{2.31}$$

The above two examples correspond respectively to the 2D topological gravity and the $CP^1$ topological sigma model. X-Mozilla-Status: 0000

# 3 Deformations of the genus zero bihamiltonian structure

We consider deformations of the genus zero bihamiltonian structure (2.12) with the following form:

$$\{w^\alpha(x), w^\beta(y)\}_i = \{w^\alpha(x), w^\beta(y)\}_i^{(0)} + \sum_{k\ge 1} \varepsilon^k \{w^\alpha(x), w^\beta(y)\}_i^{(k)}\ , \tag{3.1}$$

$$i = 1, 2;\quad \alpha, \beta = 1, \dots, n\ ,$$

where $\varepsilon$ is the deformation parameter, $\{w^\alpha(x), w^\beta(y)\}_i^{(0)}$ are defined by (2.12) with $v^\gamma, v^\gamma_x$, $\gamma = 1, \dots, n$ replaced by $w^\gamma, w^\gamma_x$, and $\{w^\alpha(x), w^\beta(y)\}_i^{(k)}$ have the form

$$\{w^\alpha(x), w^\beta(y)\}_i^{(k)} = \sum_{l=0}^{k+1} P^{\alpha\beta}_{i,k;l}(w; w_x, \dots, \partial_x^l w)\, \delta^{(k+1-l)}(x-y)\ . \tag{3.2}$$

Here $P^{\alpha\beta}_{i,k;l}(w; w_x, \dots, \partial_x^l w)$ are differential polynomials of degree $l$, i.e., they are polynomials in $\partial_x^j w^\gamma,\ j \ge 1$ with coefficients being smooth functions of $w^\gamma$, and the total degree of derivatives with respect to $x$ in each monomial equals $l$. Examples of such kind of deformations can be readily obtained by any Miura type transformation of the form

$$w^\alpha = v^\alpha + \sum_{k\geq 1} \varepsilon^k A_k(v; v_x, \dots, \partial_x^{m_k} v), \tag{3.3}$$

with $A_k$ being differential polynomials of degree $k$. We call deformation (3.1) that is obtained in this way a *trivial deformation* [14]. We will be interested in a class of nontrivial deformations of the genus zero bihamiltonian structure (2.12), these deformations have the form (3.1) and are also obtained by Miura type transformation of the form (3.3), however, instead of being differential polynomials the functions $A_k(v; v_x, \dots, \partial_x^{m_k} v)$ are smooth functions of the independent variables $v^\gamma, \partial_x^j v^\gamma,\ 1 \leq j \leq m_k$. For most of our interesting examples these coefficients are in fact rational functions in $v_x^\gamma, \dots, \partial_x^{m_k} v^\gamma$. In [14] such kind of Miura type transformation is called a *quasi-Miura transformation*, and the resulting deformation of the genus zero bihamiltonian structures is called a *quasitrivial deformation.* If a deformation (3.1) of the genus zero bihamiltonian structure is considered only at the approximation up to $\varepsilon^k$, then we call it a *genus* $\frac{k}{2}$ *deformation*; if a genus $\frac{k}{2}$ deformation is obtained by a quasi-Miura transformation at the approximation up to $\varepsilon^k$, then it is called a *genus* $\frac{k}{2}$ *quasitrivial deformation.* Two (genus $\frac{k}{2}$) quasitrivial deformations of the genus zero bihamiltonian structure (2.12) are said to be equivalent if they are related by a usual Miura type transformation.

Quasitrivial deformations of the genus zero bihamiltonian structure appear naturally when we consider the possibility of the existence of a bihamiltonian hierarchy of integrable systems that is satisfied by the full genera two point correlation functions

$$\frac{\partial^2 \mathcal{F}(t)}{\partial x \partial t^{\alpha,0}} \tag{3.4}$$

of a 2D TFT. It is conjectured in [16] that the genus $g$ free energy is a function of the genus zero two point correlation functions, this means that $\mathcal{F}_g$ should have the form

$$\mathcal{F}_g(t) = F_g(v, v_x, \dots, \partial_x^{3g-2} v)\Big|_{v=v^{(0)}(t)}\ , \tag{3.5}$$

where $F_g(v, v_x, \dots, \partial_x^{3g-2} v)$ are smooth functions of $v, v_x, \dots, \partial_x^{3g-2} v$ and $v^{(0)}(t)$ is the particular solution of the genus zero bihamiltonian hierarchy (2.21) given in the last section. On the other hand, the free energy $\mathcal{F}(t)$ can be expressed in the genus expansion form

$$\mathcal{F}(t) = \sum_{g\geq 0} \varepsilon^{2g-2} \mathcal{F}_g(t)\ , \tag{3.6}$$

the parameter $\varepsilon$ here is called the string coupling constant. So the full genera two point correlation functions (3.4) have the expressions

$$\frac{\partial^2}{\partial x \partial t^{\gamma,0}} \sum_{g\geq 0} \varepsilon^{2g} F_g(v, v_x, \dots, \partial_x^{3g-2} v)\Big|_{v=v^{(0)}(t)}\ . \tag{3.7}$$

This leads us to consider the following quasi-Miura transformation

$$w^\alpha = v^\alpha + \eta^{\alpha\gamma} \frac{\partial^2}{\partial x \partial t^{\gamma,0}} \sum_{g\geq 1} \varepsilon^{2g} F_g(v, v_x, \dots, \partial_x^{3g-2} v)\ . \tag{3.8}$$

The hypothetical bihamiltonian hierarchy that is related to the 2D TFT in full genera is expected to be a quasitrivial deformation of the genus zero one under such a quasi-Miura transformation.

Assume that we have a quasitrivial deformation (3.1) of the genus zero bihamiltonian structure, then the genus zero bihamiltonian hierarchy (2.21) also acquires a deformation. In the new coordinates $w^1, \dots, w^n$ defined by the quasi-Miura transformation (3.3) the deformed hierarchy has the form

$$\frac{\partial w^\alpha}{\partial T^{\beta,q}} = K^\alpha_{\beta,q}(w; w_x) + \sum_{k\ge 1} \epsilon^k\, W^\alpha_{k;\beta,q}(w; w_x, w_{xx}, \dots) = \{w^\alpha, H_{\beta,q}\}_1. \tag{3.9}$$

It is characterized by the deformed bihamiltonian structure and the deformation of the Hamiltonians

$$\begin{aligned} H_{\beta,q} &= \frac{1}{2\pi}\int_0^{2\pi} \theta_{\beta,q+1}(v; v_x, \dots)dx = \frac{1}{2\pi}\int_0^{2\pi} \tilde\theta_{\beta,q+1}(w; w_x, \dots)dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\theta_{\beta,q+1}(w) + \sum_{k\ge 1} \epsilon^k\, \theta^{(k)}_{\beta,q+1}(w; w_x, \dots))dx. \end{aligned} \tag{3.10}$$

The hierarchy (3.9) is called a quasitrivial deformation of the genus zero hierarchy (2.21) if $W^\alpha_{k;\beta,q}$ are differential polynomials of the $x$-derivatives of $w^\gamma$. Moreover, if we can choose the densities $\tilde\theta_{\beta,q}$ for the deformed hierarchy (3.9) so that they are differential polynomials of the $x$-derivatives of $w^\gamma$ and satisfy the properties: 1) $\tilde\theta_{\beta,0} = w_\beta = \eta_{\beta\gamma} w^\gamma$, 2) the one form $\Omega = \sum \tilde\theta_{\beta,q} dt^{\beta,q}$ is closed, then for any solution of the deformed bihamiltonian hierarchy there also exists a tau-function, in this case we call that the quasitrivial deformation of the genus zero bihamiltonian structure inherits the tau-structure of the genus zero bihamiltonian hierarchy (2.21). The deformations of the genus zero bihamiltonian structure (2.12) that we are most interested in are the quasitrivial deformations which inherits the tau-structure due to their important applications in 2D TFT.

We have the following theorem [14]:

**Theorem 1.** *If a quasitrivial deformation (3.1) inherits the tau-structure of the genus zero bihamiltonian hierarchy (2.21), then it is equivalent to a quasitrivial deformation with quasi-Miura transformation of the form*

$$w^\alpha = v^\alpha + \sum_{k\ge 1} \varepsilon^k \frac{\partial^2 F_{\frac{k}{2}}}{\partial t^{1,0}\partial t^{\alpha,0}}\ , \tag{3.11}$$

*where the functions $F_{\frac{k}{2}}$ depend at most on the variables $v^\gamma, v^\gamma_x, \dots, \partial_x^{3\,[\frac{k}{2}]-2} v^\gamma$, $\gamma = 1, \dots, n$. Conversely, if a quasitrivial deformation (3.1) corresponds to a quasi-Miura transformation of the form (3.11), and $\frac{1}{2} + \mu_\beta + q \neq 0$ for $1 \le \beta \le n$, $q \ge 1$, then it inherits the tau-structure of the genus zero bihamiltonian hierarchy.*

From the above theorem we see that a genus $\frac{1}{2}$ quasitrivial deformation (3.1) that inherits the tau-structure of the genus zero bihamiltonian hierarchy is always trivial. Now let us consider the genus one quasitrivial deformations of the genus zero bihamiltonian structure for semisimple Frobenius manifold, we require that these deformations inherit the tau-structure of the genus zero bihiamiltonian hierarchy. Due to the above theorem we can assume that the corresponding quasi-Miura transformation have the form

$$w^\alpha = v^\alpha + \varepsilon^2 \frac{\partial^2}{\partial t^{1,0} \partial t^{\gamma,0}} F_1(v; v_x) + \mathcal{O}(\varepsilon^3) \ . \tag{3.12}$$

On a semisimple Frobenius manifold, we can choose local coordinates $u^1, \dots, u^n$ such that in the basis $\frac{\partial}{\partial u^i}$, $i = 1, \dots, n$ the multiplication on $T_x M$ takes the simple form

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i} \ . \tag{3.13}$$

Such local coordinates are called the canonical coordinates of the semisimple Frobenius manifold. X-Mozilla-Status: 0000

The following theorem is proved in [14]:

**Theorem 2.** *For any semisimple Frobenius manifold, a quasi-Miura transformation of the form (3.12) defines a genus one quasitrivial deformation of the genus zero bihamiltonian structure (2.12) iff*

$$F_1(v; v_x) = \sum_{i=1}^n a_i \log(u^i_x) + g(u) \ . \tag{3.14}$$

*Here $a_i$ are arbitrary constants, and $g(u)$ is any smooth function of $u^1, \dots, u^n$.*

Theorem 2 indicates that the space of the equivalent classes of the genus one quasitrivial transformations of (2.12) that inherit the tau-structure form a finite dimensional linear space, so the property of quasitriviality and the preservation of the tau-structure imposes a rather strong restriction on the deformations of the genus zero bihamiltonian structure.

An important special case of the above theorem was proved in [6], where the coefficients $a_i$ take the value

$$a_i = \frac{1}{24}, \quad i = 1, \dots, n \ . \tag{3.15}$$

In this case the function $F_1$ has the following expression:

$$F_1 = \sum_{i=1}^n a_i \log(u^i_x) + g(u) = \frac{1}{24} \log\det(c^\alpha_{\beta\gamma} v^\gamma_x) + g(u). \tag{3.16}$$

If $g(u)$ is taken to be the G-function, a special function defined in [6, 17], then (3.16) coincides with the formula of the genus one free energy for topological sigma models. For the Frobenius manifold given in Example 1 and Example 2 of the last section, the G-functions are given by $G = 0$ and $G = -\frac{v^2}{24}$ respectively.

The genus one deformed bihamiltonian structure under the quasi-Miura transformation (3.12) with $F_1$ defined by (3.16) was explicitly given in [6]. To present them in a concise form, let's denote by $Lie_X\{\ ,\ \}^{(0)}_i$, $i = 1, 2$ the infinitesimal deformatiom of the $i$-th genus zero Poisson bracket in (2.12) under the usual Miura type transformation

$$w^\alpha = v^\alpha + \varepsilon\, X^\alpha(v; v_x, \dots) + \mathcal{O}(\varepsilon^2) \ , \tag{3.17}$$

where $X^\alpha$ are differential polynomials. That means that under the above Miura type transformation the first genus zero Poisson bracket (2.12a), for example, is tranformed to the form

$$\{w^\alpha(x), w^\beta(y)\}_1 = \eta^{\alpha\beta}\,\delta'(x-y) + \varepsilon\, Lie_X\{w^\alpha(x), w^\beta(y)\}_1^{(0)} + \mathcal{O}(\varepsilon^2)\ . \tag{3.18}$$

Under this notation, we have

**Theorem 3.** *Let's define the vector fields $X$ and $Y$ by*

$$X^\alpha = \frac{1}{24}\partial_x^2(c_\nu^{\alpha\nu}), \quad Y^\alpha = A_\gamma^\alpha\, v_{xx}^\gamma + \frac{1}{2}B_{\gamma\nu}^\alpha\, v_x^\gamma\, v_x^\nu, \tag{3.19}$$

*where*

$$A_\gamma^\alpha = \frac{1}{48}c_{\gamma\nu}^\alpha c_\lambda^{\nu\lambda} + \frac{1}{24}c_{\gamma\lambda}^\nu\,\partial_\nu g^{\alpha\lambda}, \quad B_{\gamma\nu}^\alpha = \partial_\nu A_\gamma^\alpha + \partial_\gamma A_\nu^\alpha - \eta_{\nu\xi}\, p_\gamma^{\xi\alpha} \tag{3.20}$$

*with*

$$p_\gamma^{\alpha\beta} = \frac{1}{12}(\frac{1}{2}-\mu_\beta)\, c_{\nu\xi}^{\alpha\beta}\, c_\gamma^{\nu\xi} - \frac{1}{24}c_{\nu\xi}^{\alpha\xi}\,\Gamma_\gamma^{\nu\beta} + \frac{1}{24}\Gamma_\xi^{\alpha\beta}\, c_{\gamma\nu}^{\xi\nu}. \tag{3.21}$$

*and $c_{\nu\xi}^{\alpha\beta} = \frac{\partial c_\nu^{\alpha\beta}}{\partial v^\xi}$. Then, at the approximation up to $\epsilon^2$, the genus zero bihamiltonian structure (2.12) acquires the following deformation under the quasi-Miura transformation (3.12) with $F_1$ defined by (3.16) and $g(u)=0$:*

$$\{w^\alpha(x), w^\beta(y)\}_1 = \{w^\alpha(x), w^\beta(y)\}_1^{(0)} + \epsilon^2\, Lie_X\{w^\alpha(x), w^\beta(y)\}_1^{(0)} + \mathcal{O}(\varepsilon^4) \tag{3.22}$$

$$\begin{aligned}\{w^\alpha(x), w^\beta(y)\}_2 &= \{w^\alpha(x), w^\beta(y)\}_2^{(0)} + \epsilon^2\, Lie_Y\{w^\alpha(x), w^\beta(y)\}_1^{(0)} \\ &\quad + \epsilon^2\, Lie_X\{w^\alpha(x), w^\beta(y)\}_2^{(0)} + \mathcal{O}(\varepsilon^4)\ .\end{aligned} \tag{3.23}$$

It can be shown for semisimple Frobenius manifolds that a genus $\frac{3}{2}$ quasitrivial deformation (3.1) of the genus zero bihamiltonian structure (2.12) that inherits the tau-structure is equivalent to a genus one quasitrivial deformation. Let us consider now the genus two deformation. An immediate question then arises: whether it is possible to extend the genus one quasitrivial deformations of the genus zero bihamiltonian structure (2.12) given in Theorem 2 to genus two quasitrivial deformations? Below we present some evidences that support a positive answer to the above question at least for certain class of the genus one quasitrivial deformations given in Theorem 2.

We first consider the quasitrivial deformation of the genus zero bihamiltonian structure for the one dimensional Frobenius manifold given in (2.17). For this, let us define the following quasi-Miura transformation:

$$w = v + \frac{\partial^2}{\partial x^2}\left(\epsilon^2\, F_1 + \epsilon^4\, F_2\right) + \mathcal{O}(\epsilon^6)\ , \tag{3.24}$$

where we have denoted $v^1, w^1$ simplify by $v, w$, and the functions $F_1, F_2$ are given by

$$F_1 = \frac{1}{24}\,\log(v_x),$$

$$F_2 = \frac{1}{24}\left(\frac{v_{xx}^3}{15\,v_x^4} - \frac{7\,v_{xx}\,v^{(3)}}{80\,v_x^3} + \frac{v^{(4)}}{48\,v_x^2}\right).$$

Then under the transformation (3.24) the genus zero bihamiltonian structure (2.17) is transformed to

$$\{w^1(x), w^1(y)\}_1 = \delta'(x-y) + \mathcal{O}(\varepsilon^6)\ , \tag{3.25a}$$

$$\{w^1, w^1(y)\}_2 = w^1(x)\,\delta'(x-y) + \frac{1}{2}\,w_x^1\,\delta(x-y) + \frac{\varepsilon^2}{8}\,\delta'''(x-y) + \mathcal{O}(\varepsilon^6)\ . \tag{3.25b}$$

The functions $F_1, F_2$ correspond to the genus one and genus two free energy of the pure topological gravity. By the theory of Witten [7, 8] and Kontsevich [9] we know that the bihamiltonian hierarchy that controls the topological recursion relations of the pure topological gravity is the KdV hierarchy, it has the following bihamiltonian structure

$$\{w^1(x), w^1(y)\}_1 = \delta'(x-y)\ , \tag{3.26a}$$

$$\{w^1, w^1(y)\}_2 = w^1(x)\,\delta'(x-y) + \frac{1}{2}\,w_x^1\,\delta(x-y) + \frac{\varepsilon^2}{8}\,\delta'''(x-y)\ . \tag{3.26b}$$

It is proved in [14] that this bihamiltonian structure is in fact a quasitrivial deformation of the genus zero one (2.17).

Let us finally consider the genus two quasitrivial deformation of the genus zero bihamiltonian structures for two dimensional Frobenius manifolds. In [14] it was shown that for any generic two dimensional Frobenius manifold there exists a genus two quasitrivial deformation, the quasi-Miura transformation has the form

$$w^\alpha = v^\alpha + \eta^{\alpha\gamma}\,\frac{\partial^2}{\partial x \partial t^{\gamma,0}}\left(\varepsilon^2\,F_1 + \varepsilon^4\,F_2\right), \tag{3.27}$$

where $F_1$ is defined as in (3.16) with $g(u)$ being the G-function,

and $F_2$ is defined through the genus two Virasoro constraints. It was conjectured by Eguchi, Hori and Xiong [18] that the partition function $\exp(\sum_{g\ge 0}\varepsilon^{2g-2}\mathcal{F}_g)$ of a topological sigma model is annihilated by an infinte set of linear differential operators $L_m,\ m \ge -1$ of the coupling constants $t^{\alpha,p}$, i.e.,

$$L_m\,\exp(\sum_{g\ge 0}\varepsilon^{2g-2}\mathcal{F}_g) = \left(\sum_{g\ge 0}\mathcal{A}_{m,g}\,\varepsilon^{2g-2}\right)\exp(\sum_{g\ge 0}\varepsilon^{2g-2}\mathcal{F}_g) = 0. \tag{3.28}$$

The system of equations

$$\mathcal{A}_{m,g} = 0, \quad m \ge -1 \tag{3.29}$$

is called the genus $g$ Virasoro constraints, the equation $\mathcal{A}_{m,g} = 0$ only involves the functions $\mathcal{F}_0, \dots, \mathcal{F}_g$. In [19] these Virasoro constraints are generalized to any Frobenius manifold, and it was proved that the genus zero free energy $\mathcal{F}_0$ defined in the last section for any Frobenius manifold satisfies the genus zero Virasoro constraints; under the assumption that the Frobenius manifold is semisimple, the genus one free energy $\mathcal{F}_1$ was also proved to satisfy the genus one Virasoro constraints. The validity of the genus zero Virasoro constraints for the topological sigma models was proved in [20]. Under some restrictions on the quantum cohomology of the target space, the validity of the genus one Virasoro constraints for the topological sigma models was proved in [21, 22]. The function $F_2$ in the quasi-Miura transformation (3.27) is uniquely determined by the condition that the functions $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ defined by (3.5) satisfies the genus two Virasoro constraints [14].

To illustrate the genus two quasitrivial defromations of the genus zero bihamiltonian structures for two dimensional Frobenius manifolds, let us write down here the genus two quasitrivial deformation of the bihamiltonian structure (2.20) for the $CP^1$ topological sigma model that is given in [14]. Introduce the notations

$$V_{\alpha_1,\dots,\alpha_k} = \frac{\partial^{k-1} V_{\alpha_1}}{\partial t^{\alpha_2,0}\dots\partial t^{\alpha_k,0}}, \quad k=1,2,\dots\ , \tag{3.30}$$

$$(M^\alpha_\beta) = (c^\alpha_{\beta\gamma}\, v^\gamma_x), \quad (M^{-1})^{\alpha\beta} = (M^{-1})^\alpha_\gamma\, \eta^{\gamma\beta}\ , \tag{3.31}$$

and denote

$$\begin{aligned}
Q_1 &= V_{1,\alpha_1,\alpha_2,\alpha_3,\alpha_4}\,(M^{-1})^{\alpha_1\alpha_2}\,(M^{-1})^{\alpha_3\alpha_4}\\
Q_2 &= V_{1,\alpha_1,\alpha_2,\alpha_3}\,V_{\alpha_4,\alpha_5,\alpha_6}\,(M^{-1})^{\alpha_1\alpha_4}\,(M^{-1})^{\alpha_2\alpha_5}\,(M^{-1})^{\alpha_3\alpha_6},\\
Q_3 &= V_{1,\alpha_1,\alpha_2}\,V_{\alpha_3,\alpha_4,\alpha_5,\alpha_6}\,(M^{-1})^{\alpha_1\alpha_3}\,(M^{-1})^{\alpha_2\alpha_4}\,(M^{-1})^{\alpha_5\alpha_6},\\
Q_4 &= V_{1,\alpha_1,\alpha_2}\,V_{\alpha_3,\alpha_4,\alpha_5}\,V_{\alpha_6,\alpha_7,\alpha_8}\,(M^{-1})^{\alpha_1\alpha_3}\,(M^{-1})^{\alpha_2\alpha_6}\,(M^{-1})^{\alpha_4\alpha_7}\,(M^{-1})^{\alpha_5\alpha_8}.\\
Q_5 &= V_{1,\alpha_1,\alpha_2}\,\frac{\partial^2 G}{\partial t^{\alpha_3,0}\partial t^{\alpha_4,0}}\,(M^{-1})^{\alpha_1\alpha_3}\,(M^{-1})^{\alpha_2\alpha_4}\ ,\\
Q_6 &= \frac{\partial^3 G}{\partial x\partial t^{\alpha_1,0}\partial t^{\alpha_2,0}}\,(M^{-1})^{\alpha_1\alpha_2}\ ,
\end{aligned}$$

with

$$G = -\frac{1}{24}v^2\ ,$$

then

$$F_2 = \frac{1}{1152}\,Q_1 - \frac{1}{360}\,Q_2 - \frac{1}{1152}\,Q_3 + \frac{1}{360}\,Q_4 - \frac{11}{240}\,Q_5 + \frac{1}{20}\,Q_6 + \frac{7}{5760}\,v^2_{xx},$$

and the deformed bihamiltonian structure has the expression

$$\begin{aligned}
&\{w^1(x), w^1(y)\}_1 = \{w^2(x), w^2(y)\}_1 = 0\ ,\\
&\{w^1(x), w^2(y)\}_1 = \delta' - \varepsilon^2\,\frac{1}{12}\,\delta''' + \varepsilon^4\,\frac{1}{240}\,\delta^{(5)}\ ,\\
&\{w^1(x), w^1(y)\}_2 = 2\,e^{w^2(x)}\,\delta' + w^1_x\,e^{w^2(x)}\,\delta\\
&\quad +\varepsilon^2\,e^{w^2(x)}\left[\frac{1}{6}\,\delta''' + \frac{1}{4}\,w^2_x\,\delta''\right.\\
&\quad \left. + \left(\frac{1}{12}\,(w^2_x(x))^2 + \frac{1}{4}\,w^2_{xx}\right)\delta' + \left(\frac{1}{12}\,w^2_x\,w^2_{xx} + \frac{1}{12}\,w^2_{xxx}\right)\delta\right]\\
&\quad +\varepsilon^4\,e^{w^2(x)}\left[-\frac{1}{360}\,\delta^{(5)} - \frac{1}{144}\,w^2_x\,\delta^{(4)} + \frac{1}{180}\,w^2_{xx}\,\delta'''\right.
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{120}\,(w^2_x)^2\delta''' + \frac{11}{720}\,w^2_{xxx}\delta'' + \frac{1}{240}\,w^2_{xx}w^2_x\delta'' - \frac{1}{180}\,(w^2_x)^3\delta'' + \frac{1}{90}\,\partial_x^4 w^2\delta' \\
&+\frac{1}{120}\,w^2_{xxx}w^2_x\delta' + \frac{7}{720}\,(w^2_{xx})^2\delta' - \frac{1}{720}\,w^2_{xx}(w^2_x)^2\delta' - \frac{1}{720}\,(w^2_x)^4\delta' \\
&+\left(\frac{1}{288}\,w^2_x\,(w^2_{xx})^2 + \frac{1}{360}\,w^2_x\,\partial_x^4 w^2 + \frac{1}{144}\,w^2_{xx}\,w^2_{xxx} + \frac{1}{360}\,\partial_x^5 w^2\right)\delta\Big]\ , \\
&\{w^1(x), w^2(y)\}_2 = w^1(X)\,\delta' - \frac{\varepsilon^2}{12}\left(w^1(x)\,\delta''' + w^1_x\,\delta''\right) \\
&+\varepsilon^4\,\frac{1}{240}\,w^1(x)\,\delta^{(5)} + \frac{1}{120}\,w^1_x\,\delta^{(4)} + \frac{1}{180}\,w^1_{xx}\,\delta''' + \frac{1}{720}\,w^1_{xxx}\,\delta''\ , \\
&\{w^2(x), w^2(y)\}_2 = 2\,\delta' - \varepsilon^4\,\frac{1}{120}\,\delta^{(5)}\ .
\end{aligned}
$$

Here $\delta^{(k)} = \delta^{(k)}(x-y)$. In [15] it was shown that at the approximation up to $\varepsilon^4$ the above bihamiltonian structure coincides with that of the Toda lattice hierarchy, this suggests that the bihamiltonian structure of the Toda lattice hierarchy is a quasitrivial deformation of that of the dispersionless Toda lattice hierarchy (2.20), and that the bihamiltonian hierarchy that controls the $CP^1$ topological sigma model should be the Toda lattice hierarchy, as was conjectured in [1, 11, 12, 13].

## 4 Conclusion

We have considered an important class of deformations of the genus zero bihamiltonian structure defined on the loop space of a semisimple Frobenius manifold. Such deformations are obtained by quasi-Miura transformations of the form (3.11). They have close relations with the tau-structure and Virasoro symmetries of the genus zero bihamiltonian hierarchy. It is conjectured in [14] that there exists a unique quasitrivial deformation of the genus zero bihamiltonian structure (2.12), the corresponding deformed bihamiltonian hierarchy possesses the tau-structure and admits a unique tau-function that is invariant under the the Virasoro symmetries, i.e., $L_m\tau = 0$. This tau-function plays the role of the partition function in the setting of 2D TFT. In the case of one dimensional Frobenius maifold this quasitrivial deformation of the genus zero bihamiltonian structure (2.17) is the well known bihamiltonian structure of the KdV hierarchy, the tau-function that is invariant under the Virasoro symmetries is just the partition function of the pure topological gravity. A deep exploration of the of the relations between the tau-structure, the Virasoro constraints and the quasitrivial deformations of the genus zero bihamiltonian structure on the loop space of a semisimple Frobenius manifold was given in [14].

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