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Is My ODE a Painlevé Equation in Disguise?

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Abstract

Painlevé equations belong to the class $y'' + a_1\, {y'}^3 + 3a_2\, {y'}^2 + 3a_3\, y' + a_4 = 0$, where $a_i = a_i(x, y)$. This class of equations is invariant under the general point transformation $x = \Phi(X, Y)$, $y = \Psi(X, Y)$ and it is therefore very difficult to find out whether two equations in this class are related. We describe R. Liouville's theory of invariants that can be used to construct invariant characteristic expressions (syzygies), and in particular present such a characterization for Painlevé equations I-IV.

1 Introduction

Many phenomena in Nature are modeled by ordinary differential equations (ODEs). After such an equation is derived for some physical situation, the natural question is whether that ODE is well known, or at least transformable to a well known equation. For example, one would like to know if the equation is related to a known integrable equation, e.g, in the case of a second order ODE, to one of the Painlevé equations, or to one of the equations in the Gambier[1]/Ince[2] list.

Normally these lists contain only *representative* equations, e.g., up to some group of transformations. If the equation is of the form $y'' = f(x, y, y')$ the usually considered group of transformations is

$$x = \phi(X), \quad y = \frac{\psi_1(X)Y + \psi_2(X)}{\psi_3(X)Y + \psi_4(X)}.$$

However, when we have an equation derived from some physical problem the required transformation may be more complicated.

Here we consider the following class of equations

$$y'' + a_1(x, y)\, {y'}^3 + 3a_2(x, y)\, {y'}^2 + 3a_3(x, y)\, y' + a_4(x, y) = 0. \tag{1}$$

This class is invariant under a general point transformation

$$x = \Phi(X, Y), \quad y = \Psi(X, Y), \tag{2}$$

we only need to assume that the transformation is nonsingular, i.e.,

$$\Delta := \Phi_X \Psi_Y - \Phi_Y \Psi_X \neq 0. \tag{3}$$

The point transformation (2) prolongs to

$$\begin{aligned} y' &= \frac{\Psi_X + \Psi_Y Y'}{\Phi_X + \Phi_Y Y'}, &(4)\\ y'' &= \big\{[\Psi_Y \Phi_X - \Psi_X \Phi_Y] Y'' + [\Phi_Y \Psi_{YY} - \Psi_Y \Phi_{YY}] Y'^3 \\ &\quad + (\Phi_X \Psi_{YY} - 2\Phi_{XY}\Psi_Y + 2\Phi_Y \Psi_{XY} - \Phi_{YY}\Psi_X) Y'^2 \\ &\quad + (-\Phi_{XX}\Psi_Y - 2\Phi_{XY}\Psi_X + 2\Phi_X \Psi_{XY} + \Phi_Y \Psi_{XX}) Y' \\ &\quad + \Phi_X \Psi_{XX} - \Phi_{XX}\Psi_X \big\} / [\Phi_X + \Phi_Y Y']^3, &(5) \end{aligned}$$

and when these are substituted into (1) the form of the equation (cubic in y') stays the same, but the coefficients a_i change as follows:

$$\begin{aligned} \widetilde{a_1} &= \tfrac{1}{\Delta}(\Phi_Y \Psi_{YY} - \Phi_{YY}\Psi_Y + \Phi_Y^3 A_4 + 3\Phi_Y^2 \Psi_Y A_3 + 3\Phi_Y \Psi_Y^2 A_2 + \Psi_Y^3 A_1), &(6)\\ \widetilde{a_2} &= \tfrac{1}{\Delta}[\tfrac{1}{3}(\Phi_X \Psi_{YY} - 2\Phi_{XY}\Psi_Y + 2\Phi_Y \Psi_{XY} - \Phi_{YY}\Psi_X) + \Phi_X \Phi_Y^2 A_4 \\ &\quad + (2\Phi_X \Phi_Y \Psi_Y + \Phi_Y^2 \Psi_X) A_3 + (\Phi_X \Psi_Y^2 + 2\Phi_Y \Psi_X \Psi_Y) A_2 + \Psi_X \Psi_Y^2 A_1], &(7)\\ \widetilde{a_3} &= \tfrac{1}{\Delta}[\tfrac{1}{3}(-\Phi_{XX}\Psi_Y - 2\Phi_{XY}\Psi_X + 2\Phi_X \Psi_{XY} + \Phi_Y \Psi_{XX}) + \Phi_X^2 \Phi_Y A_4 \\ &\quad + (\Phi_X^2 \Psi_Y + 2\Phi_X \Phi_Y \Psi_X) A_3 + (2\Phi_X \Psi_X \Psi_Y + \Phi_Y \Psi_X^2) A_2 + \Psi_X^2 \Psi_Y A_1], &(8)\\ \widetilde{a_4} &= \tfrac{1}{\Delta}(-\Phi_{XX}\Psi_X + \Phi_X \Psi_{XX} + \Phi_X^3 A_4 + 3\Phi_X^2 \Psi_X A_3 + 3\Phi_X \Psi_X^2 A_2 + \Psi_X^3 A_1). &(9) \end{aligned}$$

Here $A_i(X,Y) := a_i(\Phi(X,Y), \Psi(X,Y))$. Example: If we apply the transformation $x = X + Y$, $y = XY$ to the equation $y'' = 0$ (in which $a_i = 0$), then we find $\widetilde{a_1} = \widetilde{a_4} = 0$, $\widetilde{a_2} = \widetilde{a_3} = \frac{2}{3}\frac{1}{X-Y}$, i.e., the equation becomes $Y'' + \frac{2}{X-Y}(Y'^2 + Y') = 0$.

Since the coefficients of equation (1) transform in such a complicated way it is in general difficult to find a characterization of (1) that is invariant under a general point transformation (2). This classical problem was basically solved more than 100 years ago, the fundamental works being those by Liouville [3] in 1889, Tresse [4] in 1894, with more modern formulations by Cartan [5] in 1924 and Thomsen [6] in 1930. Recent wave of interest on this classical problem started with [7]. [For the restricted problem with $x = \Phi(X)$ see, e.g., [8].]

2 Relative invariants, absolute invariants and syzygies

The invariants we are looking for must be constructed from the coefficients a_i and their various derivatives. The transformation rules for a_i were given in (6-9) and we are now looking for some combinations that transform in a much simpler way. Let us consider some expression

$$I[x,y] = I(a_1, \dots, a_4, \partial_x a_1, \dots, \partial_x a_1, \partial_y a_1, \dots, \partial_y a_4, \partial_x^2 a_1, \dots).$$

Since $a_i = a_i(x,y)$ this will be a function of x, y. Under the transformation (2) the ingredients a_i transform to $\widetilde{a}_i$ and the expression constructed from the transformed quantities in exactly the same way is

$$\widetilde{I}[X,Y] = I(\widetilde{a}_1, \dots, \widetilde{a}_4, \partial_X \widetilde{a}_1, \dots, \partial_X \widetilde{a}_1, \partial_Y \widetilde{a}_1, \dots, \partial_Y \widetilde{a}_4, \partial_X^2 \widetilde{a}_1, \dots).$$

This is now a function of X, Y and if it turns out that

$$\widetilde{I}[X, Y] = \Delta^n I[\Phi(X, Y), \Psi(X, Y)],$$

i.e., that I transforms, up to some overall factor, as by *substitution* then we say that I is a **relative invariant** of weight n. If furthermore $n = 0$ or $I = 0$ we say that I is an **absolute invariant**. Normally the weight of the relative invariant is indicated by a subscript. Later we will construct several sequences of relative invariants, e.g., i_{2n}, $n = 1, 2, 3, \ldots$, then using them we can construct a sequence of absolute invariants, in this case $j_{2n} := i_{2n}/i_2^n$.

Although the absolute invariants transform by a simple substitution under (2) they are normally complicated functions of x, y and therefore in practice useless for classification. What we need are *relationships between absolute invariants*, i.e., **syzygies**.

For example, it turns out later that for one particular equation $j_6 - 6j_4 + 4 = 0$. This relationship is invariant under the transformation (2) and therefore it is an invariant characterization for that equation. Using this result we can say with certainty that any equation that does *not* satisfy this relationship cannot be transformed to first equation by *any* point transformation. The reverse is not true: several equations may satisfy the same syzygy.

Note also that the syzygy polynomials can only have numerical coefficients, because parameter values appearing in the equation can be changed with the allowed transformations.

3 Construction of Invariants

The geometric ideas behind the analysis can be seen, e.g., as follows[6]. Consider a two-dimensional geometry with the infinitesimal arch length given by

$$ds^2 = g_{ik} du^i du^j,\ i, j \in \{1, 2\},$$

where g is the metric (a nonsingular matrix possibly depending on the coordinates u^i). The equation for geodesics in this space is given by

$$\frac{d^2u^i}{ds^2} + \Gamma^i_{kl} \frac{du^k}{ds} \frac{du^l}{ds} = 0,\ i = 1, 2, \tag{10}$$

where Γ^k_{ij} are the Christoffel symbols. Instead of arch length s we could use some other independent variable, for example $x := u^1(s)$. After substituting this and eliminating $\frac{d^2u^1}{ds^2}/(\frac{du^1}{ds})^2$ between the two equations (10) we get

$$y'' + (-\Gamma^1_{22})y'^3 + (\Gamma^1_{22} - 2\Gamma^1_{12})y'^2 + (-\Gamma^1_{11} + 2\Gamma^2_{12})y' + \Gamma^2_{11} = 0,$$

($' = \frac{d}{dx}$, $y = u^2$) which is of the type (1). The transformation (2) can now be seen as a change of coordinates in this space $(u^1, u^2) \equiv (x, y)$ and the problem is to find invariants in terms of the above combinations of Christoffel symbols. We will not give the full details here, just the basic necessary formulae. (For a detailed treatment see [6], but note that the notation there is different from the present one, in particular $1 \leftrightarrow 2$ for indices.) Another formulation using four-dimensional Riemannian metric is given in [9].

The construction of the relative and absolute invariants proceeds step by step as follows: First define

$$\begin{aligned}
\Pi_{22} &= \partial_x a_1 - \partial_y a_2 + 2(a_2^2 - a_1 a_3), && (11)\\
\Pi_{12} &= \partial_x a_2 - \partial_y a_3 + a_2 a_3 - a_1 a_4, && (12)\\
\Pi_{11} &= \partial_x a_3 - \partial_y a_4 + 2(a_3^2 - a_2 a_4), && (13)
\end{aligned}$$

but these are not yet components of a true tensor. Next define

$$\begin{aligned}
L_2 &= \partial_x \Pi_{22} - \partial_y \Pi_{12} - a_1 \Pi_{11} + 2a_2 \Pi_{12} - a_3 \Pi_{22}, && (14)\\
L_1 &= \partial_x \Pi_{12} - \partial_y \Pi_{11} - a_2 \Pi_{11} + 2a_3 \Pi_{12} - a_4 \Pi_{22}. && (15)
\end{aligned}$$

Now one finds that the L_i transform as a (pseudo)vector

$$\begin{pmatrix} \widetilde{L_1} \\ \widetilde{L_2} \end{pmatrix} = \Delta \begin{pmatrix} \Phi_X & \Psi_X \\ \Phi_Y & \Psi_Y \end{pmatrix} \begin{pmatrix} L_1[\Phi(X,Y), \Psi(X,Y)] \\ L_2[\Phi(X,Y), \Psi(X,Y)] \end{pmatrix}. \qquad (16)$$

From this we get the first result (R. Liouville):

The property $L_1 = L_2 = 0$ is an absolute invariant and if it holds the equation can be transformed to $Y'' = 0$.

If we define a_i and ∂_x, ∂_y to have weight $\frac{1}{2}$, then L_i are of weight $\frac{3}{2}$. Continuing with L_i and adding one derivative or a_i we find at weight $\frac{7}{2}$ another pair

$$\begin{aligned}
Z_1 &= -L_{1y} L_1 - 3L_{1x} L_2 + 4L_1 L_{2x} + 3a_2 L_1^2 - 6a_3 L_1 L_2 + 3a_4 L_2^2, && (17)\\
Z_2 &= \quad L_{2x} L_2 + 3L_{2y} L_1 - 4L_2 L_{1y} + 3a_1 L_1^2 - 6a_2 L_1 L_2 + 3a_3 L_2^2, && (18)
\end{aligned}$$

that transforms similarly:

$$\begin{pmatrix} \widetilde{Z_1} \\ \widetilde{Z_2} \end{pmatrix} = \Delta^3 \begin{pmatrix} \Phi_X & \Psi_X \\ \Phi_Y & \Psi_Y \end{pmatrix} \begin{pmatrix} Z_1[\Phi(X,Y), \Psi(X,Y)] \\ Z_2[\Phi(X,Y), \Psi(X,Y)] \end{pmatrix} \qquad (19)$$

Using these two we get the first semi-invariant

$$\begin{aligned}
\nu_5 &= \tfrac{1}{3}[Z_1 L_2 - Z_2 L_1] \\
&= L_2(L_1 \partial_x L_2 - L_2 \partial_x L_1) + L_1(L_2 \partial_y L_1 - L_1 \partial_y L_2) \\
&\quad - a_1 L_1^3 + 3a_2 L_1^2 L_2 - 3a_3 L_1 L_2^2 + a_4 L_2^3, && (20)
\end{aligned}$$

which is of weight 5, i.e., transforms as $\widetilde{\nu_5} = \Delta^5 \nu_5$.

Observation: *For all Painlevé equations $\nu_5 = 0$.*

Our ultimate aim is to provide an invariant characterization for the Gambier/Ince list of 50 equations. It contains some equations with $L_1 = L_2 = 0$, and other equations besides Painlevé's that have $\nu_5 = 0$, but it also contains many with $\nu_5 \neq 0$. [We note in passing, that in the standard form all Painlevé equations also have the properties $a_1 = 0$ and $L_2 = 0$, but these are not (semi)invariant characterizations.]

Let us now continue with $\nu_5 = 0$, i.e. $Z_1 L_2 = Z_2 L_1$. Let us define

$$R_1 = L_1 \partial_x L_2 - L_2 \partial_x L_1 + a_2 L_1^2 - 2a_3 L_1 L_2 + a_4 L_2^2, \quad (21)$$

$$w_1 = [L_1^3(\Pi_{11} L_2 - \Pi_{12} L_1) + R_1 \partial_x (L_1^2) - L_1^2 \partial_x R_1 + L_1 R_1 (a_3 L_1 - a_4 L_2)]/L_1^4. \quad (22)$$

[If $L_1 = 0$ there is a similar expression with L_2 as divider.] The expression w_1 is a semi-invariant of weight 1.

Observation: *For all Painlevé equations* $w_1 = 0$.

The Gambier/Ince list contains also equations with $\nu_5 = 0$, $w_1 \neq 0$.

We continue further with $\nu_5 = 0$, $w_1 = 0$. A sequence of semi-invariants can now be constructed starting with

$$i_2 = 2R_1/L_1 + \partial_x L_2 - \partial_y L_1, \quad (23)$$

with higher members given recursively by

$$i_{2(m+1)} = L_1\, \partial_y i_{2m} - L_2\, \partial_x i_{2m} + 2m\, i_{2m}(\partial_x L_2 - \partial_y L_1). \quad (24)$$

If $i_2 \neq 0$ a sequence of absolute invariants is given by

$$j_{2m} = i_{2m} i_2^{-m}. \quad (25)$$

In the next section we will use the j_{2m} to characterize Painlevé equations I-IV.

At this point we can see that the classification using j_{2m} cannot be sharp. Consider the special case $y'' + a_4(x, y) = 0$. Then $\Pi_{11} = -\partial_y a_4$, $\Pi_{12} = \Pi_{22} = 0, L_1 = \partial_y^2 a_4, L_2 = 0, \nu_5 = 0$, $w_1 = 0$, and

$$i_2 = -\partial_y^3 a_4, \quad i_{2(m+1)} = L_1^{2m+1} \partial_y (i_{2m}/L_1^{2m}),$$

The semi-invariants therefore depend only on $\partial_y^2 a_4$ and its higher y-derivatives and are insensitive to the possible linear in y part in a_4.

4 Invariant characterization of $P_I - P_{IV}$

The first steps were given above: All Painlevé equations have the properties 1) at least one of L_1, L_2 is nonzero, 2) $\nu_5 = 0$, 3) $w_1 = 0$.[1] If the candidate equation fails any of these properties it cannot be transformed to a Painlevé equation using the transformation (2). Here we want to go further and derive conditions which differentiate between Painlevé equations. The following results have been derived using the symbolic algebra language REDUCE[10]. It should be noted that we give here only the lowest degree syzygy.

4.1 Painlevé I

Any equation of the form

$$y'' = 6y^2 + f_1(x)y + f_0(x), \quad (26)$$

has the property $i_2 = 0$. In principle i_2 is only a relative invariant, but since its value in this case is 0, this property is an absolute invariant. Equation (26) contains Painlevé I for which $f_1 = 0$, $f_0 = x$. For all other Painlevé equations $i_2 \neq 0$.

[1] This was first observed by V. Dryuma in late 1980's

4.2 Painlevé II

All equations of form

$$y'' = 2y^3 + f_1(x)y + f_0(x), \tag{27}$$

have the property $i_2 = 12, i_4 = 288$ and therefore $j_4 = 2$ is the syzygy for this class of equations. [In fact it is easy to see that $j_{2(m+1)} = 2^m\, m!$] Painlevé II is contained as the special case $f_1 = x,\ f_0 =$const.

4.3 Painlevé III

The following results are valid for Cases 12,13,13_1 in the Ince/Gambier classification, they all have four parameters, $\alpha, \beta, \gamma, \delta$:

$$y'' - \frac{y'^2}{y} - \gamma y^3 - \alpha y^2 - \beta - \delta/y = 0,$$
$$y'' - \frac{y'^2}{y} + \frac{y'}{x} - \gamma y^3 - \frac{1}{x}(\alpha y^2 + \beta) - \delta/y = 0,$$
$$y'' - \frac{y'^2}{y} - e^x(\alpha y^2 + \beta) - e^{2x}(\gamma y^3 + \delta/y) = 0.$$

1: If any 3 of the parameters $\alpha, \beta, \gamma, \delta$ are zero, then $j_{2(m+1)} = m!$

2a: If $\gamma = \delta = 0$ or $\alpha = \beta = 0$ then

$$j_6 - 6j_4 + 4 = 0. \tag{28}$$

These two cases are connected: $x = \frac{1}{2}X^2, y = Y^2$ takes $(\alpha, \beta, 0, 0) \to (0, 0, \alpha, \beta)$.

2b: If $\alpha = \gamma = 0$ or $\beta = \delta = 0$ then

$$2j_8 - 13j_6 - 22 + 57j_4 - 21j_4^2 = 0.$$

Transformation $x = X, y = 1/Y$ takes $(0, \beta, 0, \delta) \to (-\beta, 0, -\delta, 0)$.

2c: If $\beta = \gamma = 0$ or $\alpha = \delta = 0$ then

$$2j_8 - 23j_6 - 42 + 87j_4 - 11j_4^2 = 0.$$

Transformation $x = X, y = 1/Y$ takes $(\alpha, 0, 0, \delta) \to (0, -\alpha, -\delta, 0)$.

3: If $\alpha^2\delta + \beta^2\gamma = 0$ (which contains 2a,2b) then we get a degree 12 relation

$$4j_{12} - 76j_{10} - 1696 + 6840j_4 - 2600j_6 - 5640j_4^2$$
$$+590j_8 + 2220j_4j_6 + 450j_4^3 - 155j_4j_8 - 115j_6^2 = 0.$$

4: For the generic case we obtain

$$4j_{14} - 112j_{12} - 231j_{10}j_4 + 1274j_{10} - 385j_8j_6 + 4795j_8j_4 - 7910j_8 + 3255j_6^2$$
$$+3570j_6j_4^2 - 39060j_6j_4 + 30240j_6 - 15330j_4^3 + 78120j_4^2 - 71736j_4 + 15264 = 0.$$

4.4 Painlevé IV

Let us again consider a more general form

$$y'' = \frac{1}{2y}y'^2 + e_1\frac{3}{2}y^3 + 4(e_2x + e_3)y^2 + 2(e_4x^2 + e_5x + e_6)y + e_7\frac{1}{y},$$

Painlevé IV corresponds to $e_1 = e_2 = e_4 = 1$, $e_3 = e_5 = 0$, $e_6 = -\alpha$, $e_7 = -\beta^2/2$. The classification is sensitive only to e_1, e_2, e_3, e_7.

1: If $e_1 = e_2 = e_3 = e_7 = 0$ then $L_1 = L_2 = 0$

2a: If $e_2 = e_3 = e_7 = 0$ then $j_{2(m+1)} = \left(\frac{4}{3}\right)^m m!$

2b: If $e_2 = e_3 = e_1 = 0$ then $j_{2(m+1)} = \left(\frac{4}{5}\right)^m m!$

2c: If $e_1 = e_7 = 0$ then $j_{2(m+1)} = 2^m m!$

3a: If $e_7 = 0$ then

$$j_8 - 11j_6 - 24 + 46j_4 - 7j_4^2 = 0.$$

3b: If $e_1 = 0$ then

$$5j_8 - 95j_6 - 136 + 286j_4 + 21j_4^2 = 0.$$

3c: if $e_2 = e_3 = 0$ then

$$8125j_{10} - 2165j_8 - 2495j_6j_4 + 13096j_6 + 12872j_4^2 - 36160j_4 + 11904 = 0.$$

4: Generic case:

$$3j_{14} + 1955j_{10} - 277j_{10}j_4 - 120j_{12} + 31488 - 131008j_4 + 57952j_6 + 103600j_4^2 \\ -15062j_8 - 40486j_4j_6 - 18772j_4^3 + 5236j_4j_8 + 990j_6^2 + 3775j_4^2j_6 - 75j_6j_8 = 0.$$

5 Conclusions

We have presented here the first results of a project aiming to derive syzygies for every equation in the Gambier/Ince list. The biggest problem in finding the characteristic expressions is that we have to solve huge sets of rather complicated nonlinear algebraic equations. For Painlevé V the generic expression is of degree higher than 26 (which is as far as we checked) and for Painlevé VI presumably still much higher. When this classification is eventually finished:

- We get an *algorithmic* method for finding out if a given equation of type (1) has any change of being transformed into one of the equations in the Gambier/Ince list.
- We also get a classification of the equations into essentially different sub-cases (c.f. [11, 12]).

The first results that have already been obtained suggest some interesting open questions:

- Is the type of "integrability" different in the classes a) $\nu_5 = w_1 = 0$, b) $\nu_5 = 0$, $w_1 \neq 0$, and c) $\nu_5 \neq 0$?
- What does the degree of the minimal syzygy tell us about integrability?

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