Journal of Nonlinear Mathematical Physics 2002, Volume 9, Supplement 1, 1–13 Proceedings: Hong Kong

From Bi-Hamiltonian Geometry to Separation of Variables: Stationary Harry-Dym and the KdV Dressing Chain

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Received March 14, 2001; Revised June 19, 2001; Accepted June 25, 2001

Abstract

Separability theory of one-Casimir Poisson pencils, written down in arbitrary coordinates, is presented. Separation of variables for stationary Harry-Dym and the KdV dressing chain illustrates the theory.

1 Introduction

The separation of variables is one of the most important methods of solving nonlinear ordinary differential equations of Hamiltonian type. It is known since 19*th* century, when Hamilton and Jacobi proved that given a set of appropriate coordinates, the so called separated coordinates, it is possible to solve a related Liouville integrable dynamical system by quadratures. Unfortunately in the 19*th* century and most of the 20*th* century, for a number of models of classical mechanics the separated variables were either guessed or found by some *ad hoc* methods. A fundamental progress in this field was made in 1985, when Sklyanin adopted the method of soliton systems, i.e. the Lax representation, to systematic construction of separated variables (see his review article [1]). In his approach, the appropriate Hamiltonians appear as coefficients of the spectral curve, i.e. the characteristic equation of the Lax matrix. Recently, a new constructive separability theory was presented, based on a bi-Hamiltonian property of integrable systems. In the frame of canonical coordinates the theory was developed in a series of papers [2]-[7] (see also the review article [8]), while the general case was considered in [9] and [10].

In this paper we briefly summarize the results of the theory in the case of one-Casimir Poisson pencils and illustrate it on two examples: the stationary flow of Harry-Dym (canonical coordinates frame) and the KdV dressing chain (noncanonical coordinates frame). This last system is separated for the first time. Finally, on the basis of examples, we make a few comments on the relation between a separation curve of the bi-Hamiltonian approach and a spectral curve of the Sklyanin approach.

[1]Supported partially by KBN research grant No. 5 P03B 004 20

2 Preliminary considerations

Let M be a differentiable manifold, TM and T^*M its tangent and cotangent bundle. At any point $u \in M$, the tangent and cotangent spaces are denoted by T_uM and T^*_uM, respectively. The pairing between them is given by the map $< \cdot, \cdot >: T^*_uM \times T_uM \to \mathbb{R}$. For each smooth function $F \in C^\infty(M)$, dF denotes the differential of F. M is said to be a Poisson manifold if it is endowed with a Poisson bracket $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$, in general degenerate. The related Poisson tensor π is defined by $\{F,G\}_\pi(u) :=< dG, \pi \circ dF > (u) =< dG(u), \pi(u)dF(u) >$. So, at each point u, $\pi(u)$ is a linear map $\pi(u) : T^*_uM \to T_uM$ which is skew-symmetric and has vanishing Schouten bracket with itself, i.e. the related bracket fulfils the Jacobi identity. Any function $c \in C^\infty(M)$, such that $dc \in \ker \pi$, is called a Casimir of π. Let π_0,$\pi_1 : T^*M \to TM$ be two Poisson tensors on M. A vector field K is said to be bi-Hamiltonian with respect to π_0 and π_1 if there exist two smooth functions H,$F \in C^\infty(M)$ such that

$$K = \pi_0 \circ dH = \pi_1 \circ dF. \tag{1}$$

The Poisson tensors π_0 and π_1 are said to be compatible if the associated pencil $\pi_\lambda = \pi_1 - \lambda\pi_0$ is itself a Poisson tensor for any $\lambda \in \mathbb{R}$.

In this paper we consider a particular Poisson manifold M of $\dim M = 2n+1$ equipped with a linear Poisson pencil π_λ of maximal rank. Assuming that a Casimir of the pencil is a polynomial in λ of an order n

$$h_\lambda = h_0\lambda^n + h_1\lambda^{n-1} + ... + h_n \tag{2}$$

one gets a bi-Hamiltonian chain

$$\pi_\lambda \circ dh_\lambda = 0 \Longleftrightarrow \begin{array}{r} \pi_0 \circ dh_0 = 0 \\ \pi_0 \circ dh_1 = K_1 = \pi_1 \circ dh_0 \\ \pi_0 \circ dh_2 = K_2 = \pi_1 \circ dh_1 \\ \vdots \\ \pi_0 \circ dh_n = K_n = \pi_1 \circ dh_{n-1} \\ 0 = \pi_1 \circ dh_n. \end{array} \tag{3}$$

where $\{h_i\}_{i=1}^n$ is a set of independent functions in involution with respect to both Poisson structures, so defines a Liouville integrable system on M.

When is the system separated? Let us introduce a set of canonical coordinates $\{\lambda_i, \mu_i\}_{i=1}^n$ and a Casimir coordinate $c = h_0$. Then, let us linearize the system through a canonical transformation $(\mu, \lambda) \to (a, b)$ in the form $b_i = \frac{\partial W}{\partial a_i}$, $\mu_i = \frac{\partial W}{\partial \lambda_i}$, where $W(\lambda, a)$ is a generating function satysfying the related Hamilton-Jacobi (HJ) equations

$$h_r(\lambda, \frac{\partial W}{\partial \lambda}) = a_r, \qquad r = 1, ..., n. \tag{4}$$

In general, HJ equations (4) are nonlinear partial differential equations and to solve them is a hopeless task. Nevertheless, one can find a complete integral in some special case, when in (μ, λ) coordinates a generating function W is additively separated:

$$W(\lambda, a) = \sum_{i=1}^n W_i(\lambda_i, a). \tag{5}$$

In such a case HJ equations turn into a set of decoupled ordinary differential equations and hence, at least in principle, can be solved by quadratures. Then, in (a, b) coordinates the flow is trivial

$$(a_j)_{t_r} = 0, \quad (b_j)_{t_r} = \delta_{jr} \tag{6}$$

and the implicit form of the trajectories $\lambda_i(t_r)$ is

$$b_j(\lambda, a) = \frac{\partial W}{\partial a_j} = \delta_{jr} t_r + const, \quad j = 1, ..., n. \tag{7}$$

Such (λ, μ) coordinates are called *separated coordinates.*

Lemma 1 *A sufficient condition for (λ, μ) to be separated coordinates for the bi-Hamiltonian chain (3) is*

$$h_{\lambda_i} = f_i(\lambda_i, \mu_i), \quad i = 1, ..., n, \tag{8}$$

where

$$h_{\lambda_i} = c\lambda_i^n + h_1\lambda_i^{n-1} + ... + h_n$$

and $f_i(\lambda_i, \mu_i)$ is an arbitrary smooth function of a pair of canonically conjugate coordinates.

Proof Using the following notation

$$h = (c, h_1, ..., h_n)^T, \quad v_i = (\lambda_i^n, \lambda_i^{n-1}, ..., \lambda_i^0 = 1), \tag{9}$$

$$v = (v_1, ..., v_n)^T, \quad f = (f_1, ..., f_n)^T, \tag{10}$$

the condition (8) can be presented in the matrix form

$$v \cdot h = f \Longleftrightarrow \begin{pmatrix} \lambda_1^n & \lambda_1^{n-1} & \cdots & 1 \\ \lambda_2^n & \lambda_2^{n-1} & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_n^n & \lambda_n^{n-1} & \cdots & 1 \end{pmatrix} \begin{pmatrix} c \\ h_1 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} f_1(\lambda_1, \mu_1) \\ f_2(\lambda_2, \mu_2) \\ \vdots \\ f_n(\lambda_n, \mu_n) \end{pmatrix} \tag{11}$$

which may be called a *generalized Stäckel representation.* Indeed,

$$f_i = (v \cdot h)_i = v_i \cdot h = h_{\lambda_i}. \tag{12}$$

Multiplying the HJ equations (4), written in the matrix form

$$h = a, \quad a = (c, a_1, ..., a_n)^T, \tag{13}$$

from the left by v_i one gets

$$v_i \cdot h = v_i \cdot a \Longrightarrow f_i(\lambda_i, \frac{\partial W}{\partial \lambda_i}) = c\lambda_i^n + a_1\lambda_i^{n-1} + ... + a_n \Longrightarrow W(\lambda, a) = \sum_{i=1}^n W_i(\lambda_i, a). \tag{14}$$

3 Separated coordinates

In ref. [2] the bi-Hamiltonian chain (3) in the separated coordinates (λ, μ, c) was constructed for the first time. Actually, the Hamiltonian functions h_k take the following compact form

$$h_k(\lambda, \mu, c) = -\sum_{i=1}^{n} \frac{\partial \rho_k(\lambda)}{\partial \lambda_i} \frac{f_i(\lambda_i, \mu_i)}{\Delta_i(\lambda)} + c\rho_k(\lambda), \quad k = 1, ..., n \tag{15}$$

where $\Delta_i(\lambda) := \prod_{j \neq i}(\lambda_i - \lambda_j)$, $\rho_k(\lambda)$ are the elementary symmetric polynomials and the two Poisson structures are

$$\pi_0 = \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 0 & \Lambda & h_{1,\mu} \\ -\Lambda & 0 & -h_{1,\lambda} \\ -\left(h_{1,\mu}\right)^T & \left(h_{1,\lambda}\right)^T & 0 \end{pmatrix}, \tag{16}$$

where $\Lambda = diag(\lambda_1, ..., \lambda_n)$ and $h_{1,\mu} := \left(\frac{\partial h_1}{\partial \mu_1}, ..., \frac{\partial h_1}{\partial \mu_n}\right)^T$. Notice that all h_k are linear in c. In fact there exists a family of separated coordinates (λ', μ', c) which preserve the form (15) and (16), and are related to the set (λ, μ, c) by a canonical transformation

$$\lambda_i' = \lambda_i, \quad \mu_i' = \mu_i + \vartheta_i(\lambda_i), \quad i = 1, ..., n, \tag{17}$$

where ϑ_i are arbitrary smooth function.

If $f_i = f,\ i = 1, ..., n$, then the separated coordinates are n different points of a curve

$$f(\lambda, \mu) = h_\lambda, \qquad h_\lambda = c\lambda^n + h_1\lambda^{n-1} + ... + h_n, \tag{18}$$

called the *separation curve.*

In the separated coordinates a Poisson pencil and the chain can be trivially projected onto a symplectic leaf S of π_0 $(\dim S = 2n)$ as $\theta_\lambda = \theta_1 - \lambda\theta_0$, where

$$\theta_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \theta_1 = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix}, \tag{19}$$

is a nondegenerate Poisson pencil on S. Hence, S is Poisson-Nijenhuis manifold where the related Nijenhuis tensor N

$$N = \theta_1 \circ \theta_0^{-1} = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \tag{20}$$

and its adjoint N^* are diagonal. This is the reason why (λ, μ) are called the Darboux-Nijenhuis (DN) coordinates. On S the chain (3),(15),(16) gives rise to

$$N^* \circ d\widehat{h}_i = d\widehat{h}_{i+1} - \rho_i d\widehat{h}_1, \quad i = 1, ..., n, \tag{21}$$

where $\widehat{}$ denotes the restriction to S and $h_{n+1} = 0$. Notice that the $\rho_i(\lambda)$ are the coefficients of the minimal polynomial of the Nijenhuis tensor:

$$(\det(\lambda I - N))^{1/2} = \sum_{i=0}^{n} \rho_i \lambda^{n-i} = \prod_{i=1}^{n}(\lambda - \lambda_i), \quad \rho_0 = 1. \tag{22}$$

There exists a sequence of separable "potentials" $V_k^{(r)}, r = \pm 1, \pm 2, ...,$ which can be added to $h_k(\lambda, \mu, c)$, given by the following recursion relation [5]

$$V_k^{(r+1)} = V_{k+1}^{(r)} - V_k^{(1)} V_1^{(r)}, \quad V_k^{(1)} = \rho_k, \quad r = 1, 2, ..., \tag{23}$$

and its inverse

$$V_k^{(-r-1)} = V_{k-1}^{(-r)} - V_k^{(-1)} V_n^{(-r)}, \quad V_k^{(-1)} = \rho_{k-1}/\rho_n, \quad r = 1, 2, ... \tag{24}$$

Notice that recursion formulae are coordinate free and generate separable potentials starting from the coefficients of the minimal polynomial of the Nijenhuis tensor in arbitrary set of coordinates. Potentials $V^{(r)}$ (23) and $V^{(-s)}$ (24) entrance the separation curve in the following way

$$f(\lambda, \mu) = \lambda^{n+r-1} + c\lambda^n + h_1\lambda^{n-1} + \cdots + h_n + \lambda^{-s}. \tag{25}$$

4 Canonical coordinates

Now, let us consider an arbitrary canonical transformation on M

$$(\lambda, \mu) \to (q, p) \tag{26}$$

independent of a Casimir coordinate c (not necessarily a point transformation!). The advantage of staying inside such a class of transformations is that the clear structure of the pencil is preserved.

Applying the transformation (26) to Hamiltonian functions (15) and Poisson matrices (16) one finds that

$$h_k(q, p, c) = h_k(q, p) + c b_k(q, p), \;\; k = 1, ..., n \tag{27}$$

and

$$\begin{aligned} \pi_0 &= \begin{pmatrix} \theta_0 & 0 \\ 0 & 0 \end{pmatrix}, \; \theta_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \\ \pi_1 &= \begin{pmatrix} \theta_1 & K_1 \\ -K_1^T & 0 \end{pmatrix}, \; \theta_1 = \begin{pmatrix} D(q,p) & A(q,p) \\ -A^T(q,p) & B(q,p) \end{pmatrix}, \end{aligned} \tag{28}$$

where A, B and D are $n \times n$ matrices. The nondegenerate Poisson pencil θ_λ on S gives rise to the related Nijenhuis tensor N and its adjoint N^* in (q, p) coordinates in the form

$$N = \theta_1 \circ \theta_0^{-1} = \begin{pmatrix} A & -D \\ B & A^T \end{pmatrix}, \; N^* = \theta_0^{-1} \circ \theta_1 = \begin{pmatrix} A^T & -B \\ D & A \end{pmatrix}. \tag{29}$$

Obviously, in a real situation we start from a given bi-Hamiltonian chain (27)-(29) in canonical coordinates (q, p, c), derived by some method, and we try to find the DN coordinates which diagonalize the appropriate Nijenhuis tensor and are separated coordinates for the system considered. So now we pass to a systematic derivation of the inverse of transformation (26).

The first part of the transformation is given by

$$\rho_i(\lambda) = b_i(q,p), \quad i = 1, ..., n. \tag{30}$$

The second part can be found in a few ways. One method was presented in refs. [8] and [9]. Here we present another method suggested in [10]. Consider the vector field $Y = \pi_0 \circ db_1(q,p)$. In DN coordinates it has the following form

$$Y = \sum_{i=1}^{n} \frac{\partial}{\partial \mu_i} = \pi_0 \circ d\rho_1(\lambda), \quad \rho_1 = -\sum_{i=1}^{n} \lambda_i. \tag{31}$$

Hence, if some φ_i depends only on a pair of coordinates (λ_i, μ_i), then $Y(\varphi_i)$ also depends only on (λ_i, μ_i). So, when for some $\overline{\varphi}_i$ we have $Y(\overline{\varphi}_i) = 1$, then according to the gauge (17) it means that $\mu_i = \overline{\varphi}_i$ are an admissible DN momenta and the second part of transformation is given by

$$\mu_i = \overline{\varphi}_i(q,p), \quad i = 1, ..., n. \tag{32}$$

For this procedure our two basic objects, written in (q,p) coordinates, are

$$Y = \pi_0 \circ db_1(q,p), \quad \varphi_i = h_{\lambda_i}(q,p). \tag{33}$$

More details will be given in examples.

In a special case of a point transformation, when $b_i = b_i(q),\ i = 1, ..., n$, the first part of the transformation is of the form

$$\rho_i(\lambda) = b_i(q) \Longrightarrow q_i = \alpha_i(\lambda), \quad i = 1, ..., n \tag{34}$$

and the second part can be constructed from a generating function $G(p,\lambda) = \sum_i p_i \alpha_i(\lambda)$ in the following way

$$\mu_i = \frac{\partial G}{\partial \lambda_i} \Longrightarrow p_i = \beta_i(\lambda, \mu), \quad i = 1, ..., n. \tag{35}$$

5 Noncanonical coordinates: a general case

When the Poisson chain (3) is given in an arbitrary coordinate system $\{g_i\}_{i=1}^{2n+1}$ a clear structure of a pencil is lost and it is far from obvious whether it is projectable onto a symplectic leaf S of the first Poisson structure or not. On the other hand, such a projectibility is a necessary condition for separability of the chain, as DN coordinates are these which diagonalize an appropriate Nijenhuis tensor on S constructed from a nondegenerated Poisson pencil on S. Here we adopt a method proposed in [9], [10] to sketch the simplest case of one-Casimir Poisson pencils.

Let a vector field Z be transversal to the symplectic foliation S of π_0. Consider the class of functions $\mathcal{F}(M)$ such that

$$\mathcal{L}_Z F = Z(F) = 0, \quad \forall F \in \mathcal{F}(M), \tag{36}$$

where $\mathcal{L}$ means a Lie derivative. We can identify $\mathcal{F}$ with all functions on some leaf S_0, as for an arbitrary $f \in \mathcal{F}(S_0)$ one can define its extension F on M such that (36) is fulfilled. Hence

$$\mathcal{F}(S_0) \ni f = F_{|S_0}. \tag{37}$$

We are looking for the condition on π_λ such that

$$\forall F, G \in A \; : \; \{F, G\}_{\pi_\lambda} \in A. \tag{38}$$

Then, θ_λ defined as

$$\{f, g\}_{\theta_\lambda} := \{F, G\}_{\pi_\lambda | S_0} \tag{39}$$

is a projection of π_λ along Z on S_0.

Theorem 2 *A sufficient condition for the projectability of π_λ onto S_0 is*

$$\mathcal{L}_Z \pi_0 = 0, \quad \mathcal{L}_Z \pi_1 = Y \wedge Z, \tag{40}$$

where Y is some vector field.

Proof Let π be a Poisson tensor. Then $\forall F, G \in A$:

$$\begin{aligned} \mathcal{L}_Z \{F, G\}_\pi &= \mathcal{L}_Z < dG, \pi \circ dF > \\ &= < (\mathcal{L}_Z dG), \pi \circ dF > + < dG, (\mathcal{L}_Z \pi) \circ dF + \pi \circ (\mathcal{L}_Z dF) > \\ &= < d(\mathcal{L}_Z G), \pi \circ dF > + < dG, (\mathcal{L}_Z \pi) \circ dF + \pi \circ d(\mathcal{L}_Z F) > \\ &= < dG, (\mathcal{L}_Z \pi) \circ dF > . \end{aligned} \tag{41}$$

For $\pi = \pi_0$ under the condition (40) we have immediately $\mathcal{L}_Z \{F, G\}_{\pi_0} = 0$. For $\pi = \pi_1$ the condition (40) gives

$$\begin{aligned} \mathcal{L}_Z \{F, G\}_{\pi_1} &= < dG, (\mathcal{L}_Z \pi_1) \circ dF >=< dG, (Y \wedge Z) \circ dF > \\ &= < dG, (Y \otimes Z - Z \otimes Y) \circ dF > \\ &= (Y \otimes Z \circ dF)G - (Z \otimes Y \circ dF)G \\ &= Y(G) \cdot Z(F) - Z(G) \cdot Y(F) = 0, \end{aligned} \tag{42}$$

so $\mathcal{L}_Z \{F, G\}_{\pi_\lambda} = 0$ and the relation (38) is fulfilled.

Moreover, the following theorem can be proved.

Theorem 3 *Let a Poisson pencil be projectable in the sense of Theorem 2. If additionally*

$$\mathcal{L}_Z (\mathcal{L}_Z h_\lambda) = Z(Z(h_\lambda)) = 0 \tag{43}$$

and vector fields Z and Y are normalized in such a way that

$$\mathcal{L}_Z h_0 = Z(h_0) = 1, \quad \mathcal{L}_Y h_0 = Y(h_0) = 0, \tag{44}$$

then

(i) *h_λ is linear in a Casimir of h_0,*

(ii) $Y = \pi_0 \circ d(Z(h_1))$,

(iii) *on S_0 the chain (3) takes the form*

$$N^* \circ d\widehat{h}_i = d\widehat{h}_{i+1} - Z(h_i) d\widehat{h}_1, \;\; i = 1, ..., n. \tag{45}$$

Notice that this is a separating case from previous Sections, where now

$$\rho_i(\lambda) = Z(h_i) \tag{46}$$

is the first part of a transformation to the DN coordinates on S_0. In the language of the present Section, for arbitrary canonical coordinates considered in the previous Sections, we have

$$Z = \frac{\partial}{\partial c},\ \mathcal{L}_Z\pi_0 = 0,\ \ \mathcal{L}_Z\pi_1 = Y \wedge Z,\ \ Y = \pi_0 \circ d(Z(h_1)) = \pi_0 \circ d\frac{\partial h_1}{\partial c},$$

$$Z(Z(h_\lambda)) = 0,\ \ Z(h_0) = 1,\ \ Y(h_0) = 0. \tag{47}$$

6 Stationary flow of Harry-Dym

Here we consider the following Newton equations of motion

$$q_{1xx} = 8q_1^{-5}q_2 + \alpha q_1, \quad q_{2xx} = -2q_1^{-4} + 4\alpha q_2 - c, \qquad \alpha = const, \tag{48}$$

with x as an evolution parameter, which are the second stationary flow of the Harry-Dym hierarchy [12], [4]. The appropriate bi-Hamiltonian chain is the following

$$\begin{aligned}
h_0 &= c,\\
h_1 &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + 2q_1^{-4}q_2 - \frac{1}{2}\alpha q_1^2 - 2\alpha q_2^2 + q_2 c,\\
h_2 &= \frac{1}{2}q_2 p_1^2 - \frac{1}{2}q_1 p_1 p_2 + \frac{1}{2}q_1^{-2} + 2q_1^{-4}q_2^2 + \frac{1}{2}\alpha q_1^2 q_2 - \frac{1}{4}q_1^2 c,
\end{aligned}$$

$$\begin{aligned}
\pi_0 &= \begin{pmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\pi_1 &= \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}q_1 & p_1 \\ 0 & 0 & -\frac{1}{2}q_1 & -q_2 & p_2 \\ 0 & \frac{1}{2}q_1 & 0 & \frac{1}{2}p_1 & -8q_1^{-5}q_2 - \alpha q_1 \\ \frac{1}{2}q_1 & q_2 & -\frac{1}{2}p_1 & 0 & 2q_1^{-4} - 4\alpha q_2 + c \\ -p_1 & -p_2 & 8q_1^{-5}q_2 + \alpha q_1 & -2q_1^{-4} + 4\alpha q_2 - c & 0 \end{pmatrix},
\end{aligned} \tag{49}$$

where $p_1 = q_{1x}, p_2 = q_{2x}$. This is the case of canonical coordinates of Section 4 and the first part of the transformation (30) to DN coordinates is

$$\begin{aligned}
&\rho_1 = -\lambda_1 - \lambda_2 = q_2, \quad \rho_2 = \lambda_1\lambda_2 = -\frac{1}{4}q_1^2 \\
&\qquad\qquad \Downarrow \\
&q_1 = 2\sqrt{-\lambda_1\lambda_2},\ \ q_2 = -\lambda_1 - \lambda_2.
\end{aligned} \tag{50}$$

Evidently this is a point transformation, so the second part of the transformation can be constructed either through a generating function (35) or by a general approach presented in Section 3. Because the first method is standard we apply here the second one. As $Y = \pi_0 \circ d(q_2) = -\frac{\partial}{\partial p_2}$, we have

$$Y(h_\lambda) = -p_2\lambda + \frac{1}{2}q_1p_1,\ Y(Y(h_\lambda)) = Y^2(h_\lambda) = \lambda \Longrightarrow Y\left(\frac{Y(h_\lambda)}{Y^2(h_\lambda)}\right) = 1. \tag{51}$$

It means that

$$\mu_1 = \frac{Y(h_{\lambda_1})}{Y^2(h_{\lambda_1})} = -p_2 + \frac{1}{2}q_1p_1\frac{1}{\lambda_1},\ \mu_2 = \frac{Y(h_{\lambda_2})}{Y^2(h_{\lambda_2})} = -p_2 + \frac{1}{2}q_1p_1\frac{1}{\lambda_2} \tag{52}$$

and hence

$$p_1 = \sqrt{-\lambda_1\lambda_2}\left(\frac{\mu_1}{\Delta_1} + \frac{\mu_2}{\Delta_2}\right),\ p_2 = -\lambda_1\frac{\mu_1}{\Delta_1} - \lambda_2\frac{\mu_2}{\Delta_2},\quad \Delta_1 = -\Delta_2 = \lambda_1 - \lambda_2, \tag{53}$$

$$f(\lambda_i, \mu_i) = \frac{1}{2}\lambda_i\mu_i^2 - 2\alpha\lambda_i^3 + \frac{1}{8}\lambda_i^{-2}.$$

The separation curve takes the form

$$\frac{1}{2}\lambda\mu^2 - 2\alpha\lambda^3 + \frac{1}{8}\lambda^{-2} = c\lambda^2 + h_1\lambda + h_2. \tag{54}$$

Let us now relate the presented approach to the Sklyanin one. It is known that Liouville integrable systems can be put into the Lax form [13]

$$L_x + [L, U] = 0, \tag{55}$$

where L, U are some matrices, $[.,.]$ means the commutator and x is an evolution parameter. In the simplest case, when L is 2×2 traceless matrix

$$L = \begin{pmatrix} A(\lambda; q, p) & B(\lambda; q, p) \\ C(\lambda; q, p) & -A(\lambda; q, p) \end{pmatrix}, \tag{56}$$

i.e. in the case of the so-called Mumford systems [14], $\lambda_i, i = 1, ..., n$ are roots of $B = 0$ and $\mu_i = -A(\lambda_i; q, p), i = 1, ..., n$. The separated coordinates are different points of the *spectral curve* $\det(L - \mu I) = 0$.

A Lax pair for the stationary Harry Dym was found in [12] in the form

$$\begin{aligned} L &= \begin{pmatrix} -q_1p_1\lambda^2 + 2p_2\lambda & q_1^2\lambda^2 - 4q_2\lambda - 4 \\ -q_1^{-2}\lambda^3 - (4q_1^{-4}q_2 + p_1^2)\lambda^2 + (4\alpha q_2 - 2c)\lambda - 4\alpha & q_1p_1\lambda^2 - 2p_2\lambda \end{pmatrix}, \\ U &= \begin{pmatrix} 0 & 1 \\ -4q_1^{-4}\lambda + \alpha & 0 \end{pmatrix}. \end{aligned} \tag{57}$$

On the other hand, it is well known that the Lax representation is not unique and some admissible representation can be obtained for example via the transformation: $\lambda \to \lambda^{-1}, L(\lambda^{-1}) \to \frac{1}{2}\lambda L(\lambda^{-1})$:

$$\begin{aligned} L &= \frac{1}{2}\begin{pmatrix} -q_1p_1\lambda^{-1} + 2p_2 & q_1^2\lambda^{-1} - 4q_2 - 4\lambda \\ -q_1^{-2}\lambda^{-2} - (4q_1^{-4}q_2 + p_1^2)\lambda^{-1} + 8\alpha q_2 - 2c - 4\alpha\lambda & q_1p_1\lambda^{-1} - 2p_2 \end{pmatrix}, \\ U &= \begin{pmatrix} 0 & 1 \\ -4q_1^{-4}\lambda^{-1} + \alpha & 0 \end{pmatrix}. \end{aligned} \tag{58}$$

For this Lax representation the roots of $B(q,p;\lambda) = \frac{1}{2}q_1^2\lambda^{-1} - 2q_2 - 2\lambda = 0$ and $\mu_i = -A(q,p;\lambda_i) = \frac{1}{2}q_1p_1\lambda^{-1} - p_2$ are just the same separated coordinates (50), (52) as in the bi-Hamiltonian approach and moreover

$$\det(L-\mu I) = 0 \iff \frac{1}{2}\lambda\mu^2 - 2\alpha\lambda^3 + \frac{1}{8}\lambda^{-2} = c\lambda^2 + h_1\lambda + h_2. \tag{59}$$

Hence, in the case of stationary Harry Dym,

$$\textit{separation curve} \;=\; \textit{spectral curve}. \tag{60}$$

7 The KdV dressing chain

Consider the so-called dressing chain

$$(v_n + v_{n+1})_x = v_n^2 - v_{n+1}^2 + \alpha_n \tag{61}$$

for the Schrödinger equation

$$\Psi_{xx} = (u-\lambda)\Psi. \tag{62}$$

Actually, if u_n is a sequence of solutions of (62), generated by a chain of Darboux transformations, and $u_n = v_{nx} + v_n^2 + \beta_n$, is a Miura map to the modified fields v_n, then, as was shown in [15], the new fields v_n are related among themselves through the chain of equations (61), where $\alpha_n = \beta_n - \beta_{n+1}$. Let us close the chain (61)

$$v_i \equiv v_{i+N}, \quad \alpha_i \equiv \alpha_{i+N}, \tag{63}$$

and assume that $\sum_{i=1}^N \alpha_i = 0$, then we obtain a finite dimensional dynamical system

$$(v_i + v_{i+1})_x = v_i^2 - v_{i+1}^2 + \beta_i - \beta_{i+1}, \quad i = 1, ..., N. \tag{64}$$

As was shown by Veselov and Shabat [16], for $N = 2n+1$, it is a bi-Hamiltonian system for which the bi-Hamiltonian chain (3) can be constructed. In $g_i = v_i + v_{i+1}$ coordinates the nonzero matrix elements of both Poisson tensors are the following

$$\begin{aligned} \{g_i, g_{i-1}\}_{\pi_0} &= 1, \\ \{g_i, g_j\}_{\pi_1} &= (-1)^{j-i} g_i g_j, \quad j \neq i \pm 1, \\ \{g_i, g_{i-1}\}_{\pi_1} &= g_i g_{i-1} + \beta_i, \end{aligned} \tag{65}$$

and the Casimir of the pencil $\pi_1 - \lambda\pi_0$ is given by

$$\begin{aligned} h_\lambda &= h_0\lambda^n + h_1\lambda^{n-1} + ... + h_n \\ &= (-1)^N \left[\prod_{j=1}^N \left(1 + \zeta_{j+1}\frac{\partial^2}{\partial g_i \partial g_j}\right)\right] \prod_{k=1}^N g_k, \qquad \zeta_i = \beta_i - \lambda. \end{aligned} \tag{66}$$

As $g_i, i = 1, ..., N$, are noncanonical coordinates, to separate the system we have to apply the general formalism of Section 5. For $Z = \frac{\partial}{\partial g_N}$ one finds

$$\mathcal{L}_Z\pi_0 = 0, \quad \mathcal{L}_Z\pi_1 = Z \wedge Y,$$

$$Y = \sum_{i=1}^{N-1} (-1)^{i+1} g_i \frac{\partial}{\partial g_i} + \left(\sum_{i=1}^{N-1} (-1)^i g_i \right) \frac{\partial}{\partial g_N} = \pi_0 \circ d(Z(h_1)) \tag{67}$$

$$Z(Z(h_\lambda)) = 0, \quad Z(h_0) = 1, \quad Y(h_0) = 0.$$

The first part of the transformation to the DN coordinates is

$$c = h_0, \quad \rho_k(\lambda) = \frac{\partial h_k}{\partial g_N}, \quad k = 1, ..., n. \tag{68}$$

The following property of the Casimir h_λ

$$Y^3(h_\lambda) = Y(h_\lambda) \implies Y(\ln(Y(h_\lambda) + Y^2(h_\lambda)) = 1 \tag{69}$$

gives the second part of the transformation

$$\mu_i = \ln(Y(h_{\lambda_i}) + Y^2(h_{\lambda_i})), \quad i = 1, ..., n. \tag{70}$$

Let us illustrate the method in the case $N = 5$. We have

$$\begin{aligned} h_0 &= g_1 + g_2 + g_3 + g_4 + g_5, \\ h_1 &= -g_1g_2g_3 - g_2g_3g_4 - g_3g_4g_5 - g_4g_5g_1 - g_{51}g_1g_2 - g_1(\beta_3 + \beta_5) \\ &\quad -g_2(\beta_4 + \beta_1) - g_3(\beta_5 + \beta_2) - g_4(\beta_1 + \beta_3) - g_5(\beta_2 + \beta_4), \\ h_2 &= g_1g_2g_3g_4g_5 + \beta_1g_2g_3g_4 + \beta_2g_3g_4g_5 + \beta_3g_4g_5g_1 + \beta_4g_5g_1g_2 \\ &\quad +\beta_5g_1g_2g_3 + \beta_3\beta_5g_1 + \beta_1\beta_4g_2 + \beta_2\beta_5g_3 + \beta_1\beta_3g_4 + \beta_2\beta_4g_5, \end{aligned} \tag{71}$$

$$\pi_0 = \begin{pmatrix} 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{72}$$

$$\pi_1 = \begin{pmatrix} 0 & -g_1g_2 - \beta_2 & g_1g_3 & -g_1g_4 & g_1g_5 + \beta_1 \\ g_2g_1 + \beta_2 & 0 & -g_2g_3 - \beta_3 & g_2g_4 & -g_2g_5 \\ -g_3g_1 & g_3g_2 + \beta_3 & 0 & -g_3g_4 - \beta_4 & g_3g_5 \\ g_4g_1 & -g_4g_2 & g_4g_3 + \beta_4 & 0 & -g_4g_5 - \beta_5 \\ -g_5g_1 - \beta_1 & g_5g_2 & -g_5g_3 & g_5g_4 + \beta_5 & 0 \end{pmatrix}.$$

The transformation to DN coordinates is

$$\begin{aligned} g_1 &= \frac{1}{2} \frac{(\lambda_2 - \lambda_1)e^{\mu_1+\mu_2}}{(\lambda_2 - \beta_3)(\lambda_2 - \beta_4)(\lambda_2 - \beta_5)e^{\mu_1} - (\lambda_1 - \beta_3)(\lambda_1 - \beta_4)(\lambda_1 - \beta_5)e^{\mu_2}}, \\ g_2 &= 2\frac{[(\lambda_2 - \beta_2)(\lambda_2 - \beta_4)(\lambda_2 - \beta_5)e^{\mu_1} - (\lambda_1 - \beta_2)(\lambda_1 - \beta_4)(\lambda_1 - \beta_5)e^{\mu_2}]}{(\lambda_2 - \lambda_1)[(\lambda_2 - \beta_4)(\lambda_2 - \beta_5)e^{\mu_1} - (\lambda_1 - \beta_4)(\lambda_1 - \beta_5)e^{\mu_2}]e^{\mu_1+\mu_2}} \\ &\quad \times[(\lambda_2 - \beta_3)(\lambda_2 - \beta_4)(\lambda_2 - \beta_5)e^{\mu_1} - (\lambda_1 - \beta_3)(\lambda_1 - \beta_4)(\lambda_1 - \beta_5)e^{\mu_2}], \\ g_3 &= -\frac{1}{2} \frac{[(\lambda_2 - \beta_3)(\lambda_2 - \beta_5)e^{\mu_1} - (\lambda_1 - \beta_3)(\lambda_1 - \beta_5)e^{\mu_2}]}{[(\lambda_2 - \beta_3)(\lambda_2 - \beta_4)(\lambda_2 - \beta_5)e^{\mu_1} - (\lambda_1 - \beta_3)(\lambda_1 - \beta_4)(\lambda_1 - \beta_5)e^{\mu_2}]} \end{aligned} \tag{73}$$

$$\times \frac{[(\lambda_2-\beta_4)(\lambda_2-\beta_5)e^{\mu_1}-(\lambda_1-\beta_4)(\lambda_1-\beta_5)e^{\mu_2}]}{(\lambda_2-\lambda_1)(\lambda_1-\beta_5)(\lambda_2-\beta_5)},$$

$$g_4 = -2\frac{(\lambda_2-\lambda_1)(\lambda_1-\beta_4)(\lambda_1-\beta_5)(\lambda_2-\beta_4)(\lambda_2-\beta_5)}{(\lambda_2-\beta_4)(\lambda_2-\beta_5)e^{\mu_1}-(\lambda_1-\beta_4)(\lambda_1-\beta_5)e^{\mu_2}},$$

$$g_5 = c-g_1-g_2-g_3-g_4.$$

The appropriate function f takes the form

$$f(\lambda_i,\mu_i)=2(\lambda_i-\beta_1)...(\lambda_i-\beta_5)e^{-\mu_i}+\frac{1}{2}e^{\mu_i} \tag{74}$$

and the separation curve is

$$2(\lambda-\beta_1)...(\lambda-\beta_5)e^{-\mu}+\frac{1}{2}e^{\mu}=c\lambda^2+h_1\lambda+h_2. \tag{75}$$

The case $N=5$ suggests the general form of a separation curve for an arbitrary odd N

$$2(\lambda-\beta_1)...(\lambda-\beta_N)e^{-\mu}+\frac{1}{2}e^{\mu}=c\lambda^n+h_1\lambda^{n-1}+...+h_n. \tag{76}$$

The implicit form of the trajectories $\lambda_i(t_r)$, calculated from the solution of HJ equations, is

$$\sum_k \int^{\lambda_k} \frac{\xi_i^{n-i}}{\sqrt{h_{\xi_k}^2-\gamma(\xi_k)}}d\xi_k=\delta_{ir}t_r+const_i, \quad i=1,...,n, \tag{77}$$

where

$$h_{\xi_k} = c\xi_k^n+a_1\xi_k^{n-1}+...+a_n,$$
$$\gamma(\xi_k) = 4(\xi_k-\beta_1)...(\xi_k-\beta_N).$$

The Lax representation (55) of the KdV dressing chain (64) is given by

$$L=L_1...L_N, \quad L_i=\begin{pmatrix} v_i & 1 \\ v_i^2+\zeta_i & v_i \end{pmatrix}, \tag{78}$$

so, it is not a Mumford system as the L matrix is not a traceless one. The spectral curve takes the form

$$\det(L-\overline{\mu})=0 \iff \overline{\mu}^2-(trL)\overline{\mu}+\det L=0, \tag{79}$$

where

$$\det L = (-1)^N\zeta_1...\zeta_N=(-1)^N(\lambda-\beta_1)...(\lambda-\beta_N),$$
$$trL = \tau_N=(-1)^N h_\lambda=(-1)^N(h_0\lambda^n+h_1\lambda^{n-1}+...+h_n). \tag{80}$$

Obviously

$$\textit{separation curve} \neq \textit{spectral curve} \tag{81}$$

and points $\{\lambda_i,\overline{\mu}_i\}_{i=1}^n$ from a spectral curve are not canonical separated coordinates. Nevertheless, a simple transformation of $\overline{\mu}$

$$(-1)^N\overline{\mu}=\frac{1}{2}e^{\mu} \tag{82}$$

transforms the spectral curve (79) into the separation curve (75), making the points $\{\lambda_i,\mu_i\}_{i=1}^n$ canonical separated coordinates.

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