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Bäcklund Transformations on Coadjoint Orbits of the Loop Algebra $\tilde{gl}(r)$

Yuri FEDOROV

Department of Mathematics and Mechanics, Moscow Lomonosov University, Moscow 119 899 Russia
E-mail: fedorov@mech.math.msu.su

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Abstract

There is a wide class of integrable Hamiltonian systems on finite-dimensional coadjoint orbits of the loop algebra $\tilde{gl}(r)$ which are represented by $r \times r$ Lax equations with a rational spectral parameter. A reduced complex phase space is foliated with open subsets of Jacobians of regularized spectral curves. Under some generic restrictions on the structure of the Lax matrix, we propose an algebraic geometrical scheme of a discretization of such systems that preserve their first integrals and is represented as translations on the Jacobians. A generic discretizing map is given implicitly in the form of an intertwining relation (a discrete Lax pair) with an extra parameter governing the translation. Some special cases when the map is explicit are also considered. As an example, we consider a modified discrete version of the classical Neumann system described by a 2×2 discrete Lax pair and provide its theta-functional solution.

1 Introduction

Many finite-dimensional integrable systems, as well as finite-gap reductions of some integrable PDE's, can be regarded as flows on finite-dimensional coadjoint orbits of the loop algebra $\tilde{gl}(r)$ described by $r \times r$ Lax equations with a spectral parameter $\lambda \in \mathbb{C}$,

$$\dot{L}(\lambda) = [\, L(\lambda), \mathcal{M}(\lambda)\,], \qquad L = Y + \frac{\mathcal{N}_i}{\lambda - a_i}, \quad L, \mathcal{M} \in gl(r), \tag{1.1}$$

where $Y \in gl(r)$ is a constant matrix and $a_1, \ldots, a_n$ are arbitrary distinct constants (see [1, 3]). For simplicity, in the sequel we assume that $\operatorname{rank} \mathcal{N}_i = 1$. (The case of higher rank is only notationally more complicated).

As shown in [1], these equations naturally arise in connection with so called rank r perturbations of the constant matrix $A = \operatorname{diag}(a_1, \ldots, a_n)$, which generates Lax pairs of a series of integrable systems. In this case $L(\lambda)$ takes the form

$$L = Y + G^T(\lambda \mathbf{I}_n - A)^{-1} F \equiv Y + \sum_{i=1}^{n} \frac{\bar{p}_i \otimes \bar{q}_i}{\lambda - a_i}\,, \tag{1.2}$$

where $\mathbf{I}_n$ is the $n\times n$ unit matrix and G,F are $n\times r$ matrices of rank r,

$$G=||\bar q_1\cdots\bar q_n||,\quad F=||\bar p_1\cdots\bar p_n||,\qquad \bar q_j,\,\bar p_j\in\mathbb{C}^r\,.$$

An important class of systems which admit the Lax pair (1.1), (1.2) is given by so called *binary constrained flows* (BC flows) of various soliton equations [17]. In this case

$$\bar q_i=\Psi(a_i),\quad \bar p_i=\Psi^*(a_i),$$

where $\Psi(\lambda)\in\mathbb{C}^r$ is a vector solution to the $r\times r$ matrix spectral problems

$$\frac{\partial}{\partial x}\Psi=U(\lambda,u)\Psi,\quad \frac{\partial}{\partial t}\Psi=V(\lambda,u)\Psi,\qquad U,V\in\tilde{gl}(r),$$

whose compatibility condition generates a PDE for a set of dependent variables $u=(u_1,\dots,u_l)$, and $\Psi^*(\lambda)\in\mathbb{C}^r$ is a solution to the adjoint spectral problem. Following [1, 17], BC flows are algebraic completely integrable Hamiltonian systems in the space $(G,F)=\mathbb{C}^{2rn}$ endowed with the symplectic structure $\Omega=\operatorname{tr}(dF\wedge dG^T)$. Their generic theta-functional solutions $\bar q_i(x,t),\bar p_i(x,t)$ give rise to finite-gap solutions of the related PDE.

In particular, for the standard KdV equation one has $r=2$ and, as noticed in [18], upon the identification

$$\begin{aligned}&x=(\bar q_{11},\dots,\bar q_{n1})^T,\quad y=(\bar q_{12},\dots,\bar q_{n2})^T,\\ &\xi=(\bar p_{12},\dots,\bar q_{n2})^T,\quad \eta=-(\bar p_{11},\dots,\bar p_{n1})^T\end{aligned}$$

the binary constrained x-flow coincides with the well-known Garnier system ([11])

$$\begin{aligned}&\dot x_i=y_i,\quad \dot y_i=(a_i+u)x_i,\qquad \dot\xi_i=\eta_i,\quad \dot\eta_i=(a_i+u)x_i,\\ &u=\sum_{i=1}^n x_i\xi_i,\qquad i=1,\dots,n\end{aligned}$$

that admit the above Lax pair with

$$Y=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \mathcal{M}=\begin{pmatrix}0&1\\-\lambda-u&0\end{pmatrix}.$$

For the standard Boussinesq equation ([22, 1]) and the three wave interaction equation ([14, 6, 16]) we have $r=3$ and

$$Y=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix},\quad\text{respectively,}\quad Y=\begin{pmatrix}\gamma_1&0&0\\0&\gamma_2&0\\1&0&\gamma_3\end{pmatrix},\quad \gamma_i=\text{const}.$$

Binary constrained flows admit a Hamiltonian symmetry group $\mathcal{R}$ generated by scalings

$$\mathcal{R}_j\;:\;(\bar q_j,\bar p_j)\to(\nu_j\bar q_j,\nu_j^{-1}\bar p_j),\qquad \nu_j\in\mathbb{C}^*,\quad j=1\dots,n,$$

which leave the Lax matrix $L(\lambda)$ invariant. Following [3], by making a Marsden–Weinstein reduction of the BC flows with respect to the action of $\mathcal{R}$, namely, factoring (G,F) by $\mathcal{S}_j$

and fixing values of the Hamiltonians generating the scaling flows, one obtains integrable Hamiltonian systems on a finite-dimensional coadjoint orbit $\mathcal{O}_N$ in the loop algebra $\widetilde{gl}(r)$. The symplectic structure on $\mathcal{O}_N$ is induced by Ω. If Y is nonzero, then $\dim \mathcal{O}_N = 2(g+r-1)$, where g is the genus of the regularized spectral curve $\tilde{\mathcal{C}}$ of the Lax matrix (1.2). The entries of $\mathcal{N}_i$ play the role of abundant coordinates on $\mathcal{O}_N$. It is natural to call the reduced BC flows as *mono constrained flows.* Their complex generic invariant manifolds are open subsets of $(g+r-1)$-dimensional generalized Jacobian varieties, algebraic extensions of the customary Jacobian $\mathrm{Jac}(\tilde{\mathcal{C}})$ by the product of $r-1$ copies of $\mathbb{C}^*$, which we denote as $\mathrm{Jac}(\tilde{\mathcal{C}}, \infty)$ (see, e.g., [8, 10]). The mono constrained flows evolve linearly on the generalized Jacobians.

Next, as shown in [1, 3], the flows possess $r-1$ extra Hamiltonian linear symmetry fields that are independent and commute. Performing a further Marsden–Weinstein reduction with respect to the fields, one arrives at flows on $2g$-dimensional reduced orbit $\mathcal{O}_{\rm red}$ evolving linearly on open subsets of g-dimensional varieties $\mathrm{Jac}(\tilde{\mathcal{C}})$.

Contents of the paper. In given paper, assuming that Y is nonzero and all its eigenvalues are distinct, we propose a scheme of a discretization of the reduced mono constrained flows on $\mathcal{O}_{\rm red}$ represented as a family of maps $\mathcal{B}_{\lambda^*} : \mathcal{O}_{\rm red} \to \mathcal{O}_{\rm red}$, which preserve the first integrals and whose restriction to $\mathrm{Jac}(\tilde{\mathcal{C}})$ are translations governed by a complex parameter λ^*. In an appropriate limit $\lambda^* \to \infty$, the translation gives rise to a vector flow, whose restriction to the Jacobian is tangent to one of the infinite points of the curve $\tilde{\mathcal{C}} \subset \mathrm{Jac}(\tilde{\mathcal{C}})$. In the whole space $\mathcal{O}_{\rm red}$, the vector field coincides with one of the above reduced mono constrained flows.

Our discretization map is lifted to the coadjoint $\mathcal{O}_N$ and is described by an intertwining relation (so called discrete Lax pair)

$$\tilde{L}(\lambda)\, M(\lambda|\lambda^*) = M(\lambda|\lambda^*)\, L(\lambda), \tag{1.3}$$

where $L(\lambda)$ is defined in (1.2) and $\tilde{L}(\lambda)$ depends on new coordinates on $\mathcal{O}_N$ in the same way, as $L(\lambda)$ depends on the original ones, whereas $M(\lambda|\lambda^*)$ is a matrix polynomial in λ, whose coefficients depend on the original, as well as on the new coordinates. Hence, the map $\mathcal{B}_{\lambda^*}$ given by (1.3) is generally implicit and, as we show below, r-valued. However, it may become explicit and single-valued in some important special cases which we also consider separately.

We notice that in the case $L \in \widetilde{sl}(2)$, when the spectral curve becomes hyperelliptic, algebraic geometrical and symplectic properties of $\mathcal{B}_{\lambda^*}$ has been already discussed in detail in [13, 15], where the map was refereed to as a Bäcklund transformation due to its similarity to the analogous procedure for soliton PDE.

On the other hand, for a fixed algebraic curve $\mathcal{C}$ of arbitrary type and genus, a wide class of algebraic addition laws in a ring of meromorphic functions on $\mathrm{Jac}(\mathcal{C})$ that describe a translation has been found in [5].

In Section 2, combining the above two approaches and following the generic idea of [6, 3], we describe the structure of zeros and poles of normalized eigenvectors of matrices $L(\lambda)$ and $\tilde{L}(\lambda)$, which enables us to recover the structure of the operator $M(\lambda|\lambda^*)$ (Theorems 2.2, 2.4.). Then we consider a continuous limit of (1.3) and mention possible natural generalizations of our scheme.

In Section 3 we briefly discuss the simplest case $L \in \tilde{sl}(2)$ and, as an example, give algebraic geometrical interpretation and theta-functional solution to a modified discrete version of the classical Neumann system found in [23, 24]. A description of a more refined example corresponding to $L \in \tilde{gl}(3)$ is left for another publication.

2 Bäcklund transformation via discrete Lax pair

First we recall necessary algebraic geometrical properties of the continuous systems generated by the Lax pair (1.1), (1.2) (see e.g., [6, 3]). The Lax matrix $\hat{L}(\lambda) = a(\lambda)L(\lambda)$ is polynomial in λ, and the spectral curve $\mathcal{C} = \{|\hat{L}(\lambda) - \mu \mathbf{I}| = 0\} \subset (\lambda, \mu) = \mathbb{C}^2$ has the form

$$\mathcal{F} \equiv (-\mu)^r + (-\mu)^{r-1} \operatorname{tr} \hat{L}(\lambda) + \sum_{j=0}^{r-2} (-\mu)^j a^{j-1}(\lambda) \mathcal{H}_j(\lambda) = 0, \tag{2.4}$$

$$\begin{aligned} \mathcal{H}_j(\lambda) &= \lambda^n \mathcal{H}_{j0} + \lambda^{n-1} \mathcal{H}_{j1} + \cdots + \mathcal{H}_{jn}, \\ \operatorname{tr} \hat{L}(\lambda) &= a(\lambda) \left(\operatorname{tr} Y + \sum_{i=1}^n \frac{(\bar{q}_i, \bar{p}_i)}{\lambda - a_i} \right). \end{aligned} \tag{2.5}$$

The leading coefficients of $\operatorname{tr} \hat{L}(\lambda)$, and $\mathcal{H}_j(\lambda)$ are trivial constants determined entirely by symmetric invariants of Y, whereas n integrals $(\bar{q}_i, \bar{p}_i)$ that appear in $\operatorname{tr} \hat{L}(\lambda)$ generate the scaling transformations $\mathcal{R}_i$ described in Introduction. Now factoring the space (G, F) by the action of $\mathcal{R}$ and fixing values of the integrals, one obtains a coadjoint orbit $\mathcal{O}_N$ of dimension $2n(r-1)$. The factorization can be made by imposing the constraints $(\bar{q}_i, \bar{q}_i) = 1$. Hence

$$\mathcal{O}_N = \{(F, G) \mid (\bar{q}_i, \bar{q}_i) = 1, \ (\bar{q}_i, \bar{p}_i) = f_i = \text{const}\},$$

which is diffeomorphic to the product of n cotangent bundles of $(r-1)$-dimensional unit spheres. The symplectic structure on $\mathcal{O}_N$ is induced by the 2-form Ω on (G, F) via the Dirac formalism.

On the other hand, let $\bar{\mathcal{C}} \in \mathbb{P}^2$ be compactification of the curve $\mathcal{C}$ represented as r-fold ramified covering of $\bar{\mathbb{C}} = \{\lambda\} \cup \infty$. Under the assumption that Y is diagonalizable and its eigenvalues $\gamma_1, \ldots, \gamma_r$ are distinct, $\bar{\mathcal{C}}$ has r distinct infinite points $\infty_1, \ldots, \infty_r$ such that in a neighborhood of ∞_k with a local parameter τ ($\tau(\infty_k) = 0$), the following expansions hold

$$\lambda = 1/\tau, \quad \mu = \gamma_k \tau^{-n} + O(\tau^{-n+1}).$$

The curve $\bar{\mathcal{C}}$ is singular at some points over $\lambda = a_i$. As indicated in [3], the regularized curve $\tilde{\mathcal{C}}$ is smooth of genus $g = (r-1)(n-1)$. Thus dimension of $\mathcal{O}_N$ equals $2(g + r - 1)$.

Following the general idea introduced in [21], $\mathcal{O}_N$ can be regarded as a fiber bundle over $(g + r - 1)$-dimensional space of regularized curves (2.4) parameterized by nontrivial coefficients of the polynomials $\mathcal{H}_j(\lambda)$ whose fibers are $(g + r - 1)$-dimensional generalized Jacobians $\operatorname{Jac}(\tilde{\mathcal{C}}, \infty)$.

Further, the integrals $\mathcal{H}_{1,1}, \ldots, \mathcal{H}_{r-1,1}$ in (2.5) are bilinear in $\bar{q}_i, \bar{p}_i$ and generate $r - 1$ linear Hamiltonian symmetry fields $\mathcal{T}_j$ on $\mathcal{O}_N$. Performing Marsden–Weinstein reduction with respect to $\mathcal{T}_j$, i.e., imposing some $r - 1$ extra constraints on the components of $\mathcal{N}_i$ that are transversal to the orbits of the action and fixing some values of $\mathcal{H}_{s1}$, we arrive at *reduced* coadjoint orbit $\mathcal{O}_{\rm red}$ of dimension $2g$ foliated by customary Jacobian varieties $\mathrm{Jac}(\tilde{\mathcal{C}})$. The whole sequence of phase spaces and reductions is represented by the diagram

$$(G,F) \xrightarrow{\mathcal{R}} \mathcal{O}_N \xrightarrow{\mathcal{T}} \mathcal{O}_{\rm red}.$$

Now let $\mathcal{K}(\lambda,\mu)$ be the classical adjoint matrix of $(\hat{L}(\lambda)-\mu\mathbf{I})$ such that

$$(\hat{L}(\lambda)-\mu\mathbf{I})\,\mathcal{K}(\lambda,\mu)=\det(\hat{L}(\lambda)-\mu\mathbf{I})\,\mathbf{I}. \tag{2.6}$$

The columns of the restriction of $\mathcal{K}(\lambda,\mu)$ onto $\tilde{\mathcal{C}}$ give sections of eigenvector line bundle $\tilde{\mathcal{C}}\to\mathbb{P}^{r-1}$ solving the spectral problem $\hat{L}(\lambda)\psi(P)=\mu\psi(P)$, $P=(\lambda,\mu)\in\tilde{\mathcal{C}}$. Next, following [7], for any fixed (i.e., independent of P) vector $V\in\mathbb{C}^r$, r algebraic equations

$$\mathcal{K}(\lambda,\mu)V=0 \tag{2.7}$$

define a divisor of $g+r-1$ points $P_j=(\lambda_j,\mu_j)$ on the curve $\tilde{\mathcal{C}}$, which corresponds to a point z in $\mathrm{Jac}(\tilde{\mathcal{C}})$ via the Abel–Jacobi map

$$z=\mathcal{A}(P_1)+\cdots+\mathcal{A}(P_{g+r-1}),\qquad \mathcal{A}(P)=\int_{P_0}^{P}\omega,$$

where $\omega=(\omega_1,\ldots,\omega_g)^T$ is a basis of holomorphic differentials and P_0 is a fixed basepoint on $\tilde{\mathcal{C}}$.

In the sequel we assume $Y=\mathrm{diag}(\gamma_1,\ldots,\gamma_r)$. As noticed in [3], if V is an eigenvector of Y, then equations (2.7) have at most g *finite* solutions, whereas the rest consists of $r-1$ points over $\lambda=\infty$. In particular, for $V=(0,\ldots,0,1)^T$, the solutions of (2.7) are $P_1=(\lambda_1,\mu_1),\ldots,P_g=(\lambda_g,\mu_g)$, $\infty_1,\ldots,\infty_{r-1}$. Via the Abel–Jacobi map, the divisor $P_1,\ldots,P_g$ uniquely defines a point z in $\mathrm{Jac}(\tilde{\mathcal{C}})$ and therefore in the reduced orbit $\mathcal{O}_{\rm red}$. Moreover, according to [3], the pairs $\lambda_1,\zeta_1=\mu_1/a(\lambda_1),\ldots,\lambda_g,\zeta_g=\mu_g/a(\lambda_g)$ form a complete set of Darboux coordinates on $\mathcal{O}_{\rm red}$ with respect to the induced Poisson structure, which provide separating variables for the reduced flows.

On the other hand, as seen from (2.6), the restriction of the first row of $\mathcal{K}(\lambda,\mu)$ on $\tilde{\mathcal{C}}$ denoted as $\phi(P)=(\phi_1,\ldots,\phi_r)$ solves the transposed spectral problem

$$\phi(P)\hat{L}(\lambda)=\mu\phi(P). \tag{2.8}$$

Let us normalize $\phi(P)$ by assuming $\phi_r\equiv 1$. Then, as follows from the structure of $Y,\hat{L}(\lambda)$ and the results of [2],

$$\phi_1(P)=\left.\frac{\mathcal{K}_{11}(\lambda,\mu)}{\mathcal{K}_{1r}(\lambda,\mu)}\right|_{(\lambda,\mu)\in\tilde{\mathcal{C}}},\ \ldots,\ \phi_{r-1}(P)=\left.\frac{\mathcal{K}_{1,r-1}(\lambda,\mu)}{\mathcal{K}_{1r}(\lambda,\mu)}\right|_{(\lambda,\mu)\in\tilde{\mathcal{C}}},$$

are meromorphic functions on $\tilde{\mathcal{C}}$ whose divisors of zeros and poles have the form

$$\begin{aligned}(\phi_k)&=Q_1^{(k)}+\cdots+Q_g^{(k)}+\infty_r-P_1-\cdots-P_g-\infty_k\ ,\\ &\quad k=1,\ldots,r-1\end{aligned} \tag{2.9}$$

with some generally finite points $Q_i^{(k)}$.

Transformation of eigenvectors. Now we want to describe a (generally multi-valued) map $\mathcal{B}_{\lambda^*} : \mathcal{O}_{\rm red} \to \mathcal{O}_{\rm red}$, which leaves the fibers (Jacobians) invariant and whose restriction to each fiber is represented by a shift governed by a set of parameters λ^*. Further, we demand that the map lifts to unreduced orbits $\mathcal{O}_{\mathcal{N}}$, and the shift results in a set of algebraic addition formulas for the entries of $\mathcal{N}_i$ as meromorphic functions on the generalized Jacobians $\mathrm{Jac}(\tilde{\mathcal{C}}, \infty)$. In this connection we assume that $\mathcal{B}^*_\lambda$ admits the $r \times r$ intertwining relation (1.3), where $M(\lambda|\lambda^*)$ is a matrix polynomial in λ whose leading coefficient must commute with Y. Due to the structure of (1.3), for any nonzero scalar function χ, operators M and χM define one and the same map.

Let $\tilde{\phi}(P) = (\tilde{\phi}_1, \dots, \tilde{\phi}_r)$ be a row vector solving the transposed spectral problem associated to the new Lax matrix, $\tilde{\phi}(P)\tilde{L}(\lambda) = \mu\tilde{\phi}(P)$. Then, according to (1.3), it is related to the original row vector $\phi(P)$ in (2.8) as follows

$$\phi(P) = \varkappa\, \tilde{\phi}(P) M(\lambda) \quad \text{and} \quad \tilde{\phi}(P) = \varkappa'\, \phi(P) M^{-1}(\lambda), \tag{2.10}$$

where $\varkappa, \varkappa'$ are nonzero factors. Under the assumption $\phi_r \equiv 1$, the latter relation implies

$$\tilde{\phi}_r(P) = \alpha_1\phi_1(P) + \cdots + \alpha_{r-1}\phi_{r-1}(P) + \alpha_r, \qquad \alpha_k = \left(M^{-1}\right)_{rk}. \tag{2.11}$$

In the sequel we also assume that α_k are constants on $\tilde{\mathcal{C}}$ and without loss of generality we put $\alpha_1 = 1$, $\varkappa' = 1$. Then, in view of (2.9), (2.11), the divisor of $\tilde{\phi}_r(P)$ has the form

$$(\tilde{\phi}_r) = \mathcal{D} - P_1 - \cdots - P_g - \infty_1 - \cdots - \infty_{r-1}, \tag{2.12}$$

where $\mathcal{D}$ is a positive divisor of degree $g + r - 1$ and, as above, $P_1, \dots, P_g$ are poles of $\phi(P)$. Note that by an appropriate choice of the coefficients α_k, $r - 1$ points of $\mathcal{D}$ can be moved to any points $S_1, \dots, S_{r-1}$ chosen in advance, so that $\mathcal{D}$ takes the form $D_1 + \cdots + D_g + S_1 + \cdots + S_{r-1}$.

We assume $D_1 + \cdots + D_g$ to be a new divisor representing a new point $\tilde{z} = \mathcal{A}(D_1) + \cdots + \mathcal{A}(D_g)$ on $\mathrm{Jac}(\tilde{\mathcal{C}})$ and therefore in $\mathcal{O}_{\rm red}$.

Since $\tilde{\phi}_r(P)$ is a meromorphic function on $\mathcal{C}$, then modulo the periods of $\mathrm{Jac}(\tilde{\mathcal{C}})$,

$$\mathcal{A}(P_1 + \cdots + P_g + \infty_1 + \cdots + \infty_{r-1}) \equiv \mathcal{A}(D_1 + \cdots + D_g + S_1 + \cdots + S_{r-1}),$$

hence the new point $\tilde{z}$ is obtained from the original $z = \mathcal{A}(P_1) + \cdots + \mathcal{A}(P_g)$ by the shift by the vector

$$\mathcal{S} = -\int_{\infty_1}^{S_1} \omega - \cdots - \int_{\infty_{r-1}}^{S_{r-1}} \omega \in \mathbb{C}^g. \tag{2.13}$$

Remark 1. In view of (2.9), in the case $\{S_1, \dots, S_{r-1}\} = \{\infty_1, \dots, \infty_r\} \setminus \infty_k$, the new divisor $D_1, \dots, D_g$ coincides with $Q_1^{(k)}, \dots, Q_g^{(k)}$, the finite zeros of ϕ_k.

Given arbitrary fixed points $S_k = (\lambda_k^*, \mu_k^*)$, $k = 1, \dots, r - 1$, from the conditions

$$\tilde{\phi}_r(S_1) = \cdots = \tilde{\phi}_r(S_{r-1}) = 0$$

representing a system of linear equations with respect to $\alpha_2, \ldots, \alpha_r$, we find

$$\tilde{\phi}_r(P) = \frac{1}{\Delta \mathcal{K}_{1r}(P)} \begin{vmatrix} \mathcal{K}_{11}(\lambda_1^*, \mu_1^*) & \cdots & \mathcal{K}_{1r}(\lambda_1^*, \mu_1^*) \\ \vdots & & \vdots \\ \mathcal{K}_{11}(\lambda_{r-1}^*, \mu_{r-1}^*) & \cdots & \mathcal{K}_{1r}(\lambda_{r-1}^*, \mu_{r-1}^*) \\ \mathcal{K}_{11}(P) & \cdots & \mathcal{K}_{1r}(P) \end{vmatrix}, \tag{2.14}$$

and

$$\alpha_s = (-1)^s \Delta_s / \Delta, \qquad s = 2, \ldots, r, \tag{2.15}$$

$$\Delta = \begin{vmatrix} \mathcal{K}_{12}(\lambda_1^*, \mu_1^*) & \cdots & \mathcal{K}_{1r}(\lambda_1^*, \mu_1^*) \\ \vdots & & \vdots \\ \mathcal{K}_{12}(\lambda_{r-1}^*, \mu_{r-1}^*) & \cdots & \mathcal{K}_{1r}(\lambda_{r-1}^*, \mu_{r-1}^*) \end{vmatrix},$$

where Δ_s is obtained from the determinant Δ by replacing the $(s-1)$-th column by $\big(\mathcal{K}_{11}(\lambda_1^*, \mu_1^*), \ldots, \mathcal{K}_{11}(\lambda_{r-1}^*, \mu_{r-1}^*)\big)^T$.

Now let $\bar{\phi}(P) = (\bar{\phi}_1, \ldots, \bar{\phi}_r)$ be the *normalized* row vector associated to the new divisor $D_1, \ldots, D_g$, such that $\bar{\phi}_r \equiv 1$. By analogy with (2.9),

$$(\bar{\phi}_k) = \tilde{Q}_1^{(k)} + \cdots + \tilde{Q}_g^{(k)} + \infty_r - D_1 - \cdots - D_g - \infty_k\,, \quad k = 1, \ldots, r-1 \tag{2.16}$$

with some new positive divisors $\tilde{Q}_1^{(k)}, \ldots, \tilde{Q}_g^{(k)}$. The normalized and the non-normalized vectors represent the same point in $\mathbb{P}^{r-1}$, hence they are related as $\tilde{\phi}(P) = \tilde{\phi}_r(P)\bar{\phi}(P)$. Then, in view of (2.12), (2.16) the divisor of zeros and poles of the non-normalized component $\tilde{\psi}_k(P)$ has the form

$$\begin{aligned} (\tilde{\phi}_k) = (\bar{\phi}_k) + (\tilde{\phi}_r) = \tilde{Q}_1^{(k)} + \cdots + \tilde{Q}_g^{(k)} + S_1 + \cdots + S_{r-1} + \infty_r \\ - P_1 - \cdots - P_g - \infty_1 - \cdots - \infty_{r-1} - \infty_k. \end{aligned} \tag{2.17}$$

This divisor determines the meromorphic function $\tilde{\psi}_k(P)$ up to multiplication by a constant factor.

In the sequel we restrict ourselves to two special cases of the translation vector $\mathcal{S}$, which however generate any translation on the Jacobian by taking their appropriate composition.

In case (i) the points $S_1, \ldots, S_{r-1}$ have one and the same *finite* coordinate $\lambda = \lambda^*$, i.e., $S_k = (\lambda^*, \mu_k^*)$, and are all distinct if there are no ramification points over λ^*. If $R = (\lambda^*, \mu^+)$ is a ramification point joining m branches of $\tilde{\mathcal{C}}$, then at most m points S_k can lie at R. In case (ii) all the points are infinite as described in Remark 1.

Case (i). Let $S^* = (\lambda^*, \mu^*)$ be the extra point over finite λ^* such that

$$(\lambda - \lambda^*) = S_1 + \cdots + S_{r-1} + S^* - \infty_1 - \cdots - \infty_r. \tag{2.18}$$

Since $(\lambda - \lambda^*)$ is a divisor of a meromorphic function, modulo the period lattice of the Jacobian of $\tilde{\mathcal{C}}$ one has

$$\int_{\infty_1}^{S_1} \omega + \cdots + \int_{\infty_{r-1}}^{S_{r-1}} \omega + \int_{\infty_r}^{S^*} \omega = 0$$

and, in view of (2.13), the translation vector $\mathcal{S}$ reduces to the single integral $\int_{\infty_r}^{S^*} \omega$.

Proposition 2.1. *Under the above assumption on $S_1,\ldots,S_{r-1}$, the components of $\tilde\phi$ can be represented in form*

$$\tilde\phi_k(P)=(\gamma_k-\gamma_r)(\lambda-\lambda^*)\phi_k(P)+Z_k\tilde\phi_r(P), \qquad (2.19)$$

$$Z_k=\bar\phi_k(S^*)=\frac{\tilde{\mathcal K}_{1k}(\lambda^*,\mu^*)}{\tilde{\mathcal K}_{1r}(\lambda^*,\mu^*)},\quad k=1,\ldots,r-1,$$

where $\tilde{\mathcal K}$ is the adjoint matrix to $\tilde L(\lambda)-\mu\mathbf I$.

Proof. As follows from the structure of divisors of zeros and poles indicated in (2.9), (2.12), and (2.18), for any nonzero constant $\varkappa$, the sum $\Sigma=\varkappa(\lambda-\lambda^*)\phi_k+Z_k\tilde\phi_r$ has simple poles at $\infty_1,\ldots,\infty_{r-1}$, except ∞_k, where is has a double pole, as well as at $P_1,\ldots,P_g$, and it has no poles elsewhere. By the same reasoning, Σ has zeros at $S_1,\ldots,S_{r-1}$. Next, according to (2.9) and (2.12), the functions $(\lambda-\lambda^*)\phi_k(P)$ and $\tilde\phi_r(P)$ are both nonzero and finite at ∞_r. Then, for an appropriate choice $\varkappa^*$ of $\varkappa$, the sum Σ vanishes at ∞_r.

Hence, in view of (2.17), the sum $\Sigma^*=\varkappa^*(\lambda-\lambda^*)\phi_k+Z_k\tilde\phi_r$ has the same poles on $\tilde{\mathcal C}$ as $\tilde\phi_k(P)$ and the same zeros as $\tilde\phi_k(P)$ must have. Hence Σ^* and $\tilde\phi_k(P)$, as functions on the curve $\tilde{\mathcal C}$, are different only by a constant multiplier. On the other hand, at the point $P=S^*$ the both functions are nonzero and, due to the definition of Z_k, coincide. Finally, matching the coefficients at the highest powers of λ and μ in Σ^* and $\tilde\phi_k(P)$, we conclude that $\varkappa^*$ must equal $\gamma_k-\gamma_r$. This establishes the proposition.

Since $M^{-1}(\lambda|\lambda^*)$ and $\xi M^{-1}(\lambda|\lambda^*)$ define one and the same map, in the sequel we replace M^{-1} by the adjoint matrix of $M(\lambda|\lambda^*)$. Now comparing expressions (2.11), (2.19) with the second form of the transposed spectral problem (2.10), we obtain the following result.

Theorem 2.2. *The adjoint matrix of $M(\lambda)$ in the intertwining relation (1.3), (1.2) that defines the shift by $\int_{\infty_r}^{S^*}\omega$ on $\operatorname{Jac}(\tilde{\mathcal C})$ has the form*

$$\det M(\lambda|\lambda^*)M^{-1}(\lambda|\lambda^*)=\begin{pmatrix}1\\ \alpha_2\\ \vdots\\ \alpha_r\end{pmatrix}(Z_1\ \cdots\ Z_{r-1}\ 1)+(\lambda-\lambda^*)(Y-\gamma_r\mathbf I), \qquad (2.20)$$

where $Y=\operatorname{diag}(\gamma_1,\ldots,\gamma_r)$ and α_s, Z_k are specified in (2.15) and (2.19) respectively.

Remark 2. Formulas (2.20), (2.15), and (2.19) describing the entries of $M^{-1}(\lambda)$ include the coordinate μ^*, which depends on λ^* and on the moduli of the spectral curve. Hence, Theorem 2.2 only indicates the structure of the operators $M^{-1}(\lambda)$ and $M(\lambda)$. As soon as the structure is known, the entries of $M^{-1}(\lambda)$ can explicitly be expressed in terms of the original and new values of the phase variables on $\mathcal O_{\mathrm{red}}$ by expanding the both sides of (1.3) in λ and matching some of the leading matrix coefficients. This shows that the map $\mathcal B_{\lambda^*}$ given by the discrete Lax pair is *implicit*.

Moreover, since a generic parameter λ^* alone gives rise to r distinct points S^* on the curve $\tilde{\mathcal C}$ and therefore, via the Abel map, to r distinct translation vectors $\mathcal S^*$ on the Jacobian, the map $\mathcal B_{\lambda^*}$ is *multi-valued* (at most r-valued). The sum of all possible r vectors $\mathcal S^*$ equals $\int_{\infty_s}^{\infty_k}\omega$ for some s,k. Using this fact, one can show that the N-th iteration of $\mathcal B_{\lambda^*}$ has $(r-1)N+(N+1)N/2$ images, not r^N images as in the case of generic nonintegrable maps.

Case (ii). Expression (2.20) is obviously not applicable when $\lambda^* = \infty$. This leads us to the second case described in Remark 1, namely $\{S_1, \dots, S_{r-1}\} = \{\infty_1, \dots, \infty_r\} \setminus \infty_k$, which results in the translation vector $\mathcal{S} = \int_{\infty_r}^{\infty_k} \omega$ for a certain index $1 \le k \le r-1$. We denote the corresponding map by $\mathcal{B}_{\infty_k}$.

For concreteness, in the sequel we set $k = 1$. Then, in view of Remark 1, instead of the linear transformations (2.11), (2.14), we have a simple relation

$$\tilde{\phi}_r(P) = \phi_1(P), \tag{2.21}$$

Proposition 2.3. *In case of the map $\mathcal{B}_{\infty_1}$, the other components of $\tilde{\phi}(P)$ can be represented in the form*

$$\tilde{\phi}_1(P) = (\gamma_1 - \gamma_r)(\lambda - u)\phi_1(P) + \sum_{k=2}^{r} \delta_k \phi_k(P), \tag{2.22}$$

$$\tilde{\phi}_s(P) = \beta_s \phi_s(P) + \bar{\phi}_s(\infty_1)\phi_1(P), \qquad s = 2, \dots, r-1, \tag{2.23}$$

where u, δ_k, β_s are some uniquely defined constants on $\tilde{\mathcal{C}}$.

The proof is similar to that of Proposition 2.1. In view of (2.9), (2.21), divisors of zeros and poles of non-normalized row eigenvector $\tilde{\phi}(P) = \tilde{\phi}_r(P)\bar{\phi}(P)$ are

$$\begin{aligned} (\tilde{\phi}_1) &= (\bar{\phi}_1) + (\phi_1) = \tilde{Q}_1^{(1)} + \dots + \tilde{Q}_g^{(1)} + 2\infty_r - P_1 - \dots - P_g - 2\infty_1, \\ (\tilde{\phi}_s) &= (\bar{\phi}_s) + (\phi_1) = \tilde{Q}_1^{(s)} + \dots + \tilde{Q}_g^{(s)} + 2\infty_r - P_1 - \dots - P_g - \infty_1 - \infty_s, \end{aligned} \tag{2.24}$$

where, as before, $\tilde{Q}_1^{(k)}, \dots, \tilde{Q}_g^{(k)}$ are some positive divisors.

On the other hand, for generic constants β_s, the right hand sides of (2.23) have simple poles at $P_1, \dots, P_g, \infty_1, \infty_s$ and at least a simple zero at ∞_r. Then one can choose constants β_s such that these expressions have *a double* zero at ∞_r. These can be found from equations

$$\beta_s \frac{\partial \phi_s(P)}{\partial \tau}\bigg|_{P=\infty_r} + \frac{\partial \phi_1(P)}{\partial \tau}\bigg|_{P=\infty_r} = 0, \qquad s = 2, \dots, r-1,$$

where τ is a local coordinate on $\tilde{\mathcal{C}}$ near ∞_r. Since $\phi_1(P), \phi_s(P)$ have only a simple pole at ∞_r, the above equations have unique solutions. Then, in view of (2.24), the right hand sides of (2.23) have the same poles and zeros as $\tilde{\phi}_s$ must have, therefore they can be different from $\tilde{\phi}_s$ only by constant multipliers. These multipliers are units because quotients of both sides of (2.23) by $\phi_1(P)$ are nonzero, finite and coincide at $P = \infty_1$.

Next, according to (2.9), for generic constants u, δ_k, the right hand side of (2.22) have simple poles at $P_1, \dots, P_g, \infty_2, \dots, \infty_r$, and a double pole at ∞_1. The conditions for the sum $(\lambda - u)\phi_1(P) + \sum_{k=2}^{r} \delta_k \phi_k(P)$ to have a double zero ∞_r and *not to have* poles at $\infty_2, \dots, \infty_{r-1}$ give rise to r independent linear equations, from which u, δ_k are found uniquely. Due to (2.24), for these special values of the constants, the sum is different from $\tilde{\phi}_1$ only by multiplication by a constant factor. Finally, matching the coefficients at the highest powers of λ and μ of both sides of (2.22) proves this relation.

As above, comparing expressions (2.21)–(2.23) with the transposed spectral problem (2.10), we obtain the following result.

Theorem 2.4. *The adjoint matrix of $M(\lambda)$ in the discrete Lax pair (1.3), (1.2) defining the translation by $\int_{\infty_r}^{\infty_1}\omega$ on $\operatorname{Jac}(\tilde{\mathcal{C}})$ has the form*

$$\det M(\lambda|\lambda^*)\, M^{-1}(\lambda|\lambda^*) = \begin{pmatrix} (\gamma_1-\gamma_r)(\lambda-u) & \bar\phi_2(\infty_1) & \cdots & \bar\phi_{r-1}(\infty_1) & 1 \\ \delta_2 & & & & 0 \\ \vdots & & \mathbf{B} & & \vdots \\ \delta_{r-1} & & & & 0 \\ \delta_r & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{2.25}$$

$$\mathbf{B} = \operatorname{diag}(\beta_2,\dots,\beta_{r-1}),$$

where u,δ_k,β_s are some functions on $\mathcal{O}_{\rm red}$.

Like in the generic case $\lambda^*\neq\infty$, expression (2.25) only fixes the structure of $M^{-1}(\lambda)$ and $M(\lambda)$. Then the entries of $M(\lambda)$ can effectively be computed in terms of the original and new variables on $\mathcal{O}_{\rm red}$ by matching some leading coefficients in the expansion of (1.3) in λ.

Remark 3. In contrast to the generic map $\mathcal{B}_{\lambda^*}$, for the maps $\mathcal{B}_{\infty_k}$ the translations vectors $\int_{\infty_r}^{\infty_k}\omega$ are specified in a unique way, hence the maps themselves are *single-valued.*

Remark 4. Clearly, the inverse map $\mathcal{B}_{\lambda^*}^{-1}$ represented on $\operatorname{Jac}(\tilde{\mathcal{C}})$ as the shift by $\int_{\infty_r}^{S^*}\omega = -\int_{\infty_r}^{S^*}\omega$ is described by the discrete Lax pair (1.3) with $L(\lambda)$ and $\tilde L(\lambda)$ interchanged. Like the direct map $\mathcal{B}_{\lambda^*}$, the map $\mathcal{B}_{\lambda^*}^{-1}$ is generally r-valued. Further, for any finite numbers λ^*,ξ^*, the composition $\mathcal{B}_{\xi^*}^{-1}\circ\mathcal{B}_{\lambda^*}$ is generally r^2-valued and its restriction to $\operatorname{Jac}(\tilde{\mathcal{C}})$ is given by one of the translation vectors $\int_{(\xi*,\eta^*)}^{(\lambda^*,\mu^*)}\omega$, where $(\xi^*,\eta^*),(\lambda^*,\mu^*)\in\tilde{\mathcal{C}}$. On the contrary, the composition $\mathcal{B}_{\infty_k}^{-1}\circ\mathcal{B}_{\lambda^*}$ is only r-valued and is represented by the translation vector $\int_{\infty_k}^{(\lambda^*,\mu^*)}\omega$.

Continuous limit. Now we go back to the generic case $\lambda^*\neq\infty$ and assume that S^* tends to ∞_r. The translation vector on the Jacobian $\operatorname{Jac}(\tilde{\mathcal{C}})$ has the expansion $\mathcal{S} = \upsilon(\infty_r)\tau + O(\tau^2)$, where, as above, $\tau = 1/\lambda^*$ is the local coordinate on $\tilde{\mathcal{C}}$ near this infinite point and $\upsilon(\tau)$ is the vector of the coefficients of the holomorphic differentials ω, such that $\omega = \upsilon(\tau)d\tau$.

On the other hand, for sufficiently small τ, the new Lax matrix $\tilde L(\lambda)$ in (1.3) can be represented in the form $L(\lambda)+L_1(\lambda)\tau+O(\tau^2)$. Then one can define a limit flow on $\mathcal{O}_{\rm red}$ with an independent variable x,

$$\frac{d}{dx}L(\lambda) = \lim_{\tau\to 0}\frac{\tilde L - L}{\tau} = L_1(\lambda),$$

whose restriction onto $\operatorname{Jac}(\tilde{\mathcal{C}})$ is represented by the translationally invariant vector field $\upsilon(\infty_r)$, which is thus tangent to $\tilde{\mathcal{C}}\subset\operatorname{Jac}(\tilde{\mathcal{C}})$ at ∞_r.

In order to calculate Lax pair of the limit flow, we first take the matrix $M(\lambda|\lambda^*)/(\lambda-\lambda^*)^{r-1}$. In view of Theorem 2.2, it is finite for infinite λ^* and its expansion with respect to τ has the form

$$\frac{1}{(\lambda-\lambda^*)^{r-1}} M(\lambda|\lambda^*) = \Gamma \mathbf{I} + \mathcal{M}(\lambda)\tau + O(\tau^2),$$

where Γ is a constant. Substituting this, as well as the expansion for $\tilde{L}(\lambda)$ into (1.3), at the first order we obtain Lax equations $\frac{d}{dx}L(\lambda) = [L(\lambda), \mathcal{M}(\lambda)]$ describing the above flow. In this sense, the map $\mathcal{B}_{\lambda^*}$ can be regarded as its discretization with τ playing the role of a time step.

Possible generalizations. Our approach can be modified to construct maps on the orbits $\mathcal{O}_{red}$ for other types of matrix Y in (1.2), when the number of infinite points of the corresponding spectral curve is less than r, in particular for the case of r-gonal curves (only one infinite point), which arise in connection with finite-gap solutions of soliton equations of the KP hierarchy ([22]).

Next, since the operator $M(\lambda|\lambda^*)$ defines a transformation of an eigenvector ϕ of a $r \times r$ spectral problem, it can be used to construct Bäckund transformations of the solution u of the corresponding PDE (for $r = 2$, examples of such transformation can be found, amongst others, in [19]).

On the other hand, it is possible to construct a more general Bäcklund transformation $\mathcal{B} : \mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}}$, whose restriction to generalized Jacobians $\mathrm{Jac}(\tilde{\mathcal{C}}, \infty)$ acts nontrivially on their fibers $(\mathbb{C}^*)^{r-1}$ and which does not preserve the extra $r-1$ constraints defining $\mathcal{O}_{\mathrm{red}}$. In this case the coefficients α_k in (2.11) become dependent on the coordinates λ, μ. For the case $L \in sl(2)$, such a transformation depending on two governing parameters was described in [9].

3 The case $\tilde{sl}(2)$

Following many publications, in the simplest case $r = 2$ the coadjoint orbit $\mathcal{O}_{\mathcal{N}}$ is defines by the following $2n$ independent constraints on the $4n$-dimensional space (G, F)

$$(\bar{q}_{11}, \dots, \bar{q}_{n1}) = (\bar{p}_{12}, \dots, \bar{q}_{n2}), \quad (\bar{q}_{12}, \dots, \bar{q}_{n2}) = -(\bar{p}_{11}, \dots, \bar{p}_{n1}),$$

which are equivalent to conditions

$$(\bar{q}_i, \bar{p}_i) \equiv \bar{q}_1\bar{p}_1 + \bar{q}_2\bar{p}_2 = 0, \quad \bar{q}_{i2} + \bar{p}_{i1} = 0, \qquad i = 1, \dots, n.$$

Further, we assume $\gamma_1 = -\gamma_2 = \gamma$. Then the Lax matrix (1.2) takes the form

$$L(\lambda) = \begin{pmatrix} \gamma + \sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} & \sum_{i=1}^n \frac{q_i^2}{\lambda - a_i} \\ -\sum_{i=1}^n \frac{p_i^2}{\lambda - a_i} & -\gamma - \sum_{i=1}^n \frac{q_i p_i}{\lambda - a_i} \end{pmatrix} \in \tilde{sl}(2), \tag{3.26}$$

where we set $q = (\bar{q}_{11}, \dots, \bar{q}_{n1})^T$, $p = (\bar{p}_{11}, \dots, \bar{p}_{n1})^T$. The components of q, p play the role of local coordinates on $2n$-dimensional orbit $\mathcal{O}_{\mathcal{N}}$ and the restriction of the 2-form Ω onto $\mathcal{O}_{\mathcal{N}}$ coincides with the standard symplectic form $\sum_{i=1}^n dq_i \wedge dp_i$. Dynamical systems on $\mathcal{O}_{\mathcal{N}}$ that are described by the Lax pairs with the matrix (3.26) are sometimes refereed to as *Gaudin magnets* (see,e.g., [13]).

The spectral curve $\tilde{\mathcal{C}}$ becomes a hyperelliptic curve of genus $g = n-1$ with two infinite points ∞_1, ∞_2 and is defined by equation

$$
\begin{aligned}
\mu^2 &= a^2(\lambda)\left(-\gamma^2 - \sum_{i=1}^{n} \frac{p_i q_j}{\lambda - a_i} - \sum_{i<j}^{n} \frac{(q_i p_j - q_j p_i)^2}{(\lambda - a_i)(\lambda - a_j)}\right) \\
&\equiv a(\lambda)(-\gamma^2\lambda^n + \mathcal{H}_1\lambda^{n-1} + \cdots + \mathcal{H}_n),
\end{aligned} \tag{3.27}
$$

$\mathcal{H}_1, \ldots \mathcal{H}_n$ being integrals of motion, which are in involution with respect to the constrained Poisson bracket on the orbit $\mathcal{O}_{\mathcal{N}}$. The complexification of $\mathcal{O}_{\mathcal{N}}$ is foliated with open subsets of n-dimensional generalized Jacobians $\mathrm{Jac}(\tilde{\mathcal{C}}, \infty_{1,2}) = \mathrm{Jac}\,(\tilde{\mathcal{C}}) \times \mathbb{C}^*$.

Up to an additive constant, the integral $\mathcal{H}_1$ in (3.27) equals (p, q). A Marsden–Weinstein reduction with respect to the flow generated by the Hamiltonian $\mathcal{H}_1$ is equivalent to imposing constraints $(p, q) = f$, $(q, q) = 1$. They confine a $2g$-dimensional reduced orbit $\mathcal{O}_{\text{red}}$ foliated by open subsets of customary Jacobians $\mathrm{Jac}(\tilde{\mathcal{C}})$, which, in tun, can be regarded as reduction of $\mathrm{Jac}(\tilde{\mathcal{C}}, \infty_{1,2})$ by the above flow.

In order to describe the map $\mathcal{B}_{\lambda^*} : \mathcal{O}_{\text{red}} \to \mathcal{O}_{\text{red}}$, we represent the Lax matrix in the polynomial form

$$
\hat{L}(\lambda) = a(\lambda)L(\lambda) = \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \tag{3.28}
$$

$$
\begin{aligned}
&U(\lambda) = \lambda^g + u_1\lambda^{g-1} + \cdots + u_g, \quad V(\lambda) = \gamma\lambda^{g+1} + v_1\lambda^g + \cdots + v_{g+1}, \\
&W(\lambda) = w_0\lambda^g + w_1\lambda^{g-1} + \cdots + w_g.
\end{aligned} \tag{3.29}
$$

Then the adjoint matrix to $a(\lambda)L(\lambda) - \mu\mathbf{I}$ equals

$$
\mathcal{K} = \begin{pmatrix} -V(\lambda) - \mu & -U(\lambda) \\ -W(\lambda) & V(\lambda) - \mu \end{pmatrix},
$$

and the transposed spectral problem (2.10) has normalized solution

$$
\phi(P) = \Big((V(\lambda) + \mu)/U(\lambda), 1\Big)\Big|_{(\lambda,\mu)\in\tilde{\mathcal{C}}}.
$$

In view of (3.29) and the behavior of the coordinates λ, μ near the infinite points, the divisor of zeros and poles of the component $\phi_1(P)$ has the form

$$
\frac{V + \mu}{U}\bigg|_{(\lambda,\mu)\in\tilde{\mathcal{C}}} = Q_1 + \cdots + Q_g + \infty_1 - P_1 - \cdots - P_g - \infty_2, \tag{3.30}
$$

where $\lambda(P_i)$ and $\lambda(Q_i)$ give zeros of $U(\lambda)$ and $W(\lambda)$ respectively.

Next, introduce hyperelliptically involutive finite points $S_1 = (\lambda^*, -\mu^*)$, $S^* = (\lambda^*, \mu^*) \in \tilde{\mathcal{C}}$. Then, according to Theorem 2.2, the map $\mathcal{B}_{\lambda^*}$ representing translation by the vector $\int_{\infty_2}^{(\lambda^*,\mu^*)} \omega$ is given by the intertwining relation

$$
\tilde{L}(\lambda)M(\lambda|\lambda^*) = M(\lambda|\lambda^*)\,L(\lambda), \qquad \tilde{L}(\lambda) = \begin{pmatrix} \tilde{V}(\lambda) & \tilde{U}(\lambda) \\ \tilde{W}(\lambda) & -\tilde{V}(\lambda) \end{pmatrix}, \tag{3.31}
$$

$$
\det M(\lambda|\lambda^*)\, M^{-1}(\lambda|\lambda^*) = \begin{pmatrix} 2\gamma(\lambda - \lambda^*) + Z & 1 \\ \alpha Z & \alpha \end{pmatrix},
$$

$$
M(\lambda|\lambda^*) = \begin{pmatrix} \alpha & -1 \\ -\alpha Z & 2\gamma(\lambda - \lambda^*) + Z \end{pmatrix},
$$

where the coefficients of the polynomials $\tilde{U}(\lambda), \tilde{V}(\lambda), \tilde{W}(\lambda)$ are new coordinates on $\mathcal{O}_{\rm red}$ and, in view of (2.15), (2.19), the coefficients α, Z are defined in a rather symmetric manner

$$\alpha = \alpha_2 = \frac{-\mu^* + V(\lambda^*)}{U(\lambda^*)}, \quad Z = Z_1 = \frac{\mu^* + \tilde{V}(\lambda^*)}{\tilde{U}(\lambda^*)} \tag{3.32}$$

(that is, α and Z pass into each other under the involution $\mathcal{I}\, :\, \mu^* \to -\mu^*$, $U, V \to \tilde{U}, \tilde{V}$).

Notice that v_1 in (3.29) appears as coefficient in the equation of the spectral curve, hence it is invariant under the transformation: $v_1 = \tilde{v}_1$.

The intertwining relation (3.31) together with expressions (3.32) were already found in [15] as a result of a direct analysis of divisors of zeros and poles of the polynomials (3.29).

Now, comparing coefficients of leading powers of λ in both sides of (3.31), we obtain

$$\begin{aligned} &\alpha Z = w_0, \quad v_1 + \tilde{v}_1 + \alpha + Z = \tilde{u}_1 - \lambda^*, \\ &w_1 - w_0\lambda^* + 2\alpha Z\, v_1 = \alpha\tilde{w}_0 + Z w_0. \end{aligned} \tag{3.33}$$

which imply the following expressions of α, Z in terms of the original and new coordinates on $\mathcal{O}_{\rm red}$

$$\alpha = \frac{w_1 - w_0\tilde{u}_1}{\tilde{w}_0 - w_0}, \quad Z = \frac{w_0(w_0 - \tilde{w}_0)}{w_1 - w_0\tilde{u}_1}. \tag{3.34}$$

Relations (3.31), (3.34) describe the map $\mathcal{B}_{\lambda^*}$ in an implicit form. According to Remark 4 above, it is a two-valued map.

Limit cases. Now let S^* tend to the infinite point ∞_2 with a local coordinate $\tau = 1/\lambda^*$. In view of (3.32), (3.29), and (3.33), in a neighborhood of ∞_2 the following expansion hold

$$\alpha(\lambda^*) = \frac{-2\gamma\tau^{-1} - 2v_1 + O(\tau)}{1 + u_1\tau + O(\tau)} = -\frac{2\gamma}{\tau} + 2\gamma u_1 - 2v_1 + O(\tau), \tag{3.35}$$

$$\frac{1}{\lambda - \lambda^*} = -(\tau + \lambda\tau^2 + O(\tau^3)), \quad \frac{\alpha(\lambda^*)}{\lambda - \lambda^*} = 2\gamma + (2\gamma\lambda - 2\gamma u_1 + 2v_1)\tau + O(\tau^2),$$

$$Z(\lambda^*) = -\frac{w_0}{2\gamma}\tau + O(\tau^2). \tag{3.36}$$

As a result, we get the following expansion

$$\frac{1}{\lambda - \lambda^*} M(\lambda|\lambda^*) = 2\gamma\mathbf{I} + \mathcal{M}(\lambda)\tau + O(\tau^2), \quad \mathcal{M}(\lambda) = \begin{pmatrix} 2\gamma\lambda - 2\gamma u_1 + 2v_1 & 1 \\ w_0 & 0 \end{pmatrix}. \tag{3.37}$$

Hence, for $\tau \to 0$, the intertwining relation (3.31) has the continuous limit $\frac{d}{dx}\hat{L}(\lambda) = [\hat{L}(\lambda), \mathcal{M}(\lambda)]$ describing the integrable $x-$flow on $\mathcal{O}_{\rm red}$ given by equations

$$\begin{aligned} &\frac{d}{dx}U(\lambda) = 2V(\lambda) - (2\gamma\lambda - 2\gamma u_1 + 2v_1)U(\lambda), \\ &\frac{d}{dx}V(\lambda) = w_0 U(\lambda) - W(\lambda), \\ &\frac{d}{dx}W(\lambda) = (2\gamma\lambda - 2\gamma u_1 + 2v_1)W(\lambda) - 2w_0 V(\lambda). \end{aligned}$$

By the above derivation, the restriction of the flow onto $\mathrm{Jac}(\tilde{\mathcal{C}})$ is tangent to $\tilde{\mathcal{C}}$ at ∞_2.

In the opposite special case $S^* = \infty_1$, according to Theorem 2.4 and formula (2.25), the map $\mathcal{B}_{\infty_1}$ is described by relation

$$\tilde{L}(\lambda)M(\lambda|\infty) = M(\lambda|\infty)L(\lambda), \tag{3.38}$$

where

$$\det M(\lambda|\lambda^*)\, M^{-1}(\lambda|\infty) = \begin{pmatrix} 2\gamma(\lambda-u) & 1 \\ \delta_2 & 0 \end{pmatrix}, \quad \text{and} \quad M(\lambda|\infty) = \begin{pmatrix} 0 & -1 \\ -\delta_2 & 2\gamma(\lambda-u) \end{pmatrix}$$

with some coefficients u, δ_2. This yields

$$\tilde{U}(\lambda) = \frac{1}{\delta_2}W(\lambda), \quad \tilde{V}(\lambda) + V(\lambda) - 2\gamma(\lambda-u)\tilde{U}(\lambda) = 0.$$

Comparing these equalities with expansions (3.29), we find $2\gamma u = 2\gamma\tilde{u}_1 - v_1 - \tilde{v}_1$, $\delta_2 = w_0$. Since $v_1 = \tilde{v}_1$, we obtain

$$M(\lambda|\infty) = \begin{pmatrix} 0 & -1 \\ -w_0 & 2\gamma\lambda - (2\gamma\tilde{u}_1 - 2\tilde{v}_1) \end{pmatrix}. \tag{3.39}$$

The matrix $M(\lambda|\infty)$ describes an explicit map $(U(\lambda), V(\lambda), W(\lambda)) \to (\tilde{U}(\lambda), \tilde{V}(\lambda), \tilde{W}(\lambda))$, hence $\mathcal{B}_{\infty_1}^{-1}$ and therefore $\mathcal{B}_{\infty_1}$ are explicit and single valued maps.

Remark 5. The matrix $M(\lambda|\infty)$ can also be obtained directly from $M(\lambda|\lambda^*)$ in (3.31), (3.32) by calculating its limit at ∞_1 similarly to (3.35), (3.36). The fact that α and Z transform to each other under the involution $\mathcal{I}$ leads to the following property: $M(\lambda|\infty)$ is the adjoint of $\mathcal{M}(\lambda)$ in (3.37) with u_1 replaced by $\tilde{u}_1$.

Example. A discrete Neumann system. A first integrable discretization of the classical Neumann system describing the motion of a point on the unit sphere $S^{n-1} = \{(q,q) = 1\}$, $q = (q_1, \ldots, q_n)$ under the action of a quadratic potential was found by Veselov [27] in the form of a Lagrangian correspondence $(q(N-1), q(N)) \to (q(N), q(N+1))$, $N \in \mathbb{N}$ being the discrete time. As shown in [26], by an appropriate introduction of the momentum $p = (p_1, \ldots, p_n)$, this correspondence is represented as a two-valued canonical map $(p,q) \to (\tilde{q}, \tilde{p})$, which has the same first integrals as the classical system on T^*S^{n-1}.

Another integrable discretization of the Neumann problem, was proposed by Ragnisco in [23, 24]. It has different first integrals, although, for some limiting initial conditions, passes to its continuous counterpart. In this discretization, the canonical variables $q, p \in \mathbb{R}^n$ are subject to constraints

$$(q,q) = 1, \quad (p,q) = 1/2 \tag{3.40}$$

and the corresponding map preserving the constrained symplectic structure has the following *explicit* form

$$\tilde{q} = \frac{1}{\Lambda}(\mathbf{a}q - \mathfrak{u}q - p), \quad \tilde{p} = \Lambda q, \tag{3.41}$$

where

$$\mathfrak{u} = (q, \mathbf{a}q) - 1, \quad \Lambda^2 = (\mathbf{a}q - uq - p)^2 \qquad \mathbf{a} = \operatorname{diag}(a_1, \ldots, a_n). \tag{3.42}$$

The latter can be extended to the map $\psi : \mathcal{F} \to \mathcal{F}$ of the $(2n-1)$-dimensional manifold $\mathcal{F} = \{p, q | (q, q) = 1\}$ to itself, if we replace $\mathfrak{u}$ by expression $(q, \mathbf{a}q) - 2(p, q)$. In this case the product (p, q) becomes a first integral of ψ.

As shown in [25], up to the action of the discrete group generated by reflections $(q_i, p_i) \to (-q_i, -p_i)$, the extended mapping can be written in 2×2 discrete Lax form

$$L(\lambda)\mathfrak{M}(\lambda) = \mathfrak{M}(\lambda)\tilde{L}(\lambda), \quad \mathfrak{M}(\lambda) = \begin{pmatrix} 0 & -1 \\ \Lambda^2 & \lambda - (q, \mathbf{a}q) + 2(p, q) \end{pmatrix}, \tag{3.43}$$

where $L(\lambda)$ has the form (3.26) with $\gamma = 1/2$ and $\tilde{L}(\lambda)$ depends on the new variables $\tilde{q}, \tilde{p}$ in the same way as $L(\lambda)$ depends on q, p. It follows that generic complex invariant manifolds of the map ψ are coverings of open subsets of Jacobian varieties of hyperelliptic curves (3.27) of genus $g = n - 1$.

Now comparing the Lax matrix (3.26) for $\gamma = 1/2$ with its polynomial form (3.28), we calculate

$$u_1 - 2v_1 = (q, \mathbf{a}q) - 2(p, q), \quad -w_0 = (p, p), \quad -\tilde{w}_0 = (\tilde{p}, \tilde{p}) = \Lambda^2.$$

It follows that the above operator $\mathfrak{M}(\lambda)$ coincides with the matrix $M(\lambda|\infty)$ in the discrete Lax pair (3.38), (3.39) describing the special single-valued map $\mathcal{B}_{\infty_1}$, if we replace the new (tilded) variables with the original ones and vise versa. Since $L(\lambda)$ and $\tilde{L}(\lambda)$ in (3.43) and in (3.38) are interchanged, we conclude that the Ragnisco map ψ is the inverse of $\mathcal{B}_{\infty_1}$. Thus we arrive at the following proposition.

Proposition 3.1. *The restriction of the map ψ onto $\operatorname{Jac}(\tilde{\mathcal{C}})$ is a translation by the vector $\zeta = \mathcal{A}(\infty_1) - \mathcal{A}(\infty_2)$.*

This proposition together with some simple observations enables one to construct explicit theta-functional solutions for the discrete Neumann system (3.41). Namely, let $\lambda_1, \ldots, \lambda_{n-1}$ and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-1}$ be the roots of the polynomials $U(\lambda)$ and $W(\lambda)$ respectively, or, equivalently, spheroconic coordinates on S^{n-1}, such that

$$q_i^2 = \frac{(a_i - \lambda_1) \cdots (a_i - \lambda_{n-1})}{\prod_{j \neq i}(a_i - a_j)}, \quad p_i^2 = -w_0 \frac{(a_i - \tilde{\lambda}_1) \cdots (a_i - \tilde{\lambda}_{n-1})}{\prod_{j \neq i}(a_i - a_j)}, \tag{3.44}$$
$$i = 1, \ldots, n.$$

Let $P_k = (\lambda_k, \mu_k), Q_k = (\tilde{\lambda}_k, \tilde{\mu}_k)$, $k = 1, \ldots, g$ be the corresponding point divisors on the hyperelliptic spectral curve (3.27) and

$$z = \sum_{k=1}^{g} \mathcal{A}(P_k), \quad \tilde{z} = \sum_{k=1}^{g} \mathcal{A}(Q_k)$$

be their images in $\operatorname{Jac}(\tilde{\mathcal{C}})$ under the Abel-Jacobi map with a basepoint P_0. Suppose P_0 is a branch point. Then, comparing (3.44) with the standard theta-functional expressions for so called root functions (*Wurzelfunktionen*) in the case of even order hyperelliptic curves (see e.g., [4, 8]), we find

$$q_i^2 = \varkappa_i \frac{\theta^2[\delta+\eta_{(i)}](z)}{\theta[\delta](z-\zeta/2)\,\theta[\delta](z+\zeta/2)}, \tag{3.45}$$

$$\frac{p_i^2}{w_0} = \nu_i \frac{\theta^2[\delta+\eta_{(i)}](\tilde z)}{\theta[\delta](\tilde z-\zeta/2)\,\theta[\delta](\tilde z+\zeta/2)}, \tag{3.46}$$

where $\theta[\delta+\eta_{(i)}](z), \theta[\delta](z)$ are theta-functions with the Riemann matrix B related to the curve (3.27) and appropriate half-integer theta-characteristics

$$\delta = (\delta'', \delta')^T, \quad \eta_{(i)} = (\eta''_{(i)}, \eta'_{(i)})^T \in \mathbb{R}^{2g}/2\mathbb{R}^{2g},$$

such that modulo the period lattice of $\tilde{\mathcal{C}}$ the following relations hold

$$2\pi\sqrt{-1}\,\eta''_{(i)} + B\eta'_{(i)} = \int_{P_0}^{(a_i,0)} \omega, \quad 2\pi\sqrt{-1}\,\delta'' + B\delta' = K, \tag{3.47}$$

K being the vector of the Riemann constants. Then, for any positive divisor $\mathcal{D}$ of degree $g-1$, $\theta[\delta](\mathcal{A}(\mathcal{D})) = 0$ (see e.g., [10, 20]). Finally, $\varkappa_i, \nu_i$ are constant factors depending on the moduli of the curve only.

According to relation (3.30), the above arguments $z, \tilde z$ are different by the constant vector ζ: $\tilde z = z - \zeta$.

Since a meromorphic function on $\tilde{\mathcal{C}}$ is determined by its divisor of zeros and poles uniquely up to a constant factor, in view of (3.30) and of a known theorem on its theta-functional representation (see e.g., [20]), we have

$$\begin{aligned} \left.\frac{V(\lambda)+\mu}{U(\lambda)}\right|_{(\lambda,\mu)\in\tilde{\mathcal{C}}} &= \left.\frac{W(\lambda)}{V(\lambda)-\mu}\right|_{(\lambda,\mu)\in\tilde{\mathcal{C}}} \\ &= f(z)\frac{\theta[\delta](\mathcal{A}(P)-\tilde z)\,\theta[\delta](\mathcal{A}(P)-e-\mathcal{A}(\infty_1))}{\theta[\delta](\mathcal{A}(P)-z)\,\theta[\delta](\mathcal{A}(P)-e-\mathcal{A}(\infty_2))}, \end{aligned} \tag{3.48}$$

where $P=(\lambda,\mu)$, $f(z)$ is constant on $\tilde{\mathcal{C}}$, and $e\in\mathbb{C}^g$ is any constant vector such that $\theta[\delta](e)=0$.

The factor $w_0 = -(p,p)$ in (3.46) is a meromorphic function on $\mathrm{Jac}(\tilde{\mathcal{C}})$. In order to find its theta-functional expression, we notice that in a neighborhood of the infinite points with a local coordinate $\tau = 1/\lambda$ the following expansions hold (compare with (3.35), (3.36))

$$\text{near } \infty_1\text{:} \quad \frac{W(\lambda)}{V(\lambda)-\mu} = w_0\tau + O(\tau^2), \quad \text{near } \infty_2\text{:} \quad \frac{V(\lambda)+\mu}{U(\lambda)} = \frac{1}{\tau} + O(1).$$

Matching them with corresponding expansions of the right hand side of (3.48), we find

$$w_0 = \xi\frac{\theta[\delta](z^*-\zeta)\,\theta[\delta](z^*+\zeta)}{\theta^2[\delta](z^*)}, \qquad z^* = z-\zeta/2,$$

ξ being a constant independent of z. This, together with formulas (3.45), (3.46) and the relation between z and $\tilde z$, gives us a complete theta-functional parameterization of the squared components q_i^2, p_i^2.

Now, as above, let $q(N), p(N)$ denote the result of N-th iteration of the map ψ. Applying Proposition 3.1, we arrive at the following result.

Theorem 3.2. *Generic solution to the discrete Neumann system (3.41) has the form*

$$q_i^2(N) = \varkappa_i \frac{\theta^2[\delta+\eta_{(i)}](z+N\zeta)}{\theta[\delta](z+N\zeta-\zeta/2)\,\theta[\delta](z+N\zeta+\zeta/2)}, \tag{3.49}$$

$$p_i^2(N) = \epsilon_i \frac{\theta[\delta](z+\zeta/2+N\zeta)\,\theta^2[\delta+\eta_{(i)}](z+(N-1)\zeta)}{\theta^3[\delta](z+N\zeta-\zeta/2)}, \tag{3.50}$$

$$i=1,\dots,n, \quad N\in\mathbb{Z},$$

where now z plays the role of a constant phase vector of a discrete Neumann trajectory and $\varkappa_i, \epsilon_i = \xi\nu_i$ are constant factors depending on the moduli of the spectral curve.

The factors can be found by applying these formulas to the initial conditions. It is seen that q_i^2 has simple poles along two translates of the theta-divisor $\Theta \subset \mathrm{Jac}(\tilde{\mathcal{C}})$, whereas p_i^2 has a triple pole along one of the translates.

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