A Nonlocal Kac-van Moerbeke Equation Admitting N-Soliton Solutions

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Abstract

Using our previous work on reflectionless analytic difference operators and a nonlocal Toda equation, we introduce analytic versions of the Volterra and Kac-van Moerbeke lattice equations. The real-valued N-soliton solutions to our nonlocal equations correspond to self-adjoint reflectionless analytic difference operators with N bound states. A suitable scaling limit gives rise to the N-soliton solutions of the Korteweg-de Vries equation.

1 Introduction

In a recent paper we introduced and studied a large class of analytic difference operators admitting reflectionless eigenfunctions [1]. Subsequently, we tied in these results on reflectionless analytic difference operators (henceforth AΔOs) with several soliton systems, both finite-dimensional and infinite-dimensional [2]. In particular, we introduced an analytic version of the infinite Toda lattice [3, 4, 5]

$$\ddot{q}_n(t) = \exp\big[q_{n-1}(t) - q_n(t)\big] - \exp\big[q_n(t) - q_{n+1}(t)\big], \quad n \in \mathbb{Z}, \tag{1.1}$$

namely,

$$i\ddot{\Psi}(x,t) = \exp\big[-i\Psi(x-i,t) + i\Psi(x,t)\big] - \exp\big[-i\Psi(x,t) + i\Psi(x+i,t)\big]. \tag{1.2}$$

We showed that (1.2) admits N-soliton solutions, established their relation to the relativistic Calogero-Moser systems introduced in Ref. [6], and used this relation to clarify their long-time asymptotics.

In this paper we study the nonlocal evolution equation

$$i\dot{F}(x,t) = F(x,t)[F(x-i/2,t) - F(x+i/2,t)], \tag{1.3}$$

and its logarithmic version

$$i\dot{Y}(x,t) = \exp[Y(x-i/2,t)] - \exp[Y(x+i/2,t)], \tag{1.4}$$

along similar lines. These equations may be viewed as analytic versions of the infinite Volterra (or Langmuir) lattice equations [7, 8, 5]

$$\dot{V}_l(t) = V_l(t)[V_{l-1/2}(t) - V_{l+1/2}(t)], \quad l \in \mathbb{Z}/2, \tag{1.5}$$

and the infinite Kac-van Moerbeke lattice equations [9]

$$\dot{K}_l(t) = \exp[K_{l-1/2}(t)] - \exp[K_{l+1/2}(t)], \quad l \in \mathbb{Z}/2. \tag{1.6}$$

(See also Suris' forthcoming monograph Ref. [10] for an up-to-date and comprehensive account of these lattice systems and a host of other ones.)

Little appears to be known concerning the general theory of nonlocal evolution equations such as (1.2)–(1.4). (Some other special cases are discussed in a review by Santini [11].) In particular, we are not aware of results on existence and uniqueness of solutions to a suitably formulated Cauchy problem. Clearly, the latter should involve initial data and solutions with analyticity properties such that the shifts into the complex plane make sense. For (1.3) one can, for instance, require that $F(x, 0)$ be meromorphic and look for solutions $F(x, t)$ that are meromorphic in x for fixed t, whereas for (1.2) and (1.4) one may restrict attention to data and solutions such that $\exp[i\Psi(x, t)]$ and $\exp[Y(x, t)]$ are meromorphic in x for t fixed.

The solutions to (1.2) introduced in Ref. [2] and the solutions to (1.3) and (1.4) constructed below all have these meromorphy properties. Moreover, $\exp[i\Psi(x, t)]$ has limit 1 for $\operatorname{Re} x \to \infty$ and limit $c \in \mathbb{C}^*$ for $\operatorname{Re} x \to -\infty$, whereas $F(x, t)$ and $\exp[Y(x, t)]$ have limit 1 as $\operatorname{Re} x \to \pm\infty$. The intimate relation to our previous work on (1.2) can be expressed directly in terms of the solutions $F(x, t)$ to (1.3) at issue here. Indeed, they can all be written as

$$F(x, t) = \exp[i\Psi(x + i/4, t) - i\Psi(x - i/4, t)], \tag{1.7}$$

where $\Psi(x, t)$ is a solution to (1.2). It should be emphasized that no such relation is apparent from a comparison of (1.2) and (1.3). That is, at face value there seems to be no reason why *any* solution $\Psi(x, t)$ to (1.2) that is not i-periodic in x would yield a solution $F(x, t)$ to (1.3) via (1.7) or any other formula.

In Section 2 we define solutions to (1.3) in terms of complex numbers $r_1, \ldots, r_N$ and meromorphic i-periodic functions $\mu_1(x), \ldots, \mu_N(x)$ satisfying some further restrictions. Thus, we obtain an infinite-dimensional space of solutions already for $N = 1$. Following Section 4 in Ref. [2], we also describe how one obtains N-soliton solutions to the Volterra lattice equations (1.5) by a suitable analytic continuation of our spectral data and a subsequent x-discretization. (These soliton solutions were first obtained by Manakov [7].)

In Section 3 we introduce and study N-soliton solutions to our analytic versions (1.3) and (1.4) of the Volterra and Kac-van Moerbeke lattice equations (1.5) and (1.6). In order to get 'traveling wave' characteristics, we need to choose $\mu_n(x), n = 1, \ldots, N$, constant. It is a remarkable fact that some further reality conditions on the spectral data ensure that these nonlocal Kac-van Moerbeke solitons are real-valued for x and t real. In this connection we should stress that our solitons cannot be viewed as interpolations of the lattice solitons. (The distinction is most easily visible from the soliton speed function: In our case it takes values in $(0, 1)$, whereas it takes values in $(1, \infty)$ for the lattice solitons.)

Just as for the nonlocal Toda solitons (and the soliton solutions to a host of other equations [6, 12]), our N-soliton solutions to (1.3) and (1.4) can be parametrized by the canonical coordinates of suitable relativistic N-particle Calogero-Moser systems. This parametrization ensures that the soliton scattering transformation is a canonical map [12], and enables us to use results from Ref. [13] to study long-time asymptotics. We were however unable to prove a conjecture involving a uniform exponential decay bound, cf. (3.43) below. (Analogs of this conjectured bound hold true in various other cases [13].)

In Section 4 we show that a suitable limit of our analytic Volterra solitons yields the KdV solitons. It is however far from clear whether this is a general phenomenon. More precisely, we do not know whether a similar limit holds true for arbitrary solutions. For (a suitable interpolation of) lattice Volterra solutions, the relevant limit was rigorously established by Schwarz [14] (he attributes the key substitution for the 1-soliton limit to M. Kac). Several other aspects of the Volterra lattice $\rightarrow$ KdV limit are studied in Refs. [15, 16, 17]. Again, one may ask whether similar results hold true for our analytic versions.

To conclude this introduction, we would like to mention that this paper (as well as our previous ones Refs. [1, 2]) owes much to earlier work on the discrete difference (Jacobi) operator/lattice equation analogs of the analytic objects we study. Indeed, without the extensive knowledge gathered on the discrete versions, there would have been no clue as to what might be the case for the analytic analogs at hand. This is especially true for the result detailed in Section 4, which involves the key idea to make a time-dependent shift of x ('Galilei boost'), so as to arrive at the KdV solitons. (We learned about the relation of the Volterra lattice solitons to their KdV cousins from M. Musette.)

2   Solutions related to reflectionless AΔOs

Our main purpose in this section is to construct and study meromorphic solutions to (1.3) that depend on N numbers $r_1, \ldots, r_N$ satisfying

$$\operatorname{Im} r_n \in (0, \pi), \quad n = 1, \ldots, N, \quad r_k \neq r_l, \quad k \neq l, \tag{2.1}$$

and N meromorphic functions $\mu_1(x), \ldots, \mu_N(x)$ satisfying

$$\mu_n(x + i) = \mu(x), \quad \lim_{|\operatorname{Re} x| \to \infty} \mu_n(x) = c_n, \quad c_n \in \mathbb{C}^*, \quad n = 1, \ldots, N. \tag{2.2}$$

The key ingredient for doing so is the meromorphic solution $R(r, \mu; x)$ to the linear system

$$(D(r, \mu; x) + C(r))R(x) = (1, \ldots, 1)^t. \tag{2.3}$$

Here, C is the Cauchy matrix

$$C(r)_{mn} \equiv \frac{1}{e^{r_m} - e^{-r_n}}, \quad m, n = 1, \ldots, N, \tag{2.4}$$

and D the diagonal matrix

$$D(r, \mu; x) \equiv \operatorname{diag}(\mu_1(x) \exp(-2ir_1 x), \ldots, \mu_N(x) \exp(-2ir_N x)). \tag{2.5}$$

Using the solution R, we introduce two AΔOs of the form

$$S \equiv T_{i/2} + V(x)T_{-i/2}, \tag{2.6}$$

$$A \equiv T_i + V_a(x)T_{-i} + V_b(x), \tag{2.7}$$

where T_α is the translation over $-\alpha$,

$$T_\alpha \equiv \exp(-\alpha\partial_x), \quad \alpha \in \mathbb{C}^*. \tag{2.8}$$

Specifically, the potentials V and V_a, V_b are defined in terms of the auxiliary functions

$$\lambda(r, \mu; x) \equiv 1 + \sum_{n=1}^{N} e^{r_n} R_n(r, \mu; x), \tag{2.9}$$

$$\Sigma(r, \mu; x) \equiv \sum_{n=1}^{N} R_n(r, \mu; x), \tag{2.10}$$

namely as

$$V(r, \mu; x) \equiv \lambda(r, \mu; x)/\lambda(r, \mu; x + i/2), \tag{2.11}$$

$$V_a(r, \mu; x) \equiv V(r, \mu; x)V(r, \mu; x + i/2) = \lambda(r, \mu; x)/\lambda(r, \mu; x + i), \tag{2.12}$$

$$V_b(r, \mu; x) \equiv \Sigma(r, \mu; x - i) - \Sigma(r, \mu; x). \tag{2.13}$$

For the special case $N = 1$ one can easily verify directly that V admits an alternative representation

$$V(r, \mu; x) = \Sigma(r, \mu; x - i/2) - \Sigma(r, \mu; x) + 1. \tag{2.14}$$

As a consequence, one obtains the relation

$$S(r, \mu)^2 = A(r, \mu) + 2. \tag{2.15}$$

As a matter of fact, these equalities hold true for arbitrary N. Moreover, the reflectionless wave function

$$\mathcal{W}(r, \mu; x, p) \equiv e^{ixp}\left(1 - \sum_{n=1}^{N} \frac{R_n(r, \mu; x)}{e^p - e^{-r_n}}\right), \tag{2.16}$$

is an eigenfunction of S and A:

$$S\mathcal{W} = 2\mathrm{ch}(p/2)\mathcal{W}, \quad A\mathcal{W} = 2\mathrm{ch}(p)\mathcal{W}. \tag{2.17}$$

(This can also be readily checked for $N = 1$.)

The assertions made in the previous paragraph are proved in Theorem 3.3 of Ref. [1]. Next, we introduce time-dependent multipliers

$$\mu_n(r_n; x, t) \equiv \mu_n(x)\exp(it[e^{r_n} - e^{-r_n}]), \quad n = 1, \ldots, N, \tag{2.18}$$

and invoke results from Ref. [2]. From now on we prefix equations from the latter articles by I and II, resp.

The time dependence (2.18) is the same as that considered in Ref. [2], cf. II(2.10). (The key restriction compared to *loc. cit.* is that we require all r_j to have positive imaginary part—this is necessary for our proof of the relation (2.14).) We may therefore invoke II(2.26) to obtain the time derivative

$$\dot{\Sigma}(x,t) = i[1 - V_a(x,t)]. \tag{2.19}$$

(Here and below, the time dependence arises by taking $\mu_n(x) \to \mu_n(r_n; x, t)$.) From (2.14) and (2.12) we now deduce

$$
\begin{aligned}
\dot{V}(x,t) &= -iV_a(x - i/2, t) + iV_a(x,t) \\
&= iV(x,t)[V(x + i/2, t) - V(x - i/2, t)].
\end{aligned} \tag{2.20}
$$

We have, therefore, obtained an extensive class of solutions to the nonlocal Volterra equation (1.3). Before discussing these in more detail, it is of interest to observe that *any* solution to (2.20) gives rise to a solution of the coupled nonlocal evolution equations

$$\dot{V}_a(x,t) = iV_a(x,t)[V_b(x + i, t) - V_b(x,t)], \tag{2.21}$$

$$\dot{V}_b(x,t) = iV_a(x,t) - iV_a(x - i, t). \tag{2.22}$$

Indeed, for a given solution $V(x,t)$, we need only set

$$V_a(x,t) \equiv V(x,t)V(x + i/2, t), \quad V_b(x,t) \equiv V(x - i/2, t) + V(x,t) - 2, \tag{2.23}$$

to obtain (2.21) and (2.22) as a corollary of (2.20).

Now the 'spectral data' (r, μ) satisfying (2.1) and (2.2) not only give rise to solutions of (2.20), but also to solutions $\Psi(x,t)$ of the nonlocal Toda type evolution equation (1.2). Specifically, we may take

$$\Psi(x,t) = i \ln \lambda(x,t), \tag{2.24}$$

cf. II(2.41), (2.42). As we have just shown, a more general solution $V(x,t)$ to (2.20) gives rise to a solution to the Flaschka type system (2.21), (2.22). In turn, from solutions to the latter one can try and construct solutions to (1.2) by solving first the analytic difference equation

$$i\Psi(x + i, 0) - i\Psi(x, 0) = \ln V_a(x, 0), \tag{2.25}$$

and then setting

$$\Psi(x,t) = \int_0^t V_b(x, s)ds + \Psi(x, 0). \tag{2.26}$$

In the absence of information on the Cauchy problem for the nonlocal evolution equations at hand, it is however not clear that this yields a solution to (1.2).

Returning to the special solutions introduced above, we define

$$F(x,t) \equiv V(x - i/4, t). \tag{2.27}$$

Clearly, $F(x,t)$ obeys (1.3). In view of (2.11) it is related to the Toda solution (2.24) by (1.7). The point of the x-shift in (2.27) is that $F(x,t)$ takes values in $[0,\infty]$ for x,t real whenever r,μ satisfy

$$r_n \in i(0,\pi), \quad n = 1,\ldots,N, \tag{2.28}$$

$$i\exp(-r_n)\mu_n(x) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad n = 1,\ldots,N, \tag{2.29}$$

in addition to (2.1) and (2.2).

To explain why this remarkable feature holds true, we recall from Ref. [1] that the conditions (2.28), (2.29) entail

$$\lambda^*(x) = 1/\lambda(x), \tag{2.30}$$

cf. I(D.17). (Here, the $*$ denotes the conjugate, i.e., $\lambda^*(x) \equiv \overline{\lambda(\bar{x})}, x \in \mathbb{C}$.) From (2.11) we then have

$$F(x,t) = |\lambda(x - i/4, t)|^2, \quad (x,t) \in \mathbb{R}^2, \tag{2.31}$$

whence nonnegativity is evident.

At this point we would like to remark that the relation

$$V^*(x) = V(x - i/2), \tag{2.32}$$

which follows from (2.30), is the necessary and sufficient condition for the AΔO S (2.6) to be formally self-adjoint on $L^2(\mathbb{R}, dx)$, cf. also Appendix D in Ref. [1]. (But we can only prove self-adjointness in a rigorous, functional-analytic sense when the functions $\mu_1(x),\ldots,\mu_N(x)$ are constant in addition to the requirements (2.1), (2.2), (2.28) and (2.29), cf. [18].) Furthermore, (2.30) entails that the nonlocal Toda solution $\Psi(x,t)$ (2.24) can be chosen real-valued for x,t real.

In the next section we shall show that further restrictions on r,μ yield solutions $F(x,t)$ that are not only positive for real x,t, but that may also be viewed as N-soliton solutions. To conclude this section, we sketch how a quite different class of solutions to the nonlocal, analytic equations (1.3) gives rise to the N-soliton solutions of the Volterra lattice equations (1.5).

The pertinent class is defined by spectral data

$$r_j = e^{i\eta}\kappa_j, \quad \mu_j(x) = \exp(r_j)/\nu_j, \quad j = 1,\ldots,N. \tag{2.33}$$

Here, we fix

$$0 < \kappa_N < \cdots < \kappa_1, \quad \nu_1,\ldots,\nu_N \in (0,\infty), \tag{2.34}$$

and we make the connection to (1.5) via the $\eta \downarrow 0$ limit. Thus, $r_1,\ldots,r_N$ converge to the lower boundary of the strip in (2.1). We have already studied this limit for the (more general) Toda case in Section 4 of Ref. [2], so we only indicate how the reasoning detailed there is to be supplemented.

The main extension that is needed consists in replacing $n \in \mathbb{Z}$ in *loc. cit.* by $l \in \mathbb{Z}/2$. Since we are dealing here with the special case $N_- = 0$, all of the quantities $z_j \equiv$

$\exp(-\kappa_j), j = 1, \ldots, N$, are positive. Therefore, the matrix $L^J(l), l \in \mathbb{Z}/2$, given by II(4.17), is still similar to a positive matrix, which entails that all of our arguments extend to the lattice $\mathbb{Z}/2$. (Note the typo in II(4.20) and II(4.21): On the rhs i should be replaced by 1.)

In this way one obtains two 'non-communicating' Toda systems, namely, one for $l \in \mathbb{Z}$ and one for $l \in \mathbb{Z} + 1/2$. (Of course, the second one can just as well be viewed as living on the lattice $\mathbb{Z}$, by changing the normalization coefficients ν_j to $\nu_j z_j$, cf. II(4.19).)

The relation between these two systems arises from a consideration of the limits of $V(x)$ and S. Specifically, combining (2.11) and (the $l \in \mathbb{Z}/2$ version of) II(4.23), one gets

$$\lim_{\eta \downarrow 0} V(ie^{-i\eta}l) = \lambda^J(l)/\lambda^J(l + 1/2) \equiv V_l^2, \quad l \in \mathbb{Z}/2, \tag{2.35}$$

and the S-limit yields the discrete difference equation

$$\hat{\mathcal{W}}^J(l - 1/2, p) + V_l^2 \hat{\mathcal{W}}^J(l + 1/2, p) = 2\cos(p/2)\hat{\mathcal{W}}^J(l, p), \quad l \in \mathbb{Z}/2. \tag{2.36}$$

In view of (2.12)–(2.14) and II(4.24), (4.25), one also gets

$$2a(l) = V_l V_{l+1/2}, \quad 2b(l) = V_{l-1/2}^2 + V_l^2 - 2. \tag{2.37}$$

Making the similarity transformation II(4.37) (with $n \to l \in \mathbb{Z}/2$), one winds up with the self-adjoint discrete difference operator

$$(D_{KM}f)(l) = V_{l-1/2}f(l - 1/2) + V_l f(l + 1/2), \tag{2.38}$$

on $l^2(\mathbb{Z}/2)$. The operator $D_{KM}^2/2 - 1$ leaves the subspaces $l^2(\mathbb{Z})$ and $l^2(\mathbb{Z} + 1/2)$ invariant. Thus it gives rise to two self-adjoint Jacobi operators with nonzero diagonal and off-diagonal elements $b(l)$ and $a(l)$, where $l \in \mathbb{Z}$ and $l \in \mathbb{Z} + 1/2$, respectively.

Finally, taking the time dependence into account, one should take $t \to it$ (just as in *loc. cit.*), so as to obtain positive N-soliton solutions $V_l(t)$ to the Volterra lattice (1.5), which depend on the numbers (2.34). Their logarithms $K_l(t)$ then yield real-valued N-soliton solutions to the Kac-van Moerbeke lattice (1.6).

3   A study of the N-soliton solutions

We begin this section by studying the general $N = 1$ solution. From (2.3)–(2.5) we have

$$R_1(x, t) = \left(\mu_1(x, t)\exp(-2ir_1x) + [2\mathrm{sh}r_1]^{-1}\right)^{-1}. \tag{3.1}$$

Using (2.14) and (2.18) we now calculate

$$
\begin{aligned}
V(x, t) - 1 &= R_1(x - i/2, t) - R_1(x, t) \\
&= 2\mathrm{sh}r_1[1 + 2\mathrm{sh}(r_1)\mu_1(x - i/2)\exp(-2ir_1(x - i/2) + 2it\mathrm{sh}r_1)]^{-1} \\
&\quad -2\mathrm{sh}r_1[1 + 2\mathrm{sh}(r_1)\mu_1(x)\exp(-2ir_1x + 2it\mathrm{sh}r_1)]^{-1}. \tag{3.2}
\end{aligned}
$$

Obviously, this function is not of the traveling wave form $f(x - vt)$ unless we choose $\mu_1(x)$ constant. Doing so, we also set

$$r_1 = i\alpha, \quad \alpha \in (0, \pi), \quad \mu_1(x) = \exp(-2\alpha x_0)/2ie^{-i\alpha}\sin\alpha, \quad x_0 \in \mathbb{R}, \tag{3.3}$$

so that (2.28) and (2.29) are satisfied. Then we get

$$
\begin{aligned}
F(x,t) &\equiv V(x - i/4, t) \\
&= 1 + 2i \sin \alpha [(1 + e^{-i\alpha/2} \exp[2\alpha(x - x_0) - 2t \sin \alpha])^{-1} \\
&\quad -(1 + e^{i\alpha/2} \exp[2\alpha(x - x_0) - 2t \sin \alpha])^{-1}].
\end{aligned}
\tag{3.4}
$$

Defining a velocity function

$$
v(\alpha) \equiv \frac{\sin \alpha}{\alpha}, \quad \alpha \in (0, \pi),
\tag{3.5}
$$

this can be rewritten as

$$
F(x,t) = 1 - \frac{2 \sin(\alpha) \sin(\alpha/2)}{\cos(\alpha/2) + \mathrm{ch}[2\alpha(x - x_0 - v(\alpha)t)]}.
\tag{3.6}
$$

The upshot is that $F(x,t) - 1$ has the usual characteristics of a soliton. Moreover, the function $F(x,t), (x,t) \in \mathbb{R}^2$, takes values in $[0,1)$ for all $\alpha \in (0,\pi)$ and has no zeros for $\alpha \neq 2\pi/3$, whereas for $\alpha = 2\pi/3$ and t fixed it has a unique zero. Thus,

$$
Y(x,t) \equiv \ln F(x,t), \quad x, t \in \mathbb{R},
\tag{3.7}
$$

takes values in $[-\infty, 0)$. When we let x move into the complex plane, the function $Y(x,t)$ becomes multi-valued, but the $2\pi i$-multiples play no role in the Kac-van Moerbeke type evolution equation (1.4) it obeys.

Next, we fix $N > 1$ and specialize to spectral data

$$
r_n = i\alpha_n, \quad n = 1, \ldots, N, \quad 0 < \alpha_1 < \cdots < \alpha_N < 2\pi/3,
\tag{3.8}
$$

$$
\mu_n = -i \exp(i\alpha_n)\gamma_n, \quad \gamma_n \in (0, \infty), \quad n = 1, \ldots, N.
\tag{3.9}
$$

As we have seen in the previous section, this ensures that $F(x,t)$ is nonnegative for real x and t, cf. (2.31). In fact, this is still the case when we replace $2\pi/3$ in (3.8) by π. But we need (3.8) to guarantee absence of zeros,

$$
F(x,t) \in (0, \infty), \quad \forall (x,t) \in \mathbb{R}^2,
\tag{3.10}
$$

a feature that will be proved shortly.

Taking (3.10) for granted, we obtain a real-valued function $Y(x,t)$ via (3.7), whose (multi-valued) analytic continuation satisfies the nonlocal Kac-van Moerbeke equation (1.4). As we will show, it can be viewed as an N-soliton solution.

Turning to the details, it is convenient to trade the parameters α_n and γ_n for the generalized positions q_n and momenta θ_n of suitable relativistic Calogero-Moser systems. Comparing to the general setup in Section 5 of Ref. [2], we see that we should first set

$$
q_n \equiv \ln(\cot(\alpha_n/2)), \quad n = 1, \ldots, N.
\tag{3.11}
$$

Defining now

$$
V_n(q) \equiv \prod_{1 \leq k \leq N, k \neq n} |\coth[(q_n - q_k)/2]|,
\tag{3.12}
$$

we should choose

$$\theta_n \equiv \ln(2\gamma_n V_n(q)/\mathrm{ch}q_n), \quad n = 1, \ldots, N. \tag{3.13}$$

Next, we recall how the key quantity $\lambda(r, \mu; x)$ (2.9) can be expressed in terms of the relativistic Calogero-Moser Lax matrix

$$\mathcal{L}(q, \theta) \equiv \mathcal{C}(q)\mathcal{D}(q, \theta), \tag{3.14}$$

where

$$\mathcal{C}_{jk} \equiv 1/\mathrm{ch}[(q_j - q_k)/2], \quad j, k = 1, \ldots, N, \tag{3.15}$$

$$\mathcal{D} \equiv \mathrm{diag}(\exp(\theta_1)V_1(q), \ldots, \exp(\theta_N)V_N(q)). \tag{3.16}$$

To this end we quote the definition of the τ-function

$$\tau(r, \mu; x) \equiv |\mathbf{1}_N + C(r)D(r, \mu; x)^{-1}| \tag{3.17}$$

(cf. II(2.32)), and the relation

$$\lambda(r, \mu; x) = \tau(r, \mu; x - i)/\tau(r, \mu; x), \tag{3.18}$$

which follows from I(C.30). We now replace μ_n by $\mu_n \exp(-2t \sin \alpha_n)$, and switch from (α, γ) to (q, θ). Moreover, we set

$$\theta_n(x, t) \equiv \theta_n - 2\alpha_n(q)[x - v(\alpha_n(q))t], \quad n = 1, \ldots, N, \tag{3.19}$$

$$\alpha_n(q) \equiv 2\mathrm{Arctan}(\exp(-q_n)), \quad n = 1, \ldots, N. \tag{3.20}$$

Employing the self-adjoint Lax matrix

$$L_s(x, t) \equiv \mathcal{D}(q, \theta(x, t))^{1/2}\mathcal{C}(q)\mathcal{D}(q, \theta(x, t))^{1/2}, \tag{3.21}$$

we now get

$$\tau(x, t) = |\mathbf{1}_N + L_s(x + i/2, t)|, \tag{3.22}$$

cf. II(6.1), (6.2).

Combining (2.27), (2.11), (3.18) and (3.22), we obtain

$$F(x, t) = \frac{|\mathbf{1}_N + L_s(x + 3i/4, t)||\mathbf{1}_N + L_s(x - 3i/4, t)|}{|\mathbf{1}_N + L_s(x + i/4, t)||\mathbf{1}_N + L_s(x - i/4, t)|}. \tag{3.23}$$

Introducing

$$\mathcal{U} \equiv \mathrm{diag}(\exp(-i\alpha_1/2), \ldots, \exp(-i\alpha_N/2)), \tag{3.24}$$

this can be rewritten as

$$F(x, t) = \frac{|\mathbf{1}_N + L_s(x, t)\mathcal{U}^3||\mathbf{1}_N + \mathcal{U}^{-3}L_s(x, t)|}{|\mathbf{1}_N + L_s(x, t)\mathcal{U}||\mathbf{1}_N + \mathcal{U}^{-1}L_s(x, t)|}. \tag{3.25}$$

With these reparametrizations in effect, we proceed to prove the positivity property (3.10).

Recalling (2.31), we see that it suffices to prove invertibility of the matrices $\mathbf{1}_N + L_s(x, t)\mathcal{U}^k, k = 1, 3$. Now $L_s(x, t)$ is not only self-adjoint, but also regular. (This follows from Cauchy's identity.) Thus we are entitled to invoke the following lemma.

Lemma 3.1. *Assume L is a self-adjoint regular $N \times N$ matrix and D a diagonal matrix of the form*

$$D \equiv \mathrm{diag}(\exp(i\eta_1), \ldots, \exp(i\eta_N)), \quad \eta_n \in (-\pi, 0), \quad n = 1, \ldots, N. \tag{3.26}$$

Then the matrix $\mathbf{1}_N + LD$ is regular.

Proof. Clearly, we need only show that the matrix $L^{-1} + D$ is regular. Assume it is singular. Then there exists a nonzero $\phi \in \mathbb{C}^N$ such that $(L^{-1} + D)\phi = 0$. Since L^{-1} is self-adjoint, we obtain

$$0 = \mathrm{Im}\,(\phi, (L^{-1} + D)\phi) = \mathrm{Im}\,(\phi, D\phi) = \sum_{n=1}^{N} |\phi_n|^2 \sin \eta_n. \tag{3.27}$$

But since $\eta_n \in (-\pi, 0)$, we have $\sin \eta_n < 0$. Hence $\phi = 0$, a contradiction. $\square$

From this lemma we see that $|\mathbf{1}_N + L_s(x, t)\mathcal{U}|, (x, t) \in \mathbb{R}^2$, has no zeros even when we replace $2\pi/3$ in (3.8) by π. But we need this restriction to ensure that $|\mathbf{1}_N + L_s(x, t)\mathcal{U}^3|, (x, t) \in \mathbb{R}^2$, cannot vanish either. Indeed, for the special case $N = 1$ we have already seen that the choice $\alpha = 2\pi/3$ does give rise to zeros, cf. (3.6). With $N > 1$ and $2\pi/3$ replaced by π in (3.8), it is quite likely that zeros occur for $\alpha_N \geq 2\pi/3$. We require $\alpha_N < 2\pi/3$, since we would like to avoid such nongeneric zeros, which are hard to control.

We continue by studying the long-time asymptotics of the above positive functions $F(x, t)$. To this end we introduce the 'one-soliton function'

$$f(q, \theta; x) \equiv 1 - \frac{2\sin(\alpha)\sin(\alpha/2)}{\cos(\alpha/2) + \mathrm{ch}(2\alpha x - \theta)}, \quad \alpha \equiv 2\mathrm{Arctan}(e^{-q}), \tag{3.28}$$

and proceed along the same lines as in Section 6 of Ref. [2]. (Actually, most of the results that follow might also be obtained as corollaries of *loc. cit.* This is a consequence of the relation (1.7) between $F(x, t)$ and the nonlocal Toda N-soliton solution $\Psi(x, t)$ corresponding to the initial point (q, θ) in the relativistic Calogero-Moser phase space.)

Proposition 3.2. *Fixing $x_0, s_0 \in \mathbb{R}$, one has*

$$\lim_{\delta t \to \infty} F(x_0 + s_0 t, t) = 1, \quad s_0 \notin \{v(\alpha_1), \ldots, v(\alpha_N)\}, \tag{3.29}$$

$$\lim_{\delta t \to \infty} F(x_0 + s_0 t, t) = f(q_j, \theta_j; x_0 + \delta \Delta_j(q)/4\alpha_j), \quad s_0 = v(\alpha_j), \quad j = 1, \ldots, N, \tag{3.30}$$

where $\delta = +, -$ and

$$\Delta_j(q) \equiv \left(\sum_{k<j} - \sum_{k>j}\right) \ln(\coth^2[(q_j - q_k)/2]), \quad j = 1, \ldots, N. \tag{3.31}$$

Proof. We only detail the case $\delta = +$, since the case $\delta = -$ can be handled in the same way. We begin by introducing

$$M^+ \equiv L_s(x_0, 0)\mathcal{U}^3, \quad M^- \equiv L_s(x_0, 0)\mathcal{U}, \tag{3.32}$$

$$D \equiv \mathrm{diag}(2\alpha_1[v(\alpha_1) - s_0], \ldots, 2\alpha_N[v(\alpha_N) - s_0]). \tag{3.33}$$

Using (3.25), we then obtain

$$F(x_0 + s_0 t, t) = |Q(t)|^2, \tag{3.34}$$

with

$$Q(t) \equiv |\mathbf{1}_N + M^+ e^{tD}|/|\mathbf{1}_N + M^- e^{tD}|. \tag{3.35}$$

We are now in the position to invoke Lemma 6.2 in Ref. [2]. The pertinent principal minors can be written as

$$|M_n^+| = |L_s(x_0, 0)_n| \exp(-3i \sum_{j=1}^{n} \alpha_j/2), \tag{3.36}$$

$$|M_n^-| = |L_s(x_0, 0)_n| \exp(-i \sum_{j=1}^{n} \alpha_j/2). \tag{3.37}$$

For $s_0 \neq v(\alpha_j), j = 1, \ldots, N$, we can use (cf. II(6.25))

$$\lim_{t \to \infty} Q(t) = |M_n^+|/|M_n^-|, \tag{3.38}$$

to infer that $Q(t)$ tends to a phase as $t \to \infty$. Hence (3.29) is clear from (3.34). For $s_0 = v(\alpha_{n+1}), n \in \{0, \ldots, N-1\}$, we have (cf. II(6.27))

$$\lim_{t \to \infty} Q(t) = (|M_n^+| + |M_{n+1}^+|)/(|M_n^-| + |M_{n+1}^-|). \tag{3.39}$$

Calculating the principal minors $|L_s(x_0, 0)_j|$ via Cauchy's identity, we now arrive at (3.30) for $j = n + 1$. $\square$

For the 1-soliton solution (3.6) we have $F(x, t) < 1$ for all $(x, t) \in \mathbb{R}^2$. In view of the asymptotics just proved, this might be true for the N-soliton solution, too. In any event, we have already shown $F(x, t) > 0$ for all $(x, t) \in \mathbb{R}^2$, so that we have

$$Y(x, t) \equiv \ln F(x, t) \in \mathbb{R}, \quad \forall (x, t) \in \mathbb{R}^2. \tag{3.40}$$

Moreover, Proposition 3.2 has the corollary

$$\lim_{\delta t \to \infty} Y(x_0 + s_0 t, t) = 0, \quad s_0 \notin \{v(\alpha_1), \ldots, v(\alpha_N)\}, \tag{3.41}$$

$$\lim_{\delta t \to \infty} Y(x_0 + s_0 t, t) = \ln f(q_j, \theta_j; x_0 + \delta \Delta_j(q)/4\alpha_j), \quad s_0 = v(\alpha_j), \quad j = 1, \ldots, N. \tag{3.42}$$

Just as for the nonlocal Toda solitons studied in Section 6 of Ref. [2], we believe that the long-time asymptotics of $Y(x, t)$ can be sharpened considerably. To be specific, we expect

$$\sup_{x \in \mathbb{R}} |Y(x, t) - Y^{(\delta)}(x, t)| = O(\exp(-\delta t r)), \quad \delta t \to \infty, \quad \delta = +, -, \quad (?) \tag{3.43}$$

where

$$Y^{(\delta)}(x,t) \equiv \sum_{j=1}^{N} \ln f(q_j, \theta_j; x + \delta\Delta_j(q)/4\alpha_j - v(\alpha_j)t), \qquad (3.44)$$

$$r \equiv \min_{1 \le j < k \le N}(2\alpha_j[v(\alpha_j) - v(\alpha_k)]). \qquad (3.45)$$

We can also follow the reasoning after the proof of Prop. 6.1 in Ref. [2] to obtain global soliton space-time trajectories, with asymptotics

$$x_{\substack{j \\ N-j+1}}(t) = \frac{1}{2\alpha_j}\left(\theta_j \mp \frac{1}{2}\Delta_j(q)\right) + v(\alpha_j)t + O(\exp(\mp tr_j)), \quad t \to \pm\infty, \qquad (3.46)$$

$$r_j \equiv \min_{k \ne j}(2\alpha_k|v(\alpha_k) - v(\alpha_j)|), \quad j = 1, \dots, N. \qquad (3.47)$$

Indeed, Theorem 7.1 in our paper Ref. [13] may be invoked in the same way as for the nonlocal Toda solitons.

Just as in Ref. [2], we should add that we are not aware of '1-soliton superposition' formulas analogous to Eqs. (7.10)–(7.14) in Ref. [13]. As already mentioned in Ref. [2], the problem is that $L_s(x,t)$ is multiplied by q-dependent phase matrices, which change the spectrum in a way that is difficult to control. Indeed, it is only because we bypass this snag via Lemma 6.2 in Ref. [2] that the 'correctness' of the trajectory asymptotics (3.46) can be established, in the sense that it coincides with the locations of the minima of the pertinent 1-soliton functions for asymptotic times, cf. Prop. 3.2.

4   The relation to the KdV solitons

In Section 3 of Ref. [2] we have already detailed how a suitable scaling limit of the AΔOs S and wave functions $\mathcal{W}(x,p)$ gives rise to the well-known reflectionless Schrödinger operators

$$(Hf)(x) \equiv -f''(x) + V_H(x)f(x), \qquad (4.48)$$

and wave functions satisfying

$$(H\mathcal{W}_H)(x,p) = p^2\mathcal{W}_H(x,p), \qquad (4.49)$$

and

$$\mathcal{W}_H(x,p) \sim \begin{cases} \exp(ixp), & x \to \infty, \\ \prod_{n=1}^{N}\left(\frac{p - i\kappa_n}{p + i\kappa_n}\right) \cdot \exp(ixp), & x \to -\infty, \end{cases} \quad 0 < \kappa_N < \cdots < \kappa_1. \qquad (4.50)$$

For the time dependence in (2.18) to have a finite limit as the scale parameter β tends to 0, one should substitute $t \to t/\beta$. Indeed, starting from the spectral data

$$r_n = i\beta\kappa_n, \quad \mu_n(x,t) = \frac{\exp(i\beta\kappa_n)}{i\beta\nu_n}\exp(-2t\sin(\beta\kappa_n)), \qquad (4.51)$$

and then taking $t \to t/\beta$, the $\beta \to 0$ limit gives rise to a time dependence $\nu_n \to \nu_n \exp(2\kappa_n t)$ in II(3.18). Thus one gets a space-time dependence $x - t$, corresponding to a trivial translation type flow.

There exists however a more sophisticated starting point that leads to the N-soliton solutions of the KdV equation. As already mentioned in the introduction, the idea to obtain this relation appears to date back to Ref. [14], where the discrete Volterra equation (1.5) is tied in with the KdV equation.

Roughly speaking, this idea amounts to shifting x in a t-dependent fashion and scaling t in a different way than we did above, so as to get rid of the linear term in the β-expansion of $\sin(\beta \kappa_n)$, leaving the cubic dependence needed for KdV solitons. Turning to the details, let us start from residue functions

$$R_{\beta,n}(x,t) \equiv \beta^{-1} R_n(\beta^{-1}x - 24\beta^{-3}t, -24\beta^{-3}t), \tag{4.52}$$

where $R_n(x,t)$ denotes the solution to (2.3) with data (4.51). Thus, we have

$$(i\nu_n)^{-1} \exp\left(2\kappa_n x + i\beta\kappa_n + \frac{48\kappa_n t}{\beta^2}\left(\frac{\sin(\beta\kappa_n)}{\beta\kappa_n} - 1\right)\right) R_{\beta,n}(x,t)$$

$$+ \sum_{j=1}^{N} \frac{\beta}{\exp(i\beta\kappa_n) - \exp(-i\beta\kappa_j)} R_{\beta,j}(x,t) = 1, \quad n = 1, \dots, N. \tag{4.53}$$

From this we deduce that the functions $R_{\beta,n}(x,t)$ are holomorphic at $\beta = 0$ for generic x, t, with $\beta \to 0$ limits $R_n^H(x,t)$ satisfying

$$\frac{\exp[2\kappa_n(x - 4\kappa_n^2 t)]}{i\nu_n} R_n^H(x,t) + \sum_{j=1}^{N} \frac{1}{i\kappa_n + i\kappa_j} R_j^H(x,t) = 1, \quad n = 1, \dots, N. \tag{4.54}$$

This time-dependent $N \times N$ system amounts to the well-known system for KdV solitons, cf. e.g. [19, 20, 21]. To be specific, the function

$$V_H(x,t) \equiv -2i \sum_{n=1}^{N} \partial_x R_n^H(x,t) \tag{4.55}$$

solves the KdV equation

$$\dot{u} = 6u\partial_x u - \partial_x^3 u, \tag{4.56}$$

and is an N-soliton solution. It can be obtained directly from the nonlocal Volterra N-soliton solution in the following way. Put

$$
\begin{aligned}
F_\beta(x,t) &\equiv V(\beta^{-1}x - 24\beta^{-3}t - i/4, -24\beta^{-3}t) \\
&= \beta \sum_{n=1}^{N} [R_{\beta,n}(x - 3i\beta/4, t) - R_{\beta,n}(x - i\beta/4, t)] + 1.
\end{aligned}
\tag{4.57}
$$

For $\beta\kappa_1 < 2\pi/3$ and real x, t, this function is positive, cf. Section 3. Since $R_{\beta,n}(x,t)$ is meromorphic in x and t, $F_\beta(x,t)$ is holomorphic at $\beta = 0$ for generic x, t. In particular, it

is holomorphic at $\beta = 0$ for x, t real, and satisfies

$$
\begin{aligned}
\lim_{\beta \to 0} \frac{F_\beta(x,t) - 1}{(\beta/2)^2} &= -2i \sum_{n=1}^{N} \partial_x R_n^H(x,t) \\
&= V_H(x,t).
\end{aligned}
\tag{4.58}
$$

This is the announced relation between our nonlocal Volterra solitons and the KdV solitons.

Acknowledgments

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