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A Truncation for Obtaining all the First Degree Birational Transformations of the Painlevé Transcendents

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Abstract

A birational transformation is one which leaves invariant an ordinary differential equation, only changing its parameters. We first recall the consistent truncation which has allowed us to obtain the first degree birational transformation of Okamoto for the master Painlevé equation P6. Then we improve it by adding a preliminary step, which is to find all the Riccati subequations of the considered Pn before performing the truncation. We discuss in some detail the main novelties of our method, taking as an example the simplest Painlevé equation for that purpose, P2. Finally, we apply the method to P5 and obtain its two inequivalent first degree birational transformations.

1 Introduction

A *birational transformation* is by definition a set of two relations,

$$u = f(U', U, X),\ U = F(u', u, x), \tag{1.1}$$

with f and F rational functions, which maps an algebraic ordinary differential equation (ODE), for instance a Painlevé equation,

$$E(u) \equiv \mathrm{Pn}(u, x, \boldsymbol{\alpha}) = 0,\ \boldsymbol{\alpha} = (\alpha, \beta, \gamma, \delta), \tag{1.2}$$

into the same equation with different parameters

$$E(U) \equiv \mathrm{Pn}(U, X, \boldsymbol{A}) = 0,\ \boldsymbol{A} = (A, B, \Gamma, \Delta), \tag{1.3}$$

with some homography (usually the identity) between x and X. The parameters $(\boldsymbol{\alpha}, \boldsymbol{A})$ must obey as many algebraic relations as elements in $\boldsymbol{\alpha}$. The *degree* of a birational transformation is defined as the highest degree in U' or u' (or more generally in the $(N-1)$th derivative of U and u) of the numerator and the denominator of (1.1).

A method allowing one to derive such birational transformations was recently introduced [1, 2], and later improved [3, 4] so as to provide birational transformations which have a degree equal to one for any first degree N-th order ODE. Its application to the master Painlevé equation P6,

$$\begin{aligned}\text{P6} \ : \ u'' &= \frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right]u'^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right]u' \\ &\quad +\frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{u^2}+\gamma\frac{x-1}{(u-1)^2}+\delta\frac{x(x-1)}{(u-x)^2}\right],\end{aligned}$$

provided the birational transformation [5], already found by Okamoto [6, p. 356],

$$\frac{N}{u-U} = \frac{x(x-1)U'}{U(U-1)(U-x)}+\frac{\Theta_0}{U}+\frac{\Theta_1}{U-1}+\frac{\Theta_x-1}{U-x} \tag{1.4}$$

$$= \frac{x(x-1)u'}{u(u-1)(u-x)}+\frac{\theta_0}{u}+\frac{\theta_1}{u-1}+\frac{\theta_x-1}{u-x}, \tag{1.5}$$

$$\theta_j = \Theta_j-\frac{1}{2}\left(\sum\Theta_k\right)+\frac{1}{2},\ j,k=\infty,0,1,x, \tag{1.6}$$

$$\Theta_j = \theta_j-\frac{1}{2}\left(\sum\theta_k\right)+\frac{1}{2}. \tag{1.7}$$

The transformation is clearly birational since the l.h.s. is homographic in both u and U. In the above, the monodromy exponents $\boldsymbol{\theta}=(\theta_\infty,\theta_0,\theta_1,\theta_x)$ are defined as

$$\theta_\infty^2=2\alpha,\ \theta_0^2=-2\beta,\ \theta_1^2=2\gamma,\ \theta_x^2=1-2\delta, \tag{1.8}$$

and similarly for their uppercase counterparts, while the odd-parity constant N takes the equivalent expressions

$$N = \sum(\theta_k^2-\Theta_k^2) \tag{1.9}$$

$$= 1-\sum\Theta_k=-1+\sum\theta_k \tag{1.10}$$

$$= 2(\theta_j-\Theta_j),\ j=\infty,0,1,x. \tag{1.11}$$

The choice of the eight signs of $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ is such that the square of this transformation is the identity. We will adopt such a convention (choice of signs so as to have involutions) throughout the present paper. This will dispense us from writing the second half, e.g. (1.5), of a birational transformation. Indeed, if the l.h.s. of the first half is chosen invariant under the permutation of $(u,\boldsymbol{\theta})$ and $(U,\boldsymbol{\Theta})$, which is the case in (1.5), the second half is deduced from the first half by just permuting the lowercase and uppercase notation.

The method is an extension to ODEs of the powerful *singular manifold method* introduced by Weiss *et al.* [7]. This method in turn mainly assumes the existence of a *truncation*, i.e. a representation of the solution u of the considered ODE (1.2) by a Laurent series which terminates ("truncates"). Its current achievements are detailed in summer school proceedings, see Refs. [8, 9].

The purpose of this article is threefold. Firstly, we present a significant improvement to that method, only based on the consideration of Riccati equations. This improvement reduces the obtaining of the above birational transformation of P6 to computations which are easily feasible by hand. Secondly, using the simplest equation P2 as an example, we point out the main differences between our method and the one previously introduced [1, 2]. Thirdly, on the specific example of P5, we show that the straightforward application of our method yields the two first degree birational transformations of this ODE.

The organization of the paper is as follows. In Section 2, we recall the main difference between the truncation of an ODE and that of a partial differential equation (PDE), which is a fundamental homography between the ODE and the truncation variable, without any counterpart for a PDE.

In Section 3, we define the consistent truncation which implements this homography and we present the improvement.

The next Section 4 is devoted to a parallel processing of the simplest Painlevé equation P2, so as to clearly point out the differences between the previous method and ours.

Finally, in Section 5, we process P5 with our method, and find its two inequivalent first degree birational transformations.

Throughout this article, we discard the nongeneric cases in which the components of $\boldsymbol{\alpha}$ are constrained. One such case is the well known birational transformation between P5 with $\delta = 0$ and P3.

2 The fundamental homography, a difference with PDEs

Consider a Painlevé ODE (1.3) which admits a birational transformation, i.e. $n = 2, 3, 4, 5, 6$. There exist two Riccati equations associated to this Pn. The first one is the Painlevé equation (1.3) itself. Indeed, any N-th order, first degree ODE with the Painlevé property is necessarily [10, pp. 396–409] a Riccati equation for $U^{(N-1)}$, with coefficients depending on x and the lower derivatives of U, in our case

$$U'' = A_2(U,x)U'^2 + A_1(U,x)U' + A_0(U,x). \tag{2.1}$$

The second Riccati equation is the algebraic transform for $Z = \psi/\psi'$ of the linear second order ODE for ψ which has been built by Richard Fuchs [11] for P6, and by confluence to any other Pn,

$$Z' = 1 + z_1 Z + z_2 Z^2. \tag{2.2}$$

Since the group of invariance of a Riccati equation is the homographic group, the variables U' and Z are linked by a homography. Let us define it as

$$(U' + g_2)(Z^{-1} - g_1) - g_0 = 0,\ g_0 \neq 0, \tag{2.3}$$

or, in the affine case, as

$$(U' + G_2) - G_0 Z^{-1} = 0,\ G_0 \neq 0. \tag{2.4}$$

The coefficients g_j or G_j are rational in (U, x). We will not consider (2.4) in the present paper.

This homography allows us to compute the two coefficients z_j of the Riccati pseudopotential equation (2.2) as explicit expressions of $(g_j, \partial_U g_j, \partial_x g_j, A_2, A_1, A_0, U')$. Indeed, eliminating U' between (2.1) and (2.3) defines a first order ODE for Z, whose identification with (2.2) *modulo* (2.3) provides three relations,

$$g_0 = g_2^2 A_2 - g_2 A_1 + A_0 + \partial_x g_2 - g_2 \partial_U g_2, \tag{2.5}$$

$$z_1 = A_1 - 2g_1 + \partial_U g_2 - \partial_x \operatorname{Log} g_0 + (2A_2 - \partial_U \operatorname{Log} g_0)\, U', \tag{2.6}$$

$$z_2 = -g_1 z_1 - g_1^2 - g_0 A_2 - \partial_x g_1 - (\partial_U g_1) U'. \tag{2.7}$$

Since A_2, A_1, A_0 are given, there only remains to determine the two coefficients g_1, g_2 of the homography, which are functions of the two variables U, x. In particular, the two functions z_1, z_2 of the three variables U', U, x are not the elementary quantities to determine.

For a PDE, the homography (2.3) does not exist any more since at least (2.1) does not survive. For searching the Bäcklund transformation (see summer schools lecture notes [8, 9] and references therein), this makes the situation much easier since one does not have to assume a dependence between Z and some derivative of U. On the contrary, for ODEs, if one handles Z and U' as independent variables, this creates many difficulties.

3 The improvement to the truncation

Each Pn equation which admits a birational transformation has one or several (four for P6) couples of families of movable simple poles with opposite residues $\pm u_0$, and the assumption for the one-family truncation is

$$u = u_0 Z^{-1} + U, \;\; u_0 \neq 0, \;\; x = X, \tag{3.1}$$

in which u and U satisfy (1.2) and (1.3), and Z satisfies (2.2). After determination of the rational functions $g_1(U, x)$ and $g_2(U, x)$, the first half of the birational transformation is

$$u = U + u_0 \left(g_1(U, x) + \frac{g_0(U, x)}{U' + g_2(U, x)} \right), \tag{3.2}$$

with the restriction that its denominator should not vanish. We recently conjectured [3] that the ODE defined by this denominator,

$$U' + g_2(U, x) = 0, \tag{3.3}$$

has the Painlevé property, which restricts g_2 to an arbitrary second degree polynomial of U with coefficients depending on x.

Let us prove this conjecture and completely determine g_2 for P6, and therefore for any Pn equation thanks to the confluence.

Proof. The equation (3.2) is equivalently written

$$\frac{u_0 g_0}{u - U - u_0 g_1} = U' + g_2, \tag{3.4}$$

and the nonvanishing condition $u - U - u_0 g_1 \neq 0$ does not restrict U any more since the restriction concerns u. Therefore the equation (3.4) still holds when, simultaneously, $g_0(U, x)$ vanishes and U satisfies the equation (3.3). In the case of P6, the corresponding values $\tilde{g}_2$ and $\tilde{g}_0$ are defined by the Riccati subequation,

$$\tilde{g}_2 = \frac{U(U-1)(U-x)}{x(x-1)} \left(\frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_x - 1}{U-x} \right), \tag{3.5}$$

and by the formula (2.5) applied to $\tilde{g}_2$,

$$\tilde{g}_0 = (1 - \Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x)(1 + \Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x)\frac{U(U-1)(U-x)}{8x^2(x-1)^2}, \tag{3.6}$$

which indeed defines the constraint on $\boldsymbol{\Theta}$. The couple (g_0, g_2) in (3.4) cannot be different (*modulo* the homographies on U which preserve x and P6) from this couple $(\tilde{g}_0, \tilde{g}_2)$, since this Riccati subequation is unique. ■

Denoting R (like Riccati) the quantity

$$R = \frac{x(x-1)U'}{U(U-1)(U-x)} + \frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_x}{U-x}, \tag{3.7}$$

one has the identity, whatever be $\boldsymbol{\Theta}$,

$$\begin{aligned} & x(x-1)R' + \frac{1}{2}(1 - \Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x)(1 + \Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x) \\ & \quad - ((\Theta_1 + \Theta_x - 1)U + (\Theta_0 + \Theta_x - 1)(U-1) + (\Theta_0 + \Theta_1)(U-x))\, R \\ & \quad + \frac{1}{2}\left((U-1)(U-x) + U(U-1) + U(U-x)\right) R^2 \\ & \quad + \frac{x^2(x-1)^2}{U(U-1)(U-x)} \mathrm{P6}(U) \equiv 0, \end{aligned} \tag{3.8}$$

To summarize, the information is twofold.

1. The coefficient g_2 is determined by the fact that equation (3.3) must be a subequation of (1.3).
2. The coefficient g_0 factorizes as $g_0(U, x, A, B, \Gamma, \Delta) = f_0(A, B, \Gamma, \Delta) h_0(U, x)$, defining the condition $f_0(A, B, \Gamma, \Delta) = 0$ for the existence of the subequation.

The improved method is now the following. Before performing the truncation, one computes all the identities like (3.8) involving Riccati subequations. Each identity defines an explicit value for g_2. For each such g_2, the coefficient g_0 is explicitly given as a factorized expression by the formula (2.5). Finally, one performs the truncation to find g_1 and the algebraic relations between $\alpha, \beta, \gamma, \delta$ and A, B, Γ, Δ. Let us now describe this truncation.

The field u is represented, see (3.1), by a Laurent series in Z which terminates ("truncated series"). The l.h.s. $E(u)$ of the equation can similarly be written as a truncated series in Z. This is achieved by the elimination of u, Z', U'', U' between (1.2), (1.3), (3.1), (2.2) and (2.3), followed by the elimination of (g_0, z_1, z_2) from (2.5)–(2.7) (q denotes the singularity order of Pn written as a differential polynomial in u, it is -6 for P6),

$$E(u) = \sum_{j=0}^{-q+2} E_j(U, x, u_0, g_1, \boldsymbol{\alpha}, \boldsymbol{A}) Z^{j+q-2} = 0, \tag{3.9}$$

$$\forall j \;:\; E_j(U, x, u_0, g_1, \boldsymbol{\alpha}, \boldsymbol{A}) = 0. \tag{3.10}$$

The nonlinear *determining equations* $E_j = 0$ are independent of U', and this is the main difference with previous work [2]. Another difference is the greater number ($-q+3$ instead of $-q+1$) of equations $E_j = 0$, which is due to the additional elimination of U' with (2.3).

The $-q+3$ determining equations (3.10) in the unknown function $g_1(U, x)$ and the unknown algebraic relations between $\alpha, \beta, \gamma, \delta$ and A, B, Γ, Δ must be solved, as usual, by increasing values of their index j.

4 An elementary example: the second Painlevé equation

$$\mathrm{P2}\ :\ u'' = \delta(2u^3 + xu) + \alpha.$$

The data for the two opposite families of movable singularities of P2 are

$$p = -1,\ q = -3,\ u_0 = d^{-1},\ \delta = d^2,\ \text{ Fuchs index } 4, \tag{4.1}$$

in which d is any square root of d^2. The unique monodromy exponent θ_∞ is defined as

$$\alpha = -d\theta_\infty. \tag{4.2}$$

Let us compare the truncation of the previous section with the truncation introduced by [1] and applied by [2]. For brevity, the former will be qualified "full" and the latter "semi" for reasons to become clear soon.

4.1 Processing of P2 with the full truncation

One first computes all the identities like (3.8) involving first order first degree subequations of P2. As well as for any Pn, such a subequation can only be a Riccati equation, which is the unique such ODE with the Painlevé property,

$$R \equiv U' + a_2 U^2 + a_1 U + a_0 = 0. \tag{4.3}$$

Eliminating U' with $\mathrm{P2}(U)$, one obtains

$$\begin{aligned}\forall U\ :\ & 2(a_2^2 - D^2)U^3 + (3a_1 a_2 - a_2')U^2 \\ & +(2a_2 a_0 + a_1^2 - a_1' - D^2 x)U + (a_1 a_0 - a_0' - A) = 0,\end{aligned} \tag{4.4}$$

a system admitting the unique solution (D denotes any square root of D^2)

$$R \equiv U' + D\left(U^2 + \frac{x}{2}\right) = 0,\ 2A + D = 0, \tag{4.5}$$

i.e. the well known algebraic transform of an Airy equation. The indeterminacy on R (which, up to now, is only defined up to an additive term containing the factor $2A + D$) is removed by the explicit form of the identity between P2 and its subequation,

$$\forall (A, D) \qquad R' - \left(A + \frac{D}{2}\right) - (2DU)R + \mathrm{P2}(U, x, A, D) \equiv 0, \tag{4.6}$$

a relation valid for any A and D. By identification with (3.3) according to the proof presented in section 3, one obtains

$$g_2 = D\left(U^2 + \frac{x}{2}\right) \cdot \tag{4.7}$$

Given the coefficients of the three terms of P2

$$A_2 = 0,\ A_1 = 0,\ A_0 = D^2(2U^3 + xU) + A, \tag{4.8}$$

equation (2.5) then provides the value of g_0, which only depends on (A_2, A_1, A_0, g_2),

$$g_0 = A + \frac{D}{2} \cdot \tag{4.9}$$

Therefore, before undertaking the truncation properly said, there only remains to find g_1 and the two relations between (d, α, D, A).

The assumption for the truncation is

$$u = U + u_0 Z^{-1},\ u_0 = d^{-1}, \tag{4.10}$$
$$u'' = d^2(2u^3 + xu) + \alpha,\ U'' = D^2(2U^3 + xU) + A, \tag{4.11}$$
$$U' = -g_2 + \frac{g_0}{Z^{-1} - g_1},\ g_2 = D\left(U^2 + \frac{x}{2}\right),\ g_0 = A + \frac{D}{2}, \tag{4.12}$$
$$Z' = 1 + z_1 Z + z_2 Z^2, \tag{4.13}$$
$$z_1 = -2g_1 + 2DU,\ z_2 = -g_1 z_1 - g_1^2 - \partial_x g_1 - (\partial_U g_1)U'. \tag{4.14}$$

The elimination of $u, Z', U'', U', g_2, g_0, z_1, z_2$ generates the truncated Laurent series

$$E(u) = \sum_{j=0}^{-q+2} E_j(U, x, g_1, d, \alpha, D, A) Z^{j+q-2} = 0, \tag{4.15}$$

independent of U', and one requires its identical vanishing in Z,

$$\forall j\ :\ E_j(U, x, g_1, d, \alpha, D, A) = 0. \tag{4.16}$$

Since these six determining equations must be solved by ascending values of j, let us write each of them after insertion of the solution of the previous ones. Introducing the notation

$$\alpha = -d\theta_\infty,\ A = -D\Theta_\infty. \tag{4.17}$$

these are

$$E_0 \equiv 0, \tag{4.18}$$
$$E_1 \equiv (d - D)U + g_1 = 0, \tag{4.19}$$
$$E_2 \equiv (d^2 - D^2)x = 0, \tag{4.20}$$
$$E_3 \equiv 1 - \Theta_\infty - \theta_\infty = 0, \tag{4.21}$$
$$E_j \equiv 0,\ j = 4, 5. \tag{4.22}$$

One notices that the equation $E_4 = 0$, corresponding to the Fuchs index, is identically satisfied, and that there is no need to consider $j \ge 4$ (just like for P6, see Ref. [4]). These determining equations are solved as

$$g_1 = (D - d)U,\ D^2 = d^2,\ \theta_\infty = 1 - \Theta_\infty, \tag{4.23}$$

and the first half of the birational transformation is

$$\frac{D(1/2 - \Theta_\infty)}{du - DU} = U' + D\left(U^2 + \frac{x}{2}\right). \tag{4.24}$$

Since, whatever be the choice of sign $D = \pm d$, the relation between the parameters $(d, \theta_\infty, D, \Theta_\infty)$ is an involution, the second half is obtained by just permuting the uppercase and lowercase notation,

$$\frac{d(1/2 - \theta_\infty)}{DU - du} = u' + d\left(u^2 + \frac{x}{2}\right). \tag{4.25}$$

Although the two choices $d = \pm D$ are equally acceptable (the unique homography which conserves x, namely $(u, x) \mapsto (-u, x)$, allows one to freely reverse the sign of D), the choice $d = D$ is better because this is the one which is inherited from P6 by the confluence [4], so the final result is the involution

$$\frac{(\theta_\infty - \Theta_\infty)/2}{u - U} = U' + D\left(U^2 + \frac{x}{2}\right) = u' + d\left(u^2 + \frac{x}{2}\right), \tag{4.26}$$

$$d = D,\ \theta_\infty = 1 - \Theta_\infty. \tag{4.27}$$

Remark. When compared to the Riccati equation (2.2) for Z, the identity (4.6) shows that the values

$$Z = \frac{U' + D(U^2 + x/2)}{A + D/2},\ z_1 = 2DU,\ z_2 = 0, \tag{4.28}$$

i.e.

$$g_0 = A + \frac{D}{2},\ g_1 = 0,\ g_2 = D\left(U^2 + \frac{x}{2}\right), \tag{4.29}$$

define *a priori* a particular solution of the truncation. The computation has shown that, *modulo* the homography $(u, x, \alpha) \to (-u, x, -\alpha)$, this solution is unique.

4.2 Comparison with the semi-truncation

As already explained in summer school lecture notes [9], the method proposed in Ref. [2] is in fact not distinct from a truncation. Therefore we will adopt the truncation language to clarify its presentation and perform the comparison.

The assumption for the semi-truncation is

$$u = U + u_0 Z^{-1},\ u_0 = d^{-1}, \tag{4.30}$$

$$u'' = d^2(2u^3 + xu) + \alpha,\ U'' = D^2(2U^3 + xU) + A, \tag{4.31}$$

$$Z' = 1 + z_1 Z + z_2 Z^2. \tag{4.32}$$

The elimination of u, Z', U'' generates the truncated Laurent series

$$E(u) = \sum_{j=0}^{-q} E_j(U', U, z_1, z_2, \boldsymbol{\alpha}, \boldsymbol{A}, x) Z^{j+q} = 0,\ \boldsymbol{\alpha} = (\alpha, d),\ \boldsymbol{A} = (A, D), \tag{4.33}$$

in which the dependence on U' has been emphasized, and one does *not* require its identical vanishing in Z. Indeed, the four coefficients of this Laurent series in Z are

$$E_0 \equiv 0, \tag{4.34}$$

$$E_1 \equiv 3(2dU - z_1), \tag{4.35}$$

$$E_2 \equiv z_1' - z_1^2 - 2z_2 + d^2(6U^2 + x), \tag{4.36}$$

$$E_3 \equiv z_2' - z_1 z_2 + d(d^2 - D^2)(2U^3 + xU) + d(\alpha - A). \tag{4.37}$$

The first two coefficients $E_j, j = 0, 1$, even for cases other than P2, are independent of U' and the same in the two truncations if one remembers the correspondence (2.6) between

z_1 and (g_j, A_j). Therefore, defining the two equations $E_j = 0, j = 0, 1$ and solving them makes no difference betwen the two methods and provides

$$z_1 = 2dU. \tag{4.38}$$

As soon as $j \geq 2$, the coefficients E_j are essentially different in the two truncations, and this is because the remaining truncated Laurent series depends on both U' and Z,

$$\sum_{j=2}^{-q} E_j(U', U, z_2, \boldsymbol{\alpha}, \boldsymbol{A}, x) Z^{j+q} = 0, \tag{4.39}$$

while they are not independent but linked by the unused relation (to be found *in fine*) (2.3).

Solving the system $E_j = 0, j = 0, \dots, 3$, would indeed yield [12]

$$z_1 = 2dU,\ z_2 = dU' + (dU)^2 + \frac{1}{2} d^2 x,\ \alpha + \frac{d}{2} = 0, \tag{4.40}$$

which cannot define a birational transformation.

To overcome this first difficulty, after solving $E_1 = 0, E_2 = 0$ (hence the name semi-truncation), one eliminates Z between (2.2) and (4.39), which amounts to compute the resultant of two polynomials of Z and generically results in

$$F(U', U; z_2, \boldsymbol{\alpha}, \boldsymbol{A}, x) = 0, \tag{4.41}$$

$$Z = z(U', U; z_2, \boldsymbol{\alpha}, \boldsymbol{A}, x), \tag{4.42}$$

in which F is a differential polynomial and z a rational function of their arguments. The two equations (4.30) and (4.42) will define the first half of a birational transformation, not necessarily of degree one, after the first equation (4.41) has been solved for z_2.

Solving (4.41) for z_2 is the second difficulty of the method. Indeed, the method is to enforce the irreducibility of Pn by requiring the identical vanishing of (4.41) as a polynomial of U', U. But (4.41) depends on z_2, z_2', z_2'' and this procedure first requires an additional assumption on the explicit dependence of z_2 on (U', U). The assumption [2]

$$z_2 = f_0(x) + f_1(x) U \tag{4.43}$$

is sufficient for P2 and it leads to the expected result (4.26)–(4.27).

Finally, there exists a third difficulty, which only occurs for P6, this is the value 1 of the Fuchs index of P6, a value which cannot be changed by homography on u. In this case, the coefficient E_1 is identically zero and does not determine z_1, so one must make two assumptions for the dependence of (z_1, z_2) on (U', U). This is why, to our knowledge, the semi-truncation method has not been applied to P6 yet.

5 Processing of P5 with the present truncation

P6 has already been processed with our method [3, 4] before the present improvement, and the solution to the truncation is unique. The first reason for choosing P5 here is that, as opposed to P6, one expects at least two inequivalent solutions, a situation which only occurs for P5 and P4. The second reason is to show how the difficulty arising from the index 1 is overcome. These two inequivalent first degree birational transformations for P5 were first found by Gromak [13, Eq. (13)] and by Okamoto [14].

The definition of P5 is

$$\text{P5} : u'' = \left[\frac{1}{2u} + \frac{1}{u-1}\right] u'^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2}\left[\alpha u + \frac{\beta}{u}\right] + \gamma\frac{u}{x} + \delta\frac{u(u+1)}{u-1},$$

and the data for the two opposite families of movable singularities of u are

$$p = -1,\ q = -5,\ u_0^2 = \theta_\infty^{-2} x^2,\ \alpha = \theta_\infty^2/2,\ \text{ Fuchs index } 1, \tag{5.1}$$

with the definition of the monodromy exponents [15],

$$\theta_\infty^2 = 2\alpha,\ \theta_0^2 = -2\beta,\ d\theta_1 = -\gamma, d^2 = -2\delta. \tag{5.2}$$

The search for the Riccati subequations as explained in Section 4.1 (f_∞, f_0, f_1 denote the three functions of x to be found)

$$R = \frac{xU'}{U(U-1)^2} + \frac{f_0}{U} + \frac{f_\infty - f_0}{U-1} + \frac{f_1 x}{(U-1)^2}, \tag{5.3}$$

leads to the unique algebraic solution

$$f_0^2 = \Theta_0^2,\ f_\infty^2 = \Theta_\infty^2,\ f_1^2 = D^2,\ (1 + f_\infty - f_0) f_1 = D\Theta_1. \tag{5.4}$$

After choosing the square roots, there are two distinct identities of the type

$$R' + F_2(U,x)R^2 + F_1(U,x)R + F_0(U,x) + \text{P5}(U) \equiv 0, \tag{5.5}$$

namely

$$R = \frac{xU'}{U(U-1)^2} + \frac{\Theta_0}{U} + \frac{\Theta_1 - 1}{U-1} + \frac{Dx}{(U-1)^2}, \tag{5.6}$$

$$\begin{aligned} &xR' + \frac{1}{2}(1 - \Theta_\infty - \Theta_0 - \Theta_1)(1 + \Theta_\infty - \Theta_0 - \Theta_1) + \frac{1}{2}\left((U-1)^2 + 2U(U-1)\right)R^2 \\ &\quad - \left((\Theta_1 - 1)U + (2\Theta_0 + \Theta_1 - 1)(U-1) + dx\right)R + \frac{x^2}{U^2(U-1)^2}\text{P5}(U) \equiv 0, \end{aligned} \tag{5.7}$$

and

$$R = \frac{xU'}{U(U-1)^2} + \frac{\Theta_0}{U} + \frac{\Theta_\infty - \Theta_0}{U-1} + \frac{Dx}{(U-1)^2}, \tag{5.8}$$

$$\begin{aligned} &xR' - \frac{D(1 + \Theta_\infty - \Theta_0 - \Theta_1)x}{(U-1)^2} + \frac{1}{2}\left((U-1)^2 + 2U(U-1)\right)R^2 \\ &\quad - \left((\Theta_\infty - \Theta_0)U + (\Theta_\infty + \Theta_0)(U-1) + dx\right)R + \frac{x^2}{U^2(U-1)^2}\text{P5}(U) \equiv 0. \end{aligned} \tag{5.9}$$

The reason why they are essentially distinct is the two different factorizations of the condition on $(\Theta_\infty, \Theta_0, \Theta_1, D)$. As shown in Ref. [4], these two cases are inherited from the first degree birational transformation of Okamoto for P6 by two different confluences, which have been called respectively *normal* or *unbiased*, and *biased* in Ref. [4].

Each choice will correspond to a different solution to the truncation, defining two distinct first degree birational transformations, respectively denoted $\text{T}_{5,\text{u}}$ and $\text{T}_{5,\text{b}}$ (like unbiased, biased). We now follow exactly the steps of Section 4.1.

5.1 The normal (or unbiased) birational transformation of P5

The first possibility (5.6),

$$g_2 = \frac{U(U-1)^2}{x}\left(\frac{\Theta_0}{U} + \frac{\Theta_1 - 1}{U-1} + \frac{Dx}{(U-1)^2}\right), \quad (5.10)$$

provides the factorization (compare with the identity (5.7))

$$g_0 = -(1-\Theta_\infty-\Theta_0-\Theta_1)(1+\Theta_\infty-\Theta_0-\Theta_1)\frac{U(U-1)^2}{2x^2}\cdot \quad (5.11)$$

Denoting the residue u_0 as

$$u_0 = -\theta_\infty^{-1} x, \quad (5.12)$$

the first determining equations are

$$\begin{aligned} E_j &\equiv 0,\ j=0,1, & (5.13)\\ E_2 &\equiv x^2(\partial_x g_1 - g_1^2) - x g_2 \partial_U g_1 \\ &\quad + x\left(2(\theta_\infty - 1 + \Theta_0 + \Theta_1)U + 2 - 2\Theta_0 - \Theta_1 - \frac{4}{3}\theta_\infty + Dx\right) g_1 \\ &\quad + K_0 U^2 + K_1 U + \frac{d^2 - D^2}{6} x^2 + K_2 x + K_3 = 0, & (5.14)\end{aligned}$$

in which the K_m's are constants. The reason for the identical vanishing of E_1 is the Fuchs index 1, but this does not harm the computation. Indeed, the only possibility for g_1 to be rational in U is that it be a first degree polynomial of U, which the singularities of P5 in U suggest to define as

$$g_1 = \frac{f_0(x)U + (f_1(x) - f_0(x))(U-1)}{x}\cdot \quad (5.15)$$

Equation $E_2 = 0$ then splits into

$$E_2 \equiv \sum_{k=0}^{2} E_2^{(k)} U^k,\ \forall k\ E_2^{(k)}(f_0, f_1, x, \boldsymbol{\theta}, \boldsymbol{\Theta}) = 0, \quad (5.16)$$

$$E_2^{(2)} \equiv \Theta_\infty^2 - (2f_1 - 2\theta_\infty + 1 - \Theta_0 - \Theta_1)^2 = 0. \quad (5.17)$$

This system (5.16) of three equations is equivalent to

$$g_1 = (2\theta_\infty - 1 - \Theta_\infty + \Theta_0 + \Theta_1)\frac{3U-2}{6x},\ d^2 = D^2, \quad (5.18)$$

$$6\theta_0^2 - 2\theta_\infty^2 = (2 - \Theta_\infty + \Theta_0 - 2\Theta_1)^2 - 3(\Theta_1 - 1)^2, \quad (5.19)$$

and there only remains to find one algebraic relation between the monodromy exponents.

Since all the dependence on U is now found, the next determining equation splits even more, according to both powers of U and x,

$$E_3 \equiv (2\theta_\infty - 1 - \Theta_\infty + \Theta_0 + \Theta_1)\left[E_3^{(1,0)} U + \left(D^2 x^2 + E_3^{(0,1)} x + E_3^{(0,0)}\right)\right] = 0, \quad (5.20)$$

a system which admits as only solution the vanishing of the first factor. Therefore, the coefficient g_1 vanishes, like in the whole normal sequence [4]. This ends the resolution, therefore achieved at $j=3$, just like in the trunction for P6 [3].

This first solution $T_{5,u}$ has the affine representation

$$\text{P5} \ : \ \begin{pmatrix} \theta_\infty \\ \theta_0 \\ \theta_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} \Theta_\infty \\ \Theta_0 \\ \Theta_1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \ d = D, \tag{5.21}$$

in which the six arbitrary signs of $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ are chosen in such a way that the square of this transformation is the identity. The birational representation is

$$\text{P5} \ : \ \frac{1-\Theta_\infty-\Theta_0-\Theta_1}{u-U} = \frac{xU'}{U(U-1)^2} + \frac{\Theta_0}{U} + \frac{\Theta_1 - 1}{U-1} + \frac{Dx}{(U-1)^2}\cdot \tag{5.22}$$

This transformation for P5 was first found by Okamoto [14].

For completion, the values of z_1, z_2 are

$$z_1 = \frac{\Theta_1 U + (2\Theta_0 + \Theta_1 - 2)(U-1)}{x}, \tag{5.23}$$

$$z_2 = (-\Theta_\infty + \Theta_0 + \Theta_1 - 1)(\Theta_\infty + \Theta_0 + \Theta_1 - 1)\frac{2U(U-1)+(U-1)^2}{4x^2}\cdot \tag{5.24}$$

5.2 The biased birational transformation of P5

With the second possibility (5.8),

$$g_2 = \frac{U(U-1)^2}{x}\left(\frac{\Theta_0}{U} + \frac{\Theta_\infty - \Theta_0}{U-1} + \frac{Dx}{(U-1)^2}\right), \tag{5.25}$$

one similarly obtains

$$g_0 = (1+\Theta_\infty - \Theta_0 - \Theta_1)\frac{DU}{x}\cdot \tag{5.26}$$

In order to later make easier our involution convention, it is convenient this time to denote the residue u_0 as

$$u_0 = \theta_\infty^{-1} x. \tag{5.27}$$

The first three determining equations are

$$\begin{aligned} E_j &\equiv 0, \ j=0,1, &(5.28)\\ E_2 &\equiv -x^2(U-1)(\partial_x g_1 - g_1^2) + x(U-1)g_2\partial_U g_1 \\ &+x\left(\frac{2}{3}\theta_\infty(U-1)(3U+2) + (\Theta_\infty - \Theta_0)(U-1) + Dx(U+1)\right)g_1 \\ &+\theta_\infty(\theta_\infty - \Theta_\infty)U^3 \\ &+K_0U^2 - \frac{d^2-D^2}{6}x^2(U-1) + (K_1 x + K_2)U + K_3 x + K_4 = 0. &(5.29)\end{aligned}$$

Again, the only possibility for g_1 to be rational is to be a first degree polynomial of U, defined as in (5.15), and equation $E_2 = 0$ is equivalent to

$$g_1 = -\theta_\infty \frac{3U-2}{2x} - \frac{\Theta_\infty - \Theta_0 - \Theta_1 + 1}{6x},\ d^2 = D^2, \tag{5.30}$$

$$6\theta_0^2 - 2\theta_\infty^2 = (-\Theta_\infty + \Theta_0 - 2\Theta_1 - 1)^2 - 3\Theta_1^2. \tag{5.31}$$

Last, equation $E_3 = 0$ yields

$$\theta_\infty = \frac{-\Theta_\infty + \Theta_0 + \Theta_1 - 1}{2}. \tag{5.32}$$

This second solution $\mathrm{T}_{5,\mathrm{b}}$ is represented by the involution

$$\mathrm{P5}\ :\ \begin{pmatrix}\theta_\infty\\ \theta_0\\ \theta_1\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}1 & -1 & -1\\ -1 & 1 & -1\\ -2 & -2 & 0\end{pmatrix}\begin{pmatrix}\Theta_\infty\\ \Theta_0\\ \Theta_1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\ 1\\ 0\end{pmatrix},\ d = -D, \tag{5.33}$$

and

$$\mathrm{P5}\ :\ \frac{-2Dx}{(u-1)(U-1)} = (U-1)\left(\frac{xU'}{U(U-1)^2} + \frac{\Theta_0}{U} + \frac{\Theta_\infty - \Theta_0}{U-1} + \frac{Dx}{(U-1)^2}\right), \tag{5.34}$$

with

$$z_1 = \frac{(\Theta_\infty - \Theta_0)U + (2\Theta_0 + \Theta_1 - 1)(U-1)}{x} + \frac{2U'}{U-1}, \tag{5.35}$$

$$\frac{2x^2}{\theta_\infty} z_2 = 6xU' + 2Dx\left(\frac{2}{U-1} + U + 2\right) + 2(\Theta_\infty - \Theta_0 - 1)U(U-1) + (\Theta_\infty + 3\Theta_0 + \Theta_1 + 1)(U-1)^2. \tag{5.36}$$

This second transformation has first been obtained by Gromak [13, Eq. (13)].

Let us denote H the unique homography of P5 which conserves x,

$$\mathrm{P5}\ :\ \mathrm{H}(x, u, \theta_\infty, \theta_0, \theta_1) = (x, u^{-1}, \theta_0, \theta_\infty, \theta_1), \tag{5.37}$$

and $\mathrm{S_a}, \mathrm{S_b}, \mathrm{S_c}$ the operators which reverse the sign of, respectively, $\theta_\infty, \theta_0, \theta_1$. One has the relation

$$\mathrm{T}_{5,\mathrm{u}} = \mathrm{S_a T_{5,b} S_a S_c T_{5,b} S_a H}, \tag{5.38}$$

but we could not find an inverse relation expressing the biased transformation as powers of the unbiased one. Therefore, $\mathrm{T}_{5,\mathrm{b}}$ is more elementary than $\mathrm{T}_{5,\mathrm{u}}$.

Let us compare again with the semi-truncation. In [16], to avoid the third difficulty mentioned in Section 4.2, the authors first change the Fuchs index to 2 by performing a homography on P5. The second difficulty is handled with the extra assumption that z_2 (their τ) should be independent of U'. This then allows them to obtain the unbiased transformation $\mathrm{T}_{5,\mathrm{u}}$. The reason why they fail to find the biased one $\mathrm{T}_{5,\mathrm{b}}$ with the semi-truncation is the restricting assumption on z_2, since in this case z_2 explicitly depends on U', see expression (5.36). By looking at the ODE satisfied by Z (evidently an algebraic transform of P5), they finally obtain this second missing first degree birational transformation.

6 Conclusion

The improvement which we have presented to the truncation drastically reduces the amount of computation, and the search for first degree birational transformations of higher order ODEs by this method becomes much easier. This will be addressed in future work.

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