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Linearization of Mirror Systems

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Abstract

We demonstrate, through the fourth Painlevé and the modified KdV equations, that the attempt at linearizing the mirror systems (more precisely, the equation satisfied by the new variable θ introduced in the indicial normalization) near movable poles can naturally lead to the Schlesinger transformations of ordinary differential equations or to the Bäcklund transformations of partial differential equations.

1 Introduction

It is widely believed that a differential system being *integrable* is due to some sort of underlying *linear* structure(s). However, it has never been clear what this really means. A linear system is naturally considered as integrable. Integrability for nonlinear systems is quite ambiguous. Various properties are counted as indicators of integrability: solitons, the Lax pair, the Bäcklund transformation, the underlying Hamiltonian formulation, Hirota's bilinear representation, the Painlevé property, etc. The relations between these properties have yet to be understood.

It was recently proved [6] that for any principal balance, it is possible to introduce a variable θ through "indicial normalization" and more variables ξ, η, etc., through "truncation at resonances", so that the movable pole singularities are regularized, and the system for the new variables (called the *mirror system*) is a regular one. In this paper, with more flexible kind of indicial normalization, we attempt to make the equation satisfied by θ linearizable. As a result, the other new variables ξ, η, etc., are solved through algebraic equations. Moreover, the linearization leads to the Schlesinger transformation [2] of ordinary differential equations (ODEs) and to the Bäcklund transformation of partial differential equations (PDEs).

For a long time, people have been trying to derive the Bäcklund transformations and Lax pair from the Painlevé analysis with the techniques of "truncation method" or "singular manifold method" (See, for examples [7, 8, 9, 11] or [1, 3] for ODEs). The examples in this paper do not yet provide a comparable algorithm. Instead, our purpose here is to investigate the relation between the integrability properties, the linearizability, and the Painlevé test through the combination of the ingredients of the singular manifold method and the mirror system. Indeed, our calculation can be carried out with θ only and without using mirror systems. However, we feel the exclusion of the other variables misses important information on integrability. After all, it is the mirror system, consisting of equations for θ, ξ, η, etc., that is equivalent to principal balances.

In this paper, we demonstrate by examples that the Schlesinger (resp. Bäcklund) transformations are linearizable reductions of the mirror systems for ODEs (resp. PDEs).

We present the following two equations in this paper to demonstrate our idea: the fourth Painlevé equation and the modified Korteweg-de Vries equation. The same idea works for other integrable equations, including the second Painlevé equation, and the potential Korteweg-de Vries equation [12].

2 The fourth Painlevé equation

Consider the fourth Painlevé equation (P4)

$$\mathbf{P}_{\mathbf{IV}}(u,t;\alpha,\beta) \equiv u'' - \frac{u'^2}{2u} - \frac{3}{2}u^3 - 4tu^2 - 2(t^2-\alpha)u - \frac{\beta}{u} = 0,$$

where α and β are two constant parameters.

We will first find the mirror transformation (see [4, 5] for more details), which regularizes the movable pole singularities. We rewrite (P4) as the system of Cauchy's canonical form: $u' = v$, $v' = v^2/(2u) + (3/2)u^3 + 4tu^2 + 2(t^2-\alpha)u + \beta/u$. Since u has first order movable pole singularities, we introduce the change of dependent variables $u \leftrightarrow \theta$:

$$u = u_0\theta^{-1} + u_1, \tag{2.1}$$

where $u_j = u_j(t)$ are to be determined. We would have introduced θ by $u = \theta^{-1}$ according to [4, 5]. However, the inclusion of u_0 and u_1 here adds flexibility that will become useful in subsequent linearization. We call this change of variables as an *indicial normalization*. Then we expect the following expansions

$$\begin{aligned} \theta' &= \theta_0 + \theta_1\theta + \theta_2\theta^2 + \cdots, \\ v &= \theta^{-2}(v_0 + v_1\theta + v_2\theta^2 + \cdots). \end{aligned}$$

As in the Painlevé test, the coefficients θ_i and v_i can be determined by a recursive relation obtained by formally substituting these expansions into the differential system. It results in the existence of an arbitrary constant (called resonance parameter), namely r, in the coefficients in v. We then truncate the θ-series of v at the (first) location of r by introducing the new dependent variable $\xi : v = \theta^{-2}\left(v_0 + v_1\theta + v_2\theta^2 + \xi\theta^3\right)$. Accompany the truncated θ-series of v with the indicial normalization (2.1), we therefore obtain a specific change of variables $(u, v) \leftrightarrow (\theta, \xi)$:

$$\begin{aligned} u &= \frac{u_0}{\theta} + u_1, \\ v &= -\frac{\epsilon u_0^2}{\theta^2} - \frac{2\epsilon u_0(u_1+t)}{\theta} - 2 + 2\alpha\epsilon - \epsilon u_1(u_1+2t) + \xi\theta, \end{aligned}$$

where $\epsilon^2 = 1$. We call this transformation as a *mirror transformation*. Now, the original system for (u, v) is readily converted into a system for (θ, ξ). This is called a *mirror system* which is shown to be regular (see [4, 5, 6] for regularity). Particularly, the equation for θ' is

$$\theta' = \epsilon u_0 + \left(2\epsilon(u_1+t) + \frac{u_0'}{u_0}\right)\theta + \left(\frac{2-2\alpha\epsilon+\epsilon u_1(u_1+2t)+u_1'}{u_0}\right)\theta^2 - \frac{\xi}{u_0}\theta^3. \quad (2.2)$$

Next, we try to linearize the mirror system. Specifically, we will choose u_0 and u_1 so that (2.2) becomes a linearizable equation and ξ satisfies an algebraic equation. Naturally we postulate that (2.2) should be a Riccati equation:

$$\theta' = \epsilon u_0 + \left(2\epsilon(u_1+t) + \frac{u_0'}{u_0}\right)\theta + \left(\frac{2-2\alpha\epsilon+\epsilon u_1(u_1+2t)+u_1'}{u_0} + h\right)\theta^2, \quad (2.3)$$

where $h(t)$ is a function to be determined. By comparing (2.2) and (2.3), we may compute $\xi(t)$. Substituting the formula for ξ into the equation for ξ' in the mirror system, we obtain an equation

$$E_0 + E_1\theta + E_2\theta^2 = 0, \quad (2.4)$$

where

$$\begin{aligned} E_0 &\equiv \epsilon u_0^3 h, \\ E_1 &\equiv 2\epsilon u_0^2 u_1 h - u_0(u_0 h)', \\ E_2 &\equiv -2(\alpha-\epsilon)^2 - \beta + u_0 h\left(-2+\epsilon(2\alpha+u_1^2)\right) - (u_0 h)^2/2 - u_1(u_0 h)'. \end{aligned}$$

Previous attempts could not overcome the major difficulty caused by assuming that each coefficient of the expansion in powers of the singularity function should vanish. The results for ODEs in that case invariably led to special solutions (parabolic cylinder function equation here, for example), rather than transformations. To overcome this difficulty, we relax the constraints $E_j = 0$ by treating the equation (2.4) as a whole and need to make sure that this algebraic equation (2.4) for θ is compatible with the differential equation (2.3) (or (2.2)). The compatibility condition is obtained by eliminating θ from the two equations. If we take $u_0 = \epsilon$, as suggested by the Painlevé test, take u_1 to be a solution of (P4) with parameters A, B, and define a function $s(t)$ by $h = s - (2(\epsilon-\alpha) + u_1(u_1+2t) + \epsilon u_1')$, then the compatibility equation has the form

$$\sum_{j=0}^{12} P_j u_1^j = 0,$$

where $P_j = P_j(u_1', t, s, s', s'', s''', \alpha, \beta, A, B)$ are polynomials.

By only assuming that the coefficient function P_{12} of the expansion in powers of u_1 (not singularity function) should vanish, we obtain the following in a descending order

$$\begin{aligned} O(u_1^{12}): &\quad s = 2(A-\alpha)/3, \\ O(u_1^{11}): &\quad \text{identity}, \\ O(u_1^{10}): &\quad 3(B-\beta) = 2(A-\alpha)(A+\alpha-2\epsilon), \\ O(u_1^{9}): &\quad \text{identity}, \\ O(u_1^{8}): &\quad \text{identity}, \\ O(u_1^{7}): &\quad 9\beta + 2(\alpha+2A-3\epsilon)^2 = 0, \\ O(u_1^{j}): &\quad \text{identity}, \quad \text{for } j \le 6. \end{aligned}$$

Then the equation (2.3) becomes

$$\theta' = 1 + 2\epsilon(u_1 + t)\theta + \frac{2}{3}(A - \alpha)\theta^2, \tag{2.5}$$

and the equation (2.4) becomes

$$\hat{E}_0 + \hat{E}_1\theta + \hat{E}_2\theta^2 = 0, \tag{2.6}$$

where

$$\begin{aligned}
\hat{E}_0 &\equiv \left[\frac{2}{3}(A + 2\alpha - 3\epsilon) - 2tu_1 - u_1^2 - \epsilon u_1'\right] u_1, \\
\hat{E}_1 &\equiv -\frac{2\epsilon}{9}(A + 2\alpha - 3\epsilon)^2 + \frac{2\epsilon}{3}(A + 2\alpha - 3\epsilon)\, u_1^2 \\
&\quad -\frac{4}{3}(A - \alpha)u_1^2 + 2\epsilon t^2 u_1^2 - \frac{\epsilon}{2}u_1^4 + 2tu_1u_1' + \frac{\epsilon}{2}{u_1'}^2, \\
\hat{E}_2 &\equiv \frac{2}{9}(A - \alpha)\left[2(A + 2\alpha - 3\epsilon)^2 u_1 + 3\epsilon u_1 u_1' + 6tu_1^2 - 3u_1^3\right].
\end{aligned}$$

Substituting the indicial normalization $u = \epsilon\theta^{-1} + u_1$ into (2.6), we obtain a quadratic equation for u, with the non-trivial solution being the well-known Schlesinger transformation

$$u - u_1 = \frac{4(\alpha - A)u_1}{\epsilon(3u_1' + 6) + 3u_1^2 + 6tu_1 - 2A - 4\alpha}, \tag{2.7}$$

where A, B are in terms of α, β by

$$9\beta + 2(\alpha + 2A - 3\epsilon)^2 = 0, \quad 9B + 2(A + 2\alpha - 3\epsilon)^2 = 0$$

(the trivial solution is $u = u_1$ and $\alpha = A$). The inverse transformation

$$u - u_1 = \frac{4(\alpha - A)u}{\epsilon(3u' + 6) + 3u^2 + 6tu - 2\alpha - 4A} \tag{2.8}$$

follows from the elimination of u_1' between (2.7) and

$$\epsilon(u - u_1)' + u^2 - u_1^2 + 2t(u - u_1) + 2(A - \alpha)/3 = 0,$$

which is obtained from (2.5) by the elimination of θ from the indicial normalization. The Schlesinger transformation is therefore given by (2.7) and (2.8), for which u and u_1 satisfy (P4) with parameters (α, β) and (A, B), respectively.

Another interesting transformation is worth mentioning here.

By using the indicial normalization $u = \epsilon\theta^{-1} + u_1$ to eliminate u in (2.7) and then using (2.5) to eliminate u_1, we have a second equation for θ:

$$\theta'' = \frac{\theta'^2}{2\theta} + \frac{2}{3}(A - \alpha)^2\theta^3 + \frac{8\epsilon t}{3}(A - \alpha)\theta^2 + 2\left(t^2 - \epsilon + A + \alpha\right)\theta - \frac{1}{2\theta}.$$

This equation is actually (P4) for $(\bar{\theta}, t, \bar{\alpha}, \bar{\beta})$, where

$$\begin{aligned}
\bar{\theta} &= \frac{2\epsilon}{3}(A - \alpha)\theta, \\
\bar{\alpha} &= \epsilon - A - \alpha, \qquad \bar{\beta} = -\frac{2}{9}(A - \alpha)^2.
\end{aligned}$$

The transformation between the two copies of (P4) is obtained by eliminating u_1 between the indicial normalization and (2.5), which reads $u = \left(2(A-\alpha)+3\epsilon\bar{\theta}'\right)/(6\bar{\theta}) - \bar{\theta}/2 - t$. The inverse transformation follows immediately from (2.8). In other words, we have the following Schlesinger transformation between two copies of (P4):

$$u = \frac{2(\epsilon-\bar{\alpha}-2\alpha)+3\epsilon\bar{\theta}'}{6\bar{\theta}} - \frac{\bar{\theta}}{2} - t, \qquad \bar{\theta} = \frac{2(-\epsilon-\alpha-2\bar{\alpha})-3\epsilon u'}{6u} - \frac{u}{2} - t,$$

where u and $\bar{\theta}$ satisfy $\mathbf{P_{IV}}(u,t;\alpha,\beta)=0$ and $\mathbf{P_{IV}}(\bar{\theta},t;\bar{\alpha},\bar{\beta})=0$, respectively. The parameters are related by

$$9\beta + 2(\alpha+2\bar{\alpha}+\epsilon)^2 = 0, \qquad 9\bar{\beta} + 2(\bar{\alpha}+2\alpha-\epsilon)^2 = 0.$$

This transformation was obtained by Fokas and Ablowitz [2].

3 The modified Korteweg-de Vries equation

The idea of linearizing mirror systems can be used equally well for PDEs in finding the Bäcklund transformations. In this section, we demonstrate this through the modified Korteweg-de Vries equation (m-KdV)

$$\mathbf{mKdV}(u) \equiv u_t + \left(u_{xx} - 2\alpha^{-2}u^3\right)_x = 0.$$

Similar to the (P4), we rewrite it as the system $u_x = v$, $v_x = w$, $w_x = 6\alpha^{-2}u^2v - u_t$, introduce the indicial normalization $u = u_0\theta^{-1} + u_1 + u_2\theta$, and deduce the mirror transformation $(u,v,w) \leftrightarrow (\theta,\xi,\eta)$:

$$\begin{aligned}
u &= \frac{u_0}{\theta} + u_1 + u_2\theta, \\
v &= -\frac{\epsilon u_0^2}{\alpha}\theta^{-2} - \frac{2\epsilon u_0 u_1}{\alpha}\theta^{-1} + \left(\frac{\alpha^2\theta_t}{2u_0} - \frac{2\epsilon u_0 u_2}{\alpha} - \frac{\epsilon u_1^2}{\alpha}\right) + \xi\theta, \\
w &= \frac{2u_0^3}{\alpha^2}\theta^{-3} + \frac{6u_0^2 u_1}{\alpha^2}\theta^{-2} + \left(\frac{6u_0 u_1^2}{\alpha^2} + \frac{6u_0^2 u_2}{\alpha^2} - \epsilon\alpha\theta_t\right)\theta^{-1} \\
&\quad + \left(\frac{2u_1^3}{\alpha^2} + \frac{10u_0u_1u_2}{\alpha^2} - \frac{\epsilon u_0\xi}{\alpha} + \frac{\epsilon\alpha u_{0t}}{2u_0}\right) + \eta\theta,
\end{aligned}$$

where $\epsilon^2 = 1$. The sign parameter ϵ characterizes two principal balances of Laurent series for θ_x, v and w. The mirror transformation converts the m-KdV equation into a regular system, in which the equation for θ_x is

$$\begin{aligned}
\theta_x &= (u_0 - u_2\theta^2)^{-1}\left[\frac{\epsilon u_0^2}{\alpha} + \left(\frac{2\epsilon u_0 u_1}{\alpha} + u_{0x}\right)\theta \right. \\
&\quad \left. + \left(\frac{\epsilon u_1^2}{\alpha} + \frac{2\epsilon u_0 u_2}{\alpha} - \frac{\alpha^2\theta_t}{2u_0} + u_{1x}\right)\theta^2 + (u_{2x} - \xi)\,\theta^3\right].
\end{aligned} \tag{3.1}$$

For small θ, the right-hand side of the last equation has the following expansion

$$\frac{\epsilon u_0}{\alpha} + \left(\frac{2\epsilon u_1}{\alpha} + \frac{u_{0x}}{u_0}\right)\theta + \left(\frac{\epsilon u_1^2}{\alpha u_0} + \frac{3\epsilon u_2}{\alpha} - \frac{\alpha^2\theta_t}{2u_0^2} + \frac{u_{1x}}{u_0}\right)\theta^2 + \cdots.$$

Thus we postulate θ to satisfy a Riccati equation of the following form

$$\theta_x = \frac{\epsilon u_0}{\alpha} + \left(\frac{2\epsilon u_1}{\alpha} + \frac{u_{0x}}{u_0}\right)\theta + \left(\frac{\epsilon u_1^2}{\alpha u_0} + \frac{3\epsilon u_2}{\alpha} + \frac{u_{1x}}{u_0} + h\right)\theta^2, \tag{3.2}$$

where h is a function to be determined. By comparing the difference between (3.1) and (3.2), we may compute the θ-series for ξ. Substituting this θ-series into the equation for ξ_x in the mirror system, we may compute the θ-series for η. Further substituting the θ-series for ξ and η into the equation for η in the mirror system, we may compute the θ-series for θ_t:

$$\begin{aligned}\theta_t &= -\frac{2u_0^2 h}{\alpha^2} + \left(\frac{-4u_0u_1h}{\alpha^2} + \frac{u_{0t}}{u_0} + \frac{2\epsilon(u_0h)_x}{\alpha}\right)\theta \\ &\quad + \left(\frac{-2\epsilon u_0 h^2}{\alpha} - \frac{2u_1^2h}{\alpha^2} - \frac{6u_0u_2h}{\alpha^2} + \frac{u_{1t}}{u_0} + \frac{2\epsilon u_1(u_0h)_x}{\alpha u_0} - \frac{(u_0h)_{xx}}{u_0}\right)\theta^2 + \cdots.\end{aligned}$$

Motivated by what happened generally for the behaviour of same order derivatives for PDEs, we would like to have θ_t also satisfying a linearizable equation. Thus we postulate

$$\begin{aligned}\theta_t &= -\frac{2u_0^2 h}{\alpha^2} + \left(\frac{-4u_0u_1h}{\alpha^2} + \frac{u_{0t}}{u_0} + \frac{2\epsilon(u_0h)_x}{\alpha}\right)\theta \\ &\quad + \left(\frac{-2\epsilon u_0 h^2}{\alpha} - \frac{2u_1^2h}{\alpha^2} - \frac{6u_0u_2h}{\alpha^2} + \frac{u_{1t}}{u_0} + \frac{2\epsilon u_1(u_0h)_x}{\alpha u_0} - \frac{(u_0h)_{xx}}{u_0} + g\right)\theta^2,\end{aligned} \tag{3.3}$$

by introducing a function $g(t)$. In order for (3.2) and (3.3) to be compatible, we need the following to vanish

$$\begin{aligned}(\theta_x)_t - (\theta_t)_x &\equiv -\frac{2\epsilon u_0 g}{\alpha}\theta + \left(\frac{(u_0h)_t}{u_0} + \frac{3\epsilon(u_0u_2)_t}{\alpha u_0} - \frac{2\epsilon u_1 g}{\alpha} - \frac{(u_0g)_x}{u_0}\right. \\ &\quad \left. + \frac{6\epsilon h(u_0h)_x}{\alpha} + \frac{12u_2(u_0h)_x}{\alpha^2} + \frac{6h(u_0u_2)_x}{\alpha^2} + \frac{(u_0h)_{xxx}}{u_0}\right)\theta^2.\end{aligned}$$

Now we try to choose appropriate u_0, u_1, g, h so that the compatibility condition is satisfied. As suggested by the Painlevé test, we choose $u_0 = \epsilon\alpha$. Then we simply choose $g = 0$, introduce $s(x,t)$ by $h = s - \epsilon u_0^{-1}u_1^2/\alpha - 3\epsilon u_2/\alpha - u_0^{-1}u_{1x}$ and obtain

$$\begin{aligned}(\theta_x)_t - (\theta_t)_x &= \theta^2\left[-\frac{2u_1}{\alpha^2}\mathbf{mKdV}(u_1) - \frac{\epsilon}{\alpha}\left(\mathbf{mKdV}(u_1)\right)_x\right. \\ &\quad + 6\left(u_2 - \epsilon\alpha s\right)\left(\frac{u_{1xx}}{\alpha^2} + \frac{2\epsilon u_1u_{1x}}{\alpha^3}\right) + s_t + s_{xxx} - \frac{3\epsilon u_{2xxx}}{\alpha} \\ &\quad \left. + 6\,s_x\left(s - \frac{u_1^2}{\alpha^2} - \frac{\epsilon u_2}{\alpha} - \frac{\epsilon u_{1x}}{\alpha}\right) + 12u_{2x}\left(\frac{u_{1x}}{\alpha^2} - \frac{\epsilon s}{\alpha} + \frac{\epsilon u_1^2}{\alpha^3}\right)\right].\end{aligned}$$

A simple choice for the above to vanish is

$$\mathbf{mKdV}(u_1) = 0, \quad u_2 - \epsilon\alpha s = 0, \quad s_t = 0, \quad s_x = 0.$$

In summary, with the following choices

(i) $u_0 = \epsilon\alpha$;

(ii) u_1 is a solution of the m-KdV equation;

(iii) $u_2 = -\epsilon\alpha\lambda^2$, with λ a constant;

(iv) $g = 0$;

(v) $h = 2\lambda^2 - u_2/\alpha^2 - \epsilon u_{1x}/\alpha$,

we have the following compatible system

$$\left\{\begin{array}{rcl} \theta_x & = & 1 + \dfrac{2\epsilon u_1}{\alpha}\theta - \lambda^2\theta^2, \\ \theta_t & = & -4\lambda^2 + \dfrac{2u_1^2}{\alpha^2} + \dfrac{2\epsilon u_{1x}}{\alpha} \\ & & + \left(-\dfrac{8\epsilon\lambda^2 u_1}{\alpha} + \dfrac{4\epsilon u_1^3}{\alpha^3} - \dfrac{2\epsilon u_{1xx}}{\alpha}\right)\theta + \left(4\lambda^4 - \dfrac{2\lambda^2 u_1^2}{\alpha^2} + \dfrac{2\epsilon\lambda^2 u_{1x}}{\alpha}\right)\theta^2. \end{array}\right. \tag{3.4}$$

Now, in contrast to the treatment of ODEs, in the last system, we relax the condition on the function u_1 to allow it arbitrary instead of a solution of the modified KdV equation. Then the system is naturally not compatible in general. In fact, we have

$$(\theta_x)_t - (\theta_t)_x = \frac{2\epsilon}{\alpha}\theta \cdot \mathbf{mKdV}(u_1).$$

This indicates that we can obtain the auto-Bäcklund transformation. Indeed, the indicial normalization $u = \epsilon\alpha\theta^{-1} + u_1 - \epsilon\alpha\lambda^2\theta$ and the first equation in (3.4) imply

$$u + u_1 = \frac{\epsilon\alpha\theta_x}{\theta}.$$

Write $u = U_x$ and $u_1 = U_{1x}$. We can solve θ from the last equation. A particular solution is given by

$$\theta = \lambda^{-1}\exp\left[\epsilon\alpha^{-1}(U + U_1)\right].$$

Substituting this into system (3.4) and simplifying, we have the Bäcklund transformation

$$\left\{\begin{array}{rcl} (U - U_1)_x & = & -2\alpha\lambda\sinh\left[\alpha^{-1}(U + U_1)\right], \\ (U - U_1)_t & = & -8\lambda^2 U_{1x} + 4\lambda U_{1xx}\cosh\left[\alpha^{-1}(U + U_1)\right] \\ & & + \left(8\alpha\lambda^3 - 4\alpha^{-1}\lambda U_{1x}^2\right)\sinh\left[\alpha^{-1}(U + U_1)\right], \end{array}\right. \tag{3.5}$$

where U and U_1 are two solutions of the potential m-KdV equation $U_t + U_{xxx} - 2\alpha^{-2}U_x^3 = 0$. The symmetric form of (3.5) was given by [10]. In terms of a new dependent variable $f = \tanh\left[(2\alpha)^{-1}(U + U_1)\right]$, the x-derivative equation of (3.5) takes the form

$$f_x - \left(\frac{u_1}{\alpha}\right)(1 - f^2) = -2\lambda f.$$

Substituting $f = v_2/v_1$, we obtain the coupled pair of linear differential equations:

$$\begin{array}{rcl} v_{1x} - \lambda v_1 & = & \alpha^{-1}u_1 v_2, \\ v_{2x} + \lambda v_2 & = & \alpha^{-1}u_1 v_1. \end{array}$$

Similarly we obtain from the t-derivative equation of (3.5)

$$
\begin{aligned}
v_{1t} &= Av_1 + Bv_2, \\
v_{2t} &= Cv_1 - Av_2,
\end{aligned}
$$

where

$$
\begin{aligned}
A &= -4\lambda^3 + 2\alpha^{-2}\lambda u_1^2, \\
B &= -4\alpha^{-1}\lambda^2 u_1 - 2\alpha^{-1}\lambda u_{1x} + 2\alpha^{-3}u_1^3 - \alpha^{-1}u_{1xx}, \\
C &= -4\alpha^{-1}\lambda^2 u_1 + 2\alpha^{-1}\lambda u_{1x} + 2\alpha^{-3}u_1^3 - \alpha^{-1}u_{1xx}.
\end{aligned}
$$

4 Conclusions

The recent discovery of the mirror systems for integrable equations has provided a new tool to study integrability. It has been proved rigorously the equivalence between passing the Painlevé test and being regular for the mirror systems. The regularity indicates that integrable equations are linear near their movable poles. Extension of this local result to the global region is the key technical step to understand integrability.

In this paper, we use classical integrable equations to demonstrate that the Bäcklund transformations as well as the Schlesinger transformations are natural consequences of the linearizable mirror systems. The crucial observation here for these examples is that the θ function, defined explicitly by the indicial normalization, satisfies a linearizable equation.

Although it is not clear how many terms needed *a priori* for recovering the transformations, our investigations reveal that the θ functions through different indicial normalizations always yield some interesting results. The function could play an important role in understanding the global structures of integrability through the local Painlevé analysis.

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